Imprints of the Frequency-Domain Source Function on Black Hole Ringdown

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Most black hole ringdown studies treat the driving source as known and use the signal to measure the mass and spin of the final black hole. This report begins to look at whether the ringdown itself has information about that source. The Green's function is built off of the radial Teukolsky equation using the numerically stable Generalized Sasaki–Nakamura formalism. The Wronskian is evaluated both from the radial solutions and from their asymptotic amplitudes as a consistency check. A Dirac-delta source is used as a test case. The next step will be to introduce more complex source functions and to construct rational and full quasinormal-mode filters that isolate chosen modes. This framework may make it possible to uncover the perturbing source behavior and whether ringdown filters could improve parameter estimation and provide a new test of general relativity.

I. INTRODUCTION

A. Background

When two black holes merge, they heavily distort spacetime and this disturbance creates ripples in spacetime that propagate outwards as gravitational waves. General relativity(GR) predicts that a binary black hole merger will produce a gravitational wave signal with three phases: inspiral, merger, and ringdown. The ringdown is the final stage where the remnant black hole settles down into its final state via oscillations known as quasinormal modes (QNMs), which dominate this phase. However, additional components such as power-law tails may appear at late times. Like a plucked guitar string vibrating, the black hole quasinormal oscillations occur with specific frequencies and decay times. These QNM frequencies are completely determined by the final black hole's mass and spin, which is consistent with the no hair theorem [7].

The first observation of gravitational waves came from a binary black hole merger on September 14, 2015 (GW150914) [1]. This detection confirmed the existence of the ringdown phase through detailed modeling of the waveform. Since then, several studies have focused specifically on analyzing the ringdown of GW150914, using it to test general relativity and extract the properties of the remnant black hole [2, 3, 6].

One such technique for modeling the ringdown waveform is using black hole perturbation theory. The final black hole is treated as perturbation to the Kerr spacetime metric. The gravitational radiation emitted by this black hole can be described by the Teukolsky equation[12]. The Teukolsky equation is a second-order linear partial differential equation equation that describes how a perturbation to a rotating black hole evolves. The equation can be decomposed into a radial and angular part. The radial part is shown in equation 1.

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0 \tag{1}$$

In this equation, $\Delta = r^2 - 2Mr + a^2$ is the horizon function of the Kerr metric, where M is the black hole mass and a is its spin parameter. The function $K(r) = (r^2+a^2)\omega - am$, where ω is the Fourier frequency and m is the azimuthal quantum number. The separation constant λ arises from angular separation and is related to the eigenvalue of the spin-weighted spheroidal harmonics. s is the spin weight of the perturbation. $s = -2$ in later calculations to describe outgoing gravitational radiation (i.e., perturbations in ψ_4). The function $R(r)$ is the radial part of the perturbation. Due to this separation, the waveforms can be decomposed into QNMs which can allow for implementing filters to remove any chosen mode, revealing the subdominant effects. Such filters were proposed by Sizheng Ma et al [9]. They construct a full and rational filter. The full filter is made using the transmissivity of the remnant black hole, while the rational filter is built on a set of QNMs. These filters allow for further analysis of the ringdown including the visualization of existing subdominant effects such as mode mixing, second-order modes, and retrograde modes.

B. Motivation

In most ringdown analysis, the focus was on inferring the remnant black hole's properties[5], understanding how the QNMs are excited[10], and testing GR [1, 5]. These analysis typically assume a known source of perturbation from numerical simulations, such as a plunging particle or a merging black hole, and aim to compute the resulting waveform rather than infer details about the source itself. Therefore, the question arises to whether we can extract information about the source itself from the ringdown. Specifically, intermediate sources that are not studied in current literature and are more complicated than the Dirac delta function but less complex than one describing a black hole merger. Extracting the behavior of the source from the ringdown could provide a better understanding of the environmental signature around a black hole and could offer a much stronger test of GR.

Similarly, there has been little investigation into the use of ringdown filters. This study will help determine whether ringdown filters could be a useful tool in ringdown data analysis for improving parameter estimation and testing general relativity.

II. CURRENT PROGRESS

In order to understand the behavior of a source from ringdown, first the sourced ringdown waveform must be computed using the Green's function. In the wave zone ($r \gg \lambda$), the observed gravitational waveform in equation 2 is related to the second time derivatives of the metric perturbation via $h(t,r) = \ddot{h}_+ - i\ddot{h}_\times$. Here, h_+ and $h_\times$ are the two independent polarizations of the gravitational wave signal, corresponding to the "plus" and "cross" polarization modes predicted by GR. The Green's function is the response of the black hole spacetime to a source, $S_{lm}(r', \omega)$.

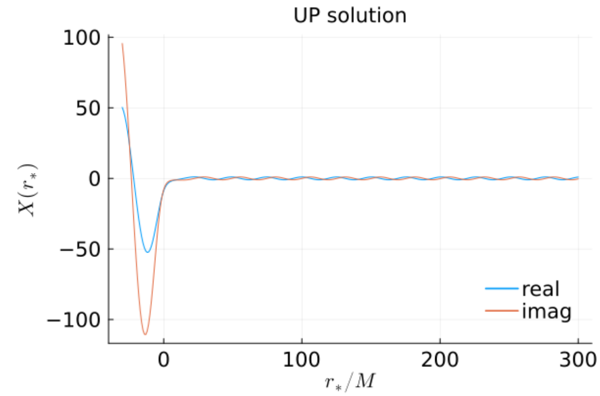
$$h(t,r) = \sum_{l,m} \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int dr'\, \frac{r'^2 + a^2}{\Delta} \times G_{lm}(r, r', \omega)\, S_{lm}(r', \omega). \tag{2}$$

The Green's function can be computed using the following:

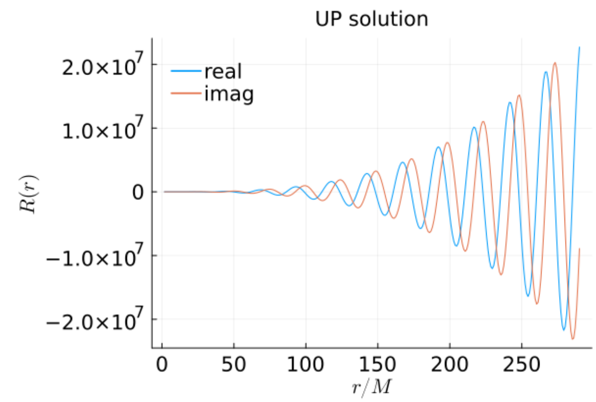
$$G_l(\omega, r, r') = \frac{1}{W_l(\omega)} \Big[\theta(r - r')\, R_l^{\rm in}(\omega, r')\, R_l^{\rm up}(\omega, r) + \theta(r' - r)\, R_l^{\rm in}(\omega, r)\, R_l^{\rm up}(\omega, r') \Big]. \tag{3}$$

Where θ is the Heaviside function. $R^{\rm in}$ and $R^{\rm up}$ are the radially incoming and outgoing solutions to the Teukolsky equation respectively. These solutions are obtained via Rico K. Lo's package that uses the Generalized Sasaki Nakamura(GSN) formalism to solve the radial part of the homogeneous Teukolsky equation from equation 1 [8]. The GSN formalism transforms the Teukolsky equation into an equation that excludes the case for outgoing radiation. As a result, the equation is more numerically stable as seen in figure 1 where the amplitude in the GSN solution 1a retains a constant amplitude while the Teukolsky solution 1b grows rapidly.

I created a table that included R^{in} and R^{up} for various frequencies and coordinates for the possibility of needing to use a different platform to carry out further calculations. Luckily, so far the calculations have been working out well in Julia. With these solution, the Wronskian, $W_l(\omega)$, can be calculated in two ways:



(a) GSN Solution



(b) Teukolsky Solution

FIG. 1: GSN and Teukolsky Solution Comparison [8].

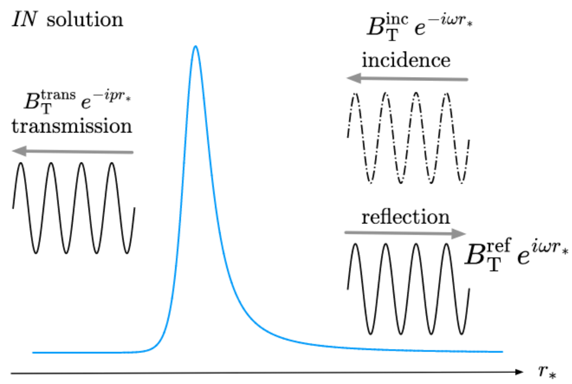
$$\mathcal{W}_R = \Delta^{s+1} \left(R^{\rm in} R^{\rm up'} - R^{\rm up} R^{\rm in'} \right) \tag{4}$$

$$\mathcal{W}_R = 2i\omega C_{\rm T}^{\rm trans} B_{\rm T}^{\rm inc} \tag{5}$$

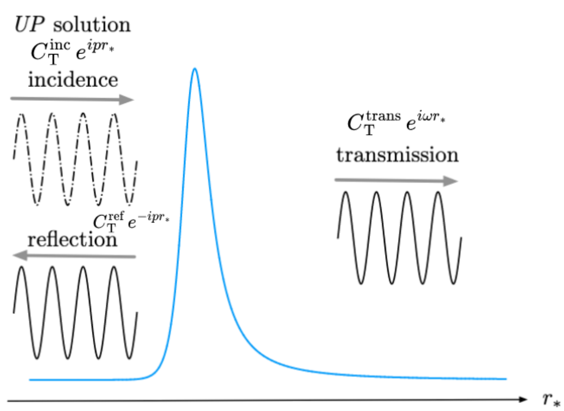
Where $C_T^{trans} = 1$. In equation 4, the radial solutions to the Teukolsky equations and their derivatives with respect to r can be used. These derivatives were computed using the ForwardDiff Julia package [11]. In equation 5, the incidence and transmission amplitudes of the incoming and outgoing waves are used. A physical depiction of these amplitudes is shown in figure 2.

I was able to compute the Wronskian using both equations in Julia successfully using Rico's solutions directly and from my generated table. Finding a way to compute the derivatives from my generated table was a bit tricky as it wasn't a function like Rico's solutions. However, the ForwardDiff package worked well and aligned much better with the results of obtaining the Wronskian directly from Rico's code.

To test that our Green function is correct and the $R^{\rm in}$ and $R^{\rm up}$ solutions are indeed homogeneous solutions to



(a) IN solution



(b) UP solution

FIG. 2: Amplitudes of the radial Teukolsky solutions for (a) IN and (b) UP solutions [8].

the Teukolsky equation, we use a simple source function such as the Dirac Delta function $\delta(r - r')$. This serves as a toy model, allowing us to validate the framework before introducing more complex sources. Since $\delta(r - r')$ is 1 only at $r = r'$ and 0 otherwise, this allows us to skip the integration over r in equation 2 and can directly check that the Green's function constructed in equation 6 satisfies the homogeneous Teukolsky equation.

$$G(r, r', \omega) = \frac{R^{in}(r')R^{up}(r)}{W(\omega)} \tag{6}$$

Ensuring that $r > r'$ because we want to see how the localized source at r' affects the waveform at some r, whereas $r' > r$ describes the influence far from the black hole. $R^{in}(r)$ and $W(\omega)$ are constants since we are evaluating at the one r that satisfies the Dirac delta function and the Wronskian is constant for all r. Therefore, we are really checking if R^{up} satisfies the homogeneous Teukolsky equation, which it does.

I tested a fast Fourier transform(FFT) package [4] in Julia using a simple sine function to ensure everything was working correctly before proceeding to the next stage of the project.

III. OUTLOOK

The work done so far has been focused on setting up the codebase and validating key functions and numerical values required for the next stage of the project. In this upcoming part, I will begin calculating the sourced waveforms described by equation 2. To make sure the FFT package works, I will continue to use the dirac delta function as the source. After that, the goal will be to determine the morphology of more complicated source functions and compute the corresponding sourced waveforms, which I believe will be computationally challenging due to the complexity of the sources. I will also construct the rational and full filters, which requires locating the QNMs and computing the transmissivity of the black hole. I anticipate the full filter to be more challenging because it depends directly on that transmissivity calculation. After applying the filters, I will analyze the filters on the sourced waveform in the frequency domain. A successful outcome would be if the Fourier transform of the filtered waveform resembles that of the Fourier transform of the original source function. Another indicator of success would be observing consistent patterns in how specific features of different source functions appear in the filtered waveforms.

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