Imprints of the Frequency-Domain Source Function on Black Hole Ringdown

Aleyna Koro
Northeastern University, Boston, MA, USA

Mentors: Andrew Laeuger and Colin Weller
LIGO Laboratory, California Institute of Technology, Pasadena, CA, USA

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Most black hole ringdown studies treat the driving source as known and use the signal to measure the mass and spin of the final black hole. This project explores whether the ringdown can reveal information about that source itself. Using black hole perturbation theory, the frequency-domain Green's function for a rotating black hole from the radial Teukolsky equation is constructed and solved in the numerically stable Generalized Sasaki–Nakamura form. The Green's function is validated with a Dirac-delta source, for which the expected relationship between source position and ringdown onset is reproduced. Two types of quasinormal-mode filters are implemented: a full filter that removes all modes and a rational filter that selectively removes chosen modes. These behave as expected, cleanly suppressing the targeted portions of the signal. The method is extended to a Gaussian source, producing both a prompt response and a ringdown. Unexpectedly, the prompt response timing shifts with source position, a behavior that is not yet understood. This framework establishes the tools needed to test whether filtered ringdown waveforms can isolate the imprint of the source, with the goal of extending the analysis to more realistic astrophysical perturbations.

I. INTRODUCTION

A. Background

When two black holes merge, they heavily distort spacetime and this disturbance creates ripples in spacetime that propagate outwards as gravitational waves. General relativity(GR) predicts that a binary black hole merger will produce a gravitational wave signal with three phases: inspiral, merger, and ringdown. The ringdown is the final stage where the remnant black hole settles down into its final state via oscillations known as quasinormal modes (QNMs), which dominate this phase. However, additional components such as power-law tails may appear at late times. Like a plucked guitar string vibrating, the black hole quasinormal oscillations occur with specific frequencies and decay times. These QNM frequencies are completely determined by the final black hole's mass and spin, which is consistent with the no hair theorem [7, 9, 10]. When a source perturbs a black hole, the resulting radiation includes both a prompt response then the ringdown after. The prompt response, also known as the precursor, is the initial burst of radiation that travels directly from the source to the observer, while the ringdown arises from radiation that backscatters off the black hole's curved spacetime geometry (the effective potential barrier) before reaching the observer [11, 14].

The first observation of gravitational waves came from a binary black hole merger on September 14, 2015 (GW150914) [1]. This detection confirmed the existence of the ringdown phase through detailed modeling of the waveform. Since then, several studies have focused specifically on analyzing the ringdown of GW150914, using it to test general relativity and extract the properties of the remnant black hole [3, 4, 6]. Such analysis often model the ringdown using black hole perturbation theory where the remnant is treated as a perturbed Kerr spacetime and the gravitational radiation is described by the Teukolsky equation [19]. Sizheng Ma et al. [13] introduced frequency-domain filters designed to aid in the analysis of black hole ringdown signals. Two filters are considered in their work: the rational filter, which targets specific quasinormal modes, and the full filter, which removes the entire QNM content of the signal.

B. Motivation

In most ringdown analysis, the focus was on inferring the remnant black hole's properties[5], understanding how the QNMs are excited[16], and testing GR [1, 5]. These analysis typically assume a known source of perturbation from numerical simulations, such as a plunging particle or a merging black hole, and aim to compute the resulting waveform rather than infer details about the source itself. Therefore, the question arises to whether we can extract information about the source itself from the ringdown. Specifically, intermediate sources that are not studied in current literature and are more complicated than the Dirac delta function but less complex than one describing a black hole merger. Extracting the behavior of the source from the ringdown could provide a better understanding of the environmental signature around a black hole and could offer a much stronger test of GR.

Similarly, there has been little investigation into the use of ringdown filters. This study will help determine whether ringdown filters could be a useful tool in ringdown data analysis for improving parameter estimation and testing general relativity.

II. PERTURBATIONS TO ROTATING BLACK HOLES

A. Homogeneous Solutions to the Teukolsky and Sasaki-Nakamura Equations

One such technique for modeling the ringdown waveform is using black hole perturbation theory. The final black hole is treated as a perturbation to the Kerr spacetime metric. The gravitational radiation emitted by this black hole can be described by the Teukolsky equation[19]. The Teukolsky equation is a second-order linear partial differential equation that describes how a perturbation to a rotating black hole evolves. The equation can be decomposed into a radial and angular part shown in equation 1 and 2.

Due to the separability and the symmetries of the Kerr background this reduces the problem to two coupled ordinary differential equations: an angular equation for the spin-weighted spheroidal harmonics and a radial equation for the wave propagation. The angular equation is of Sturm–Liouville type, with the spin-weighted spheroidal harmonics as eigenfunctions and the separation constant λ as the corresponding eigenvalue. The radial equation, supplemented with boundary conditions of purely ingoing waves at the horizon and purely outgoing waves at infinity allows solutions for only a discrete set of complex frequencies ω_{lmn} [15]. These frequencies correspond to the black hole's QNMs.

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right)R=0 \tag{1}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right)+\left(a^2\omega^2\cos^2\theta-\frac{m^2}{\sin^2\theta}-2a\omega s\cos\theta -\frac{2ms\cos\theta}{\sin^2\theta}-s^2\cot^2\theta+s+A\right)S=0 \tag{2}$$

In equation 1, $\Delta = r^2-2Mr+a^2$ is the horizon function of the Kerr metric, where M is the black hole mass and a is its spin parameter. The function $K(r) = (r^2+a^2)\omega - am$, where ω is the Fourier frequency and m is the azimuthal quantum number. The separation constant $\lambda = A + a^2\omega^2 - 2am\omega$ arises from angular separation(A) and is related to the eigenvalue of the spin-weighted spheroidal harmonics(S). The function $R(r)$ is the radial part of the perturbation. s is the spin weight of the perturbation. $s=-2$ is used in later calculations to describe outgoing gravitational radiation.

B. Sourced Solutions

In order to understand the behavior of a source from ringdown, first the sourced waveform must be computed using the Green's function. The Green's function is the response of the black hole spacetime to a source, $S_{lm}(r',\omega)$. The frequency-domain solution $R(r)$ is converted to the time domain via an inverse Fourier transform, yielding $R(t,r)_{\text{inhom}}$, the time-domain Teukolsky radial function. In our case, $R(t,r)_{\text{inhom}}$ is the direct output of the Green's-function inversion. It is not directly the metric perturbation $h_{\mu\nu}$. A metric-reconstruction procedure, such as the Chrzanowski-Cohen-Kegeles(CCK) formalism, is required to obtain the observable gravitational wave strain, $h(t,r)$[2, 8].

$$R(t,r)_{\text{inhom}} = \sum_{l,m}\int\frac{d\omega}{2\pi}\,e^{-i\omega t}\int dr'\,\frac{r'^2+a^2}{\Delta} \times G_{lm}\big(r,r',\omega\big)\,S_{lm}\big(r',\omega\big). \tag{3}$$

Where $dr'\frac{r'^2+a^2}{\Delta}$ is the conversion from tortoise coordinates to Boyd-Lindquist coordinates. The Green's function, $G_l(\omega,r,r')$, can be computed using the following:

$$G_l(\omega,r,r') = \frac{1}{W_l(\omega)}\Big[\theta(r-r')\,R_l^{\text{in}}(\omega,r')\,R_l^{\text{up}}(\omega,r) + \theta(r'-r)\,R_l^{\text{in}}(\omega,r)\,R_l^{\text{up}}(\omega,r')\Big]. \tag{4}$$

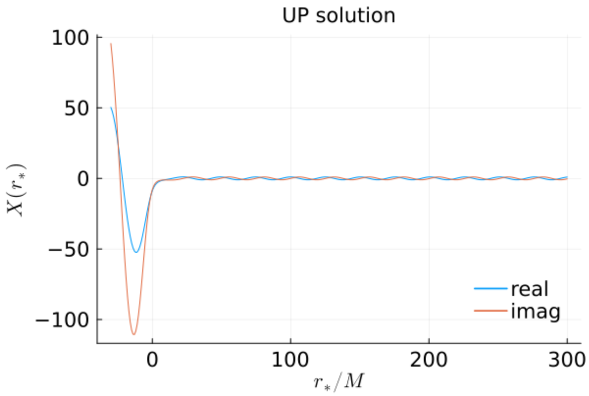
Where θ is the Heaviside function. R^{in} and R^{up} are the radially incoming and outgoing solutions to the Teukolsky equation respectively. These solutions are obtained via Rico K. Lo's package[12] that uses the Generalized Sasaki Nakamura(GSN) formalism to solve the radial part of the homogeneous Teukolsky equation from equation 1 [12]. The GSN formalism transforms the Teukolsky equation into an equation that excludes the case for outgoing radiation. As a result, the equation is more numerically stable as seen in figure 1 where the amplitude in the GSN solution 1a retains a constant amplitude while the Teukolsky solution 1b grows rapidly.

With these solutions the Wronskian, $W_l(\omega)$, can be calculated in two ways:

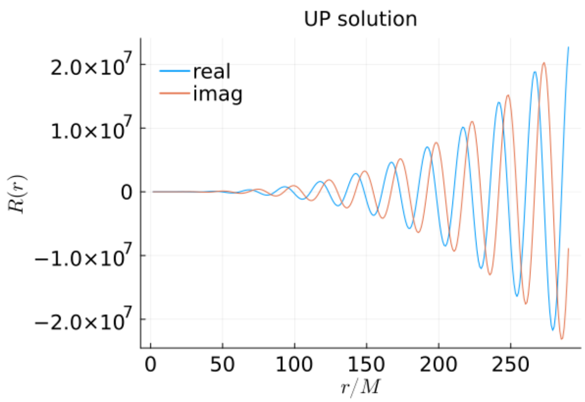
$$\mathcal{W}_R = \Delta^{s+1}\left(R^{\text{in}}\frac{R^{\text{up}}}{dr} - R^{\text{up}}\frac{R^{\text{in}}}{dr}\right) \tag{5}$$

$$\mathcal{W}_R = 2i\omega C_{\text{T}}^{\text{trans}} B_{\text{T}}^{\text{inc}} \tag{6}$$

Where $C_{\text{T}}^{\text{trans}} = 1$. In equation 5, the radial solutions to the Teukolsky equations and their derivatives with respect to r can be used. These derivatives were computed using the ForwardDiff Julia package [17]. In equation 6, the incidence and transmission amplitudes of the incoming and outgoing waves are used. A physical depiction of these amplitudes is shown in figure 2. For ease, in waveform computations equation 6 is used.

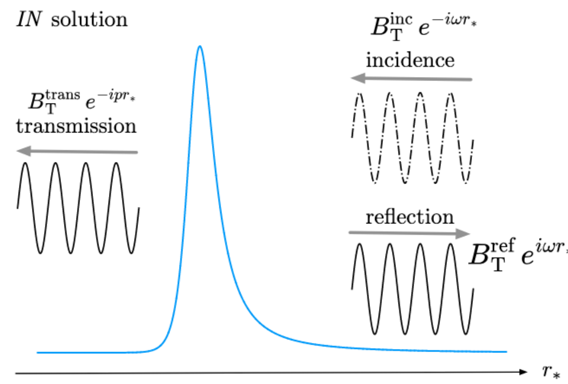


(a) GSN Solution

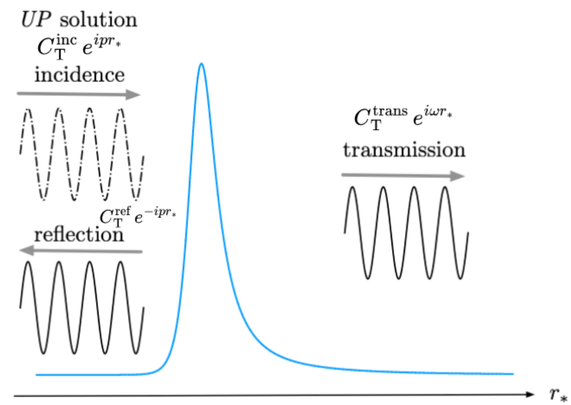


(b) Teukolsky Solution

FIG. 1: GSN and Teukolsky Solution Comparison [12].



(a) IN solution



(b) UP solution

FIG. 2: Amplitudes of the radial Teukolsky solutions for (a) IN and (b) UP solutions [12].

C. Filters

QNM filters are frequency domain filters that help analyze the ringdown of black hole mergers. The two filters are the rational and full filter. As stated in the introduction, the rational filter is built on a set of QNM frequencies of the remnant black hole. The rational filter has the form:

$$\mathcal{F}_{lmn} = \frac{\omega - \omega_{lmn}}{\omega - \omega^*_{lmn}} \tag{7}$$

Where $\omega_{\rm lmn}$ and its complex conjugate $\omega^*_{\rm lmn}$ are the specific QNM frequencies we wish to remove from the waveform. These QNM frequencies correspond to the poles of the Green's function, i.e., values of ω for which $G(r, r', \omega) = 0$, and can be computed using Leo Stein's Python package [18]. The rational filter is applied by multiplying it with the unfiltered frequency-domain solution $R(t, r)_{\rm inhom}$, followed by an inverse Fourier transform. This filtering process allows us to isolate the contribution from other modes, potentially revealing subdominant features in the ringdown that would otherwise be obscured.

The full filter is constructed using the transmissivity of the remnant black hole with the form:

$$\mathcal{F}^D_{lm} = \frac{D^{\rm out}_{lm}}{D^{\rm out\,*}_{lm}} \tag{8}$$

Where $D^{\rm out}_{\rm lm}$ and its complex conjugate $D^{\rm out*}_{\rm lm}$ are the transmissivity amplitude of the outgoing gravitational waveform which is obtained from the radial Teukolsky solutions. The transmissivity is able to be used as a filter because it is 0 at the QNM frequencies, so the transmissivity can cancel out the ringing by zeroing out at the QNM poles. Therefore, the full filter removes the whole set of the QNM frequencies. Instead of using the transmissivity, we use the incidence amplitude of the incoming wave, $B^{\rm inc}_{\rm T}$, since it is also 0 at the QNM frequencies. The full filter is applied the same way as the rational filter.

III. ANALYSIS OF DIRAC DELTA SOURCE FUNCTION WAVEFORMS

To test that the Green function from equation 4 is correct and the $R^{\rm in}$ and $R^{\rm up}$ solutions are indeed homogeneous solutions to the Teukolsky equation, we use a simple source function such as the Dirac delta function $\delta(r-r')$. This serves as a toy model, allowing us to validate the framework before introducing more complex sources. Since $\delta(r-r')$ is 1 only at $r=r'$ and 0 otherwise, this allows us to skip the integration over r in equation 3 and can directly check that the Green's function constructed in equation 9 satisfies the homogeneous Teukolsky equation.

$$G(r,r',\omega)=\frac{R^{\rm in}(r')R^{\rm up}(r)}{W(\omega)} \tag{9}$$

Ensuring that $r>r'$ because we want to see how the localized source at r' affects the waveform at some r, whereas $r'>r$ describes the influence far from the black hole. $R^{\rm in}(r)$ and $W(\omega)$ are constants since we are evaluating at the one r that satisfies the Dirac delta function and the Wronskian is constant for all r. Therefore, we are really checking if R^{up} satisfies the homogeneous Teukolsky equation, which it does.

A. Unfiltered

Using $a=0.7, l=2, m=2, s=-2$, the sourced waveform in the time domain from equation 3 is computed via manual inverse Fourier Transform for a Dirac delta source function. The plots in figure 3 show $R(t,r)$ for a various range of source locations.

Since the source here is modeled as a Dirac delta function which is point-like in space and instantaneous in time, the prompt response is very short lived and the ringdown dominates. When the source is moved farther away from the black hole, we expect the onset of the ringdown to be delayed by the radial shift. This is because the inward-traveling part of the perturbation must first propagate from the source to the near-horizon region to excite the quasinormal modes, and then the resulting radiation must travel back out to the observer. The plots confirm this expectation: for example, shifting the source location from $r'=40$ to $r'=50$ delays the ringdown by 10 in the retarded time ($u=t-r^*$).

B. Filtered

To investigate how specific quasinormal modes contribute to the overall ringdown signal, we apply the filters described in Section II C to the unfiltered waveforms obtained above for a delta source. The goal is to remove the targeted modes from the signal. Two types of filters are considered: the full filter, which is constructed from the transmissivity of the black hole, and the rational filter, which is built from a set of QNMs.

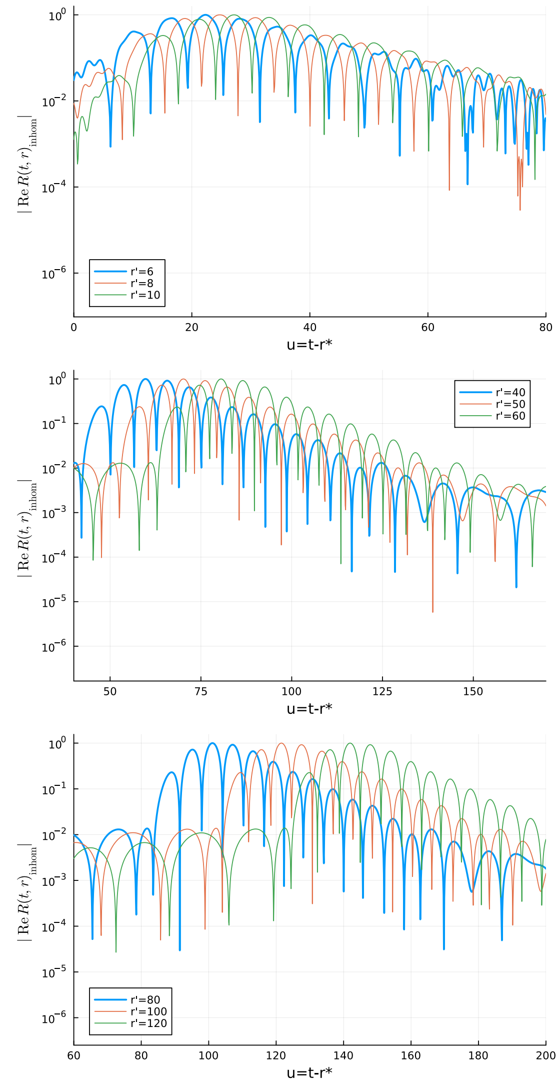


FIG. 3: Comparison of waveforms for various source locations for delta source function.

1. *Full Filter*

Using the unfiltered waveforms computed for $a=0.7, l=2, m=2, s=-2$, we apply the full filter described in equation 8 to remove the full set of QNMs as seen in figure 4.

As expected, applying the full filter removes the full set of QNMs entirely, leaving the waveform zero. The residual signal corresponds to the non-physical time reversed counterpart of the removed QNM, which is a mathematical consequence of the filtering procedure.

2. *Rational Filter*

In figure 5, the rational filter is applied to remove specific quasinormal modes from the delta source waveform.

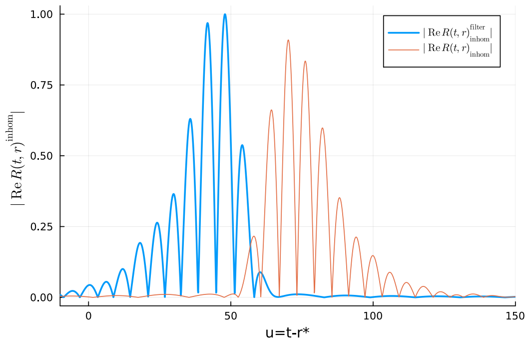


FIG. 4: Full Filter applied to delta sourced waveform for r'=50

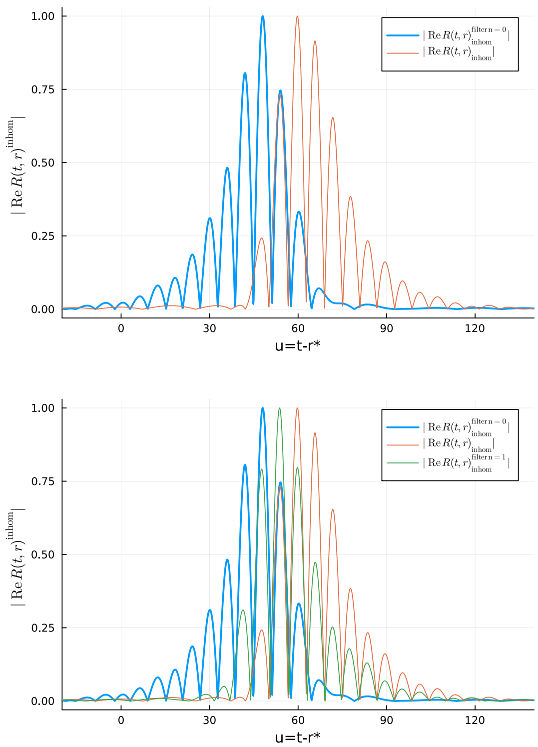


FIG. 5: Rational Filter applied to delta source waveforms for n=0 and n=1 QNM for r'=40

In the top panel, filtering out only the fundamental mode (n=0) suppresses the late-time portion of the signal. This is as expected, since the fundamental mode dominates the end of the ringdown. In the bottom panel, we remove the first overtone (n=1) in addition to the fundamental mode. The resulting waveform shows that the overtone contribution has been effectively suppressed, while the n=0 filtered case remains unaffected in the late-time region. These results confirm that the rational filter correctly targets and removes the chosen QNM from the signal.

IV. ANALYSIS OF EXTENDED SOURCE FUNCTION WAVEFORMS

Having constructed the Green's function and filtering framework for a Dirac delta source function successfully, we now investigate a spatially extended perturbation source. We consider a Gaussian function shown in equation 10.

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(r'-r_0)^2}{\sigma^2}\right) \tag{10}$$

Where r_0 is the center of the Gaussian, r' is the location of the source, and σ is the width of the Gaussian.

A. Unfiltered

For the Gaussian source function, the unfiltered waveforms exhibit a clear prompt response followed by the black hole ringdown. The prompt response appears as the initial burst of radiation, arising from the portion of the perturbation that propagates directly from the source to the observer without significant scattering. The subsequent ringdown is generated by radiation that first travels inward toward the black hole, is scattered by the curvature potential barrier, and then propagates outward to the observer.

As expected, shifting the source location r' produces a corresponding delay in the onset of the ringdown, consistent with the additional travel time for the inward and outward propagating waves. In principle, shifting the source location r' should affect only the arrival time of the ringdown component, since the prompt response propagates directly to the observer and should therefore remain fixed in time for a given observer location. However, in the computed Gaussian source waveforms shown in figure 6, we observe that the prompt response arrival time also shifts with r'. The reason for this behavior is not yet understood. Further investigation is required to determine whether this shift is a physical or a computational issue.

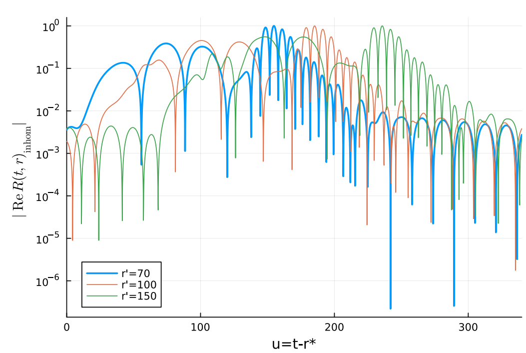


FIG. 6: Comparison of waveforms for various source locations for Gaussian source function.

V. CONCLUSION

In this work so far, we developed a framework to investigate whether information about the perturbing source can be extracted from the black hole ringdown signal. The approach is based on constructing the homogeneous Green's function from the radial Teukolsky equation using the numerically stable GSN formalism, and validating it through consistency checks on the Wronskian. This framework was first applied to a simple Dirac delta source function. As expected, moving the source location delayed the onset of the ringdown by approximately the radial shift in retarded time.

Next, the QNM filters: full filter, which removes the entire QNM spectrum, and the rational filter, which removes specific modes were applied to the delta-source waveforms. The full filter successfully eliminated all QNM content, while the rational filter effectively suppressed the chosen modes and left the rest of the signal untouched. This demonstrated the ability of these tools to isolate subdominant features in the ringdown. Extending this analysis, we use a Gaussian source with finite spatial width. This produced waveforms with a clear prompt response followed by the ringdown. Unexpectedly, the arrival time of the prompt response shifted with the source location, contrary to theoretical expectations for a fixed observer position. This behavior will require further investigation to determine whether it is a physical effect of the extended source model or a numerical issue.

Overall, these results validate the Green's function construction and filtering procedure for simple sources and set the stage for applying the framework to more complex, astrophysical motivated source models.

VI. ACKNOWLEDGMENTS

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