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Optical Pathlength Fluctuations in an Interferometer Due to Residual Gas

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Planning is now underway for large gravitational wave antennae to be constucted in the next 5 years. These antennae will use ultra-precise laser interferometers to monitor gravitationally-induced motions of test mass over multikilometer baselines with a precision of 10^{-17} m or better. With such a high precision, these interferometers are sensitive to a number of noise sources which are normally ignored in other applications of optical interferometry. This letter discusses one such noise source, the fluctuations in optical pathlength (equivalent to fluctuations in the index of refraction of the residual gas) due to the statistical variation of the number of atoms in the laser beam. Since the cost of the vacuum system is the major expense associated with the large antennae, it is important to estimate the vacuum requirements as accurately as possible.

Assume that the measurement is to be made along the z axis from $z=0$ to $z=L$ using a laser beam with a gaussian intensity profile (in cylindrical coordinates)

$$I(r,z) = \frac{2I_0}{\pi w(z)^2} \exp\left[-\frac{2r^2}{w(z)^2}\right]$$

where $w(z)$ is the waist parameter and I_0 is the total energy flowing in the beam. In what follows, it is assumed that $w \ll L$ and $\frac{dw}{dz} \ll 1$

The most useful formulation of the problem is to calculate the spectral density of the optical pathlength fluctuations. The optical pathlength $l(t)$ fluctuates about its mean $<l>$ as molecules move across the beam. These fluctuations can be characterized by the spectral noise density ($f \neq 0$)

$$G_l(f) = \lim_{T \to \infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} l(t) \exp(i 2\pi f t) dt \right|^2$$

where $l(t)$ is the optical pathlength as a function of time. $G_l(f)$ has units of $m^2\ Hz^{-1}$.

As shown below, the precision needed for gravitational-wave detection requires gas pressures which yield mean free paths long compared to the transverse dimensions of the beam. Thus the molecules can be considered to cross the beam independently and without collisions. Our approach will be to calculate the effect of a single molecule crossing the beam, as a function of its velocity and position of closest approach to the beam axis, and to subsequently integrate over the equilibrium distribution of velocities and positions to get the spectral density of the optical pathlength fluctuations.

The presence of each molecule in the laser beam has two effects: a small distortion of the phase fronts and a small change in the average phase of the light. The net phase shift is proportional to the number of molecules in the beam, while the wave front distortions grow proportional to the square root of the number of molecules and can thus be ignored. The phase shift due to a particular molecule (and the corresponding addition to the optical pathlength) is proportional to the intensity of the laser light at the position of that molecule divided by the total energy flux of the beam; that is, if the position of the i^{th} molecule is (r_i, z_i), its presence makes a contribution to the optical pathlength

$$\delta l_i = \beta \exp\left[- \frac{2 r_i^2}{w(z_i)^2} \right]$$

where β depends among other things on the type of the molecule. Assuming there is only a single molecular species present, β can be calculated by summing the contributions to the optical pathlength from a uniform distribution of molecules in the beam and comparing the result with that predicted using the classical index of refraction. This procedure gives

$$\beta = \frac{2(n-1)}{\eta_0 \pi w^2} = \frac{2\alpha}{\pi w^2}$$

where η_0 is the number density of molecules, n is the mean index of refraction at that density, and $\alpha = (n-1)/\eta_0$ depends only on the type of molecule.

In fact, the molecules move through the beam randomly and the contribution to the total optical pathlength of the i^{th} molecule is a function of time $\delta l_i(t)$. If the molecule makes its closest approach to the beam axis with impact parameter ρ_i at time t_i and with speed perpendicular to the beam axis v_i, then

$$\delta l_i(t) = \beta \exp\left[-2\,\frac{\rho^2 + (t-t_i)^2 v_i^2}{w(z_i)^2}\right]$$

The total optical pathlength as a function of time is obtained by summing over all molecules

$$l(t) = L + \sum_i \delta l_i(t)$$

where L is the physical pathlength. This yields a spectral density ($f \neq 0$)

$$G_l(f) = \lim_{T\to\infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} \sum_i \delta l_i\,(t) \exp(i2\pi f t) dt \right|^2$$

$$= \lim_{T\to\infty} \frac{2}{T} \sum_i \left| \int_{-T/2}^{T/2} \delta l_i\,(t) \exp(i2\pi f t) dt \right|^2$$

The cross terms in $G_l(f)$ average to zero because they depend on the transit times of unrelated molecules. The individual integrals are straightforward. The remaining sum and limit may be evaluated using the distribution function for ρ and v.

$$G_l(f) = 2\int\int\int \eta(\rho,v) \left| \int_{-\infty}^{\infty} \delta l(\rho,v) \exp(i2\pi f t)\, dt \right|^2 d\rho\, dv\, dz$$

where $\eta(\rho,v)d\rho\, dv$ is the number of particles per unit length per second making their closest approach to the beam axis with impact parameter ρ and perpendicular speed v. Assuming a Maxwell-Boltzmann velocity distribution, η is given by

$$\eta(\rho,v)\, d\rho\, dv\, dz = \frac{4\eta_0 v^2}{v_0^2} \exp\left[-\frac{v^2}{v_0^2}\right] d\rho\, dv\, dz$$

where $v_0 = (2kT/m)^{1/2}$. Integrating over ρ and v gives the result

$$G_l(f) = \int_0^L \frac{2\eta_0 \alpha^2}{v_0 w(z)} \exp\left[-\frac{2\pi f w(z)}{v_0}\right] dz \tag{1}$$

As a practical example of the usage of (1), consider a gravitational wave antenna with two 5 km long arms. A reasonable goal for the strain sensitivity is $\Delta l / l \leq 3\times10^{-23}\ \mathrm{Hz}^{-1/2}$ at a frequency of 1 kHz. One possible configuration for the gravitational wave interferometer uses a pair of optical cavities for the two arms. The strain noise of such an interferometer due to the gas

$$\frac{\Delta l}{l} = \frac{[2G_l(f)]^{1/2}}{L}$$

is found by looking at the (incoherent) difference between the two arms. The most economical design, in terms of total evacuated volume and mirror size, is a cavity configuration near confocal. In this case the beam has a roughly constant diameter $w \approx (\lambda L / 2\pi)^{1/2}$. For an antenna with 5 km long arms operating at

$\lambda = 500\,\mathrm{nm}$, a sensitivity of $3\times10^{-23}\ \mathrm{Hz}^{-1/2}$ at 1 kHz requires a density $<2\times10^{11}\mathrm{cm}^{-3}$ (assuming the residual gas is N_2), corresponding to a pressure $<6\times10^{-6}$ torr. In some vacuum systems, more than 95% of the residual gas is hydrogen. If this is the case, then the pressure could be higher by a factor of ~10 because of the lower refractive index and higher speed of hydrogen.

An alternative configuration for the interferometer is a Michelson interferometer using Herriott delay lines in the two arms. The geometry of the Heriott delay line is such that the b one-way paths through the arms are nearly independent (non-overlapping) with a minimum beam diameter approximately the same as for the optical cavities discussed above. It follows that the strain sensitivity of such an interferometer is given by

$$\frac{\Delta l}{l} = \frac{[2b\,G_1(f)]^{1/2}}{bL}$$

The pressure required for a delay line is a factor of b higher than for an interferometer using optical cavities. This result is simple to understand: since the delay line uses a b times larger volume of gas than the cavity, it can have a b times higher density to get the same fractional fluctuations in the column density.