Note on the Hamiltonian, gravitational modes and equations of motion of the SEOBNRv5 waveform model

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This note is based on one of the papers we are writing on the SEOBNRv5 waveform model. It explains the derivation of the Hamiltonian and PN equations of motion for precessing spins, and lists the gravitational modes used to build the SEOBNRv5HM model. Please keep in mind, when reading this note, that citations are not complete. The note has been written with the only scope of facilitating the review of the SEOBNRv5 model for O4.

NOTATION

We use use units in which $c = G = 1$.

We consider a binary with masses m_1 and m_2, with $m_1 \geq m_2$, and spins $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$. We define the following combinations of the masses:

$$M \equiv m_1 + m_2, \qquad \mu \equiv \frac{m_1 m_2}{M}, \qquad \nu \equiv \frac{\mu}{M}, \qquad \delta \equiv \frac{m_1 - m_2}{M}, \qquad q \equiv \frac{m_1}{m_2}, \qquad X_\mathrm{i} \equiv \frac{m_\mathrm{i}}{M}, \tag{1}$$

where $\mathrm{i} = 1, 2$, and define the dimensionless spin vectors

$$\boldsymbol{\chi}_\mathrm{i} \equiv \frac{\boldsymbol{a}_\mathrm{i}}{m_\mathrm{i}} = \frac{\boldsymbol{S}_\mathrm{i}}{m_\mathrm{i}^2}, \tag{2}$$

along with the intermediate definition for $\boldsymbol{a}_\mathrm{i}$. The spin magnitudes χ_i vary between -1 and 1, with positive spins being in the direction of the angular momentum, and the following combinations of spins:

$$\boldsymbol{a}_\pm \equiv \frac{\boldsymbol{a}_1 \pm \boldsymbol{a}_2}{M} = \frac{m_1}{M}\chi_1 \pm \frac{m_2}{M}\chi_2. \tag{3}$$

Note that, unlike $\boldsymbol{a}_\mathrm{i}$, we define $\boldsymbol{a}_\pm$ to be dimensionless by dividing $\boldsymbol{a}_\mathrm{i}$ by the total mass.

The spin quadrupole, octupole, and hexadecapole constants are denoted C_{iES^2}, C_{iBS^3}, and C_{iES^4}, respectively. These constants equal one for black holes (BHs), but are greater than one for neutron stars (NSs). We define

$$\begin{aligned}
&\widetilde{C}_{1ES^2} \equiv C_{1ES^2} - 1, \qquad \widetilde{C}_{1BS^3} \equiv C_{1BS^3} - 1, \qquad \widetilde{C}_{1ES^4} \equiv C_{1ES^4} - 1,\\
&\widetilde{C}_{2ES^2} \equiv C_{2ES^2} - 1, \qquad \widetilde{C}_{2BS^3} \equiv C_{2BS^3} - 1, \qquad \widetilde{C}_{2ES^4} \equiv C_{2ES^4} - 1,
\end{aligned} \tag{4}$$

such that expressions for BHs can be easily recovered by setting $\widetilde{C}_{\dots} \to 0$. To simplify the expressions for NSs, we define the following combinations of spins and multipole constants:

$$\begin{aligned}
C_\pm^{a^2} &\equiv \widetilde{C}_{1ES^2}\frac{a_1^2}{M^2} \pm \widetilde{C}_{2ES^2}\frac{a_2^2}{M^2},\\
C_\pm^{n\cdot a^2} &\equiv \widetilde{C}_{1ES^2}\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2}{M^2} \pm \widetilde{C}_{2ES^2}\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2}{M^2},\\
C_\pm^{a^3} &\equiv \widetilde{C}_{1BS^3}\frac{a_1^3}{M^3} \pm \widetilde{C}_{2BS^3}\frac{a_2^3}{M^3},\\
C_\pm^{a^4} &\equiv \widetilde{C}_{1ES^4}\frac{a_1^4}{M^4} \pm \widetilde{C}_{2ES^4}\frac{a_2^4}{M^4},
\end{aligned} \tag{5}$$

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which are zero for BHs (see below the definition of $\boldsymbol{n}$).

The relative position and momentum vectors are denoted $\boldsymbol{R}$ and $\boldsymbol{P}$, with

$$\boldsymbol{P}^2 = P_R^2 + \frac{L^2}{R^2}, \qquad P_R = \boldsymbol{n}\cdot\boldsymbol{P}, \qquad \boldsymbol{L} = \boldsymbol{R}\times\boldsymbol{P}, \tag{6}$$

where $\boldsymbol{n} = \boldsymbol{R}/R$, and $\boldsymbol{L}$ is the orbital angular momentum with magnitude L. The total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}_1 + \boldsymbol{S}_2$. For precessing spins, we use the spherical-coordinates phase-space variables $(R, \theta, \phi, P_R, P_\theta, P_\phi)$, where θ is the polar angle, ϕ is the azimuthal angle, and P_ϕ and P_θ are their conjugate momenta. For equatorial orbits (aligned-spins), the angular momentum $L = P_\phi$.

For precessing spins, we consider two frames: one with unit vectors $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{\lambda})$, where $\boldsymbol{l}$ is in the direction of $\boldsymbol{L}$ and $\boldsymbol{\lambda} \equiv \boldsymbol{l}\times\boldsymbol{n}$, while the unit vectors in the other frame are denoted $(\boldsymbol{l}_N, \boldsymbol{n}, \boldsymbol{\lambda}_N)$, where $\boldsymbol{\lambda}_N \equiv \boldsymbol{l}_N \times \boldsymbol{n}$, and $\boldsymbol{l}_N$ is in the direction of $\boldsymbol{L}_N \equiv \mu\boldsymbol{R}\times\mathbf{v}$, where the velocity $\mathbf{v} \equiv \mathrm{d}\boldsymbol{R}/\mathrm{d}T$ with T being the time variable. The orbital frequency is denoted Ω, and we define the dimensionless frequency parameter $v \equiv (M\Omega)^{1/3}$.

We use the rescaled dimensionless variables

$$\begin{gathered} t \equiv \frac{T}{M}, \qquad \boldsymbol{r} \equiv \frac{\boldsymbol{R}}{M}, \qquad u \equiv \frac{1}{r}, \qquad \boldsymbol{p} \equiv \frac{\boldsymbol{P}}{\mu}, \qquad p_r \equiv \frac{P_R}{\mu}, \\ p_\phi \equiv \frac{P_\phi}{M\mu}, \qquad \tilde{\boldsymbol{L}} \equiv \frac{\boldsymbol{L}}{M\mu}, \qquad \tilde{H} \equiv \frac{H}{\mu}, \end{gathered} \tag{7}$$

where we use either a lowercase symbol or a tilde to indicate the dimensionless quantities.

I. HAMILTONIAN

The EOB formalism [1, 2] maps the dynamics of a binary to that of a test mass in a deformed Schwarzschild or Kerr background, with the deformation parameter being the symmetric mass ratio ν. The effective Hamiltonian H_{eff} is related to the two-body EOB Hamiltonian H_{EOB} via the energy map

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}. \tag{8}$$

For nonspinning binaries, in the $\nu \to 0$ limit, the effective Hamiltonian reduces to that of a (nonspinning) test mass in a Schwarzschild background. The nonspinning EOB Hamiltonian was first derived in Refs. [1, 2] with 2PN information, and then extended to 3PN [3] and 4PN [4], with partial information at 5PN [5, 6] and 6PN orders [7, 8].

For spinning binaries, one can map the binary dynamics to that of a test body in a Kerr background. The first spinning EOB Hamiltonian [9] was constructed based on the Hamiltonian for the geodesic motion of a test mass in Kerr spacetime, while including leading-order (LO) spin-orbit (SO) and spin-spin (SS) effects. This was later extended to the next-to-leading (NLO) [10] and next-to-NLO (NNLO) [11] SO levels, in addition to the NLO SS level for aligned [12–14] and precessing spins [15], then to NNLO SS for aligned spins and circular orbits [16]. The complete 4PN conservative dynamics for precessing spins and generic orbits was incorporated in EOB Hamiltonians in Ref. [17], and the 4.5PN SO dynamics in Ref. [18].

Another category of spinning EOB Hamiltonians is based on the Hamiltonian for a *spinning* test-body (test spin) in a Kerr background [19, 20], first developed with NLO SO and LO SS [21], then to NNLO SO [22]. Such a Hamiltonian is applicable for generic (precessing) spins, and has been extended to 4PN in Ref. [17].

The `SEOBNRv2`, `SEOBNRv3`, and `SEOBNRv4` waveform models use an effective Hamiltonian that is a deformation of a test spin in a Kerr background. Here, for `SEOBNRv5`, we take the effective Hamiltonian to be a deformation of the test-mass Kerr Hamiltonian. The masses of the background BH and test mass are identified to be $M = m_1 + m_2$ and $\mu = m_1 m_2/M$, respectively, while the Kerr spin $\boldsymbol{a}_{\text{Kerr}}$ is mapped to be

$$\boldsymbol{a}_{\text{Kerr}} = \boldsymbol{a}_1 + \boldsymbol{a}_2 = M\boldsymbol{a}_+. \tag{9}$$

An advantage of this map is that the Kerr Hamiltonian reproduces all even-in-spin leading PN orders for BHs [23].

To include post-Newtonian (PN) information beyond some leading orders in the EOB Hamiltonian, we write an ansatz for the coefficients of H_{eff} and solve for the unknowns such that H_{EOB} is related to a PN-expanded Hamiltonian in another gauge by a canonical transformation. To obtain that transformation, we write an ansatz for a generating function $\mathcal{G}$, perform the transformation using Poisson brackets

$$H_{\text{EOB}}^{\text{nPN}} = H^{\text{nPN}} + \{\mathcal{G}, H^{\text{nPN}}\} + \frac{1}{2!}\{\mathcal{G}, \{\mathcal{G}, H^{\text{nPN}}\}\} + \ldots, \tag{10}$$

where each bracket introduces a factor of $1/c^2$, and finally match the right and left hand sides of the above equation to solve for the unknown coefficients in $H_{\rm eff}$ and the generating function. (See Ref. [17] for a detailed explanation.)

We include in the Hamiltonian all PN information up to 4PN order for precessing spins, in addition to most of the 5PN nonspinning contributions [6]. We start from the 4PN precessing-spin Hamiltonian in the gauge of Ref. [24], then perform a canonical transformation to EOB coordinates. Thus, our EOB Hamiltonian includes spin-orbit (SO) information up to the next-to-next-to-leading order (NNLO), spin-spin (SS) information to NNLO, as well as cubic- and quartic-in-spin terms at leading orders (LO).

In the following subsections, we write the Kerr Hamiltonian, and by building on it, we construct the effective Hamiltonian, which we first present for nonspinning binaries, then for aligned and precessing spins. In Sec. (II A), we obtain a more computationally efficient precessing-spin Hamiltonian in an orbit average for circular orbits. Table I summarizes the Hamiltonians presented in this paper.

TABLE I. Summary of the Hamiltonians presented in this paper and their relations to each other.

Hamiltonian	description
$H^{\rm Kerr}$, Eq. (11)	Kerr Hamiltonian for a nonspinning test mass in a *generic* orbit
$H^{\rm Kerr,eq}$, Eq. (14)	Kerr Hamiltonian for a nonspinning test mass in an *equatorial* orbit
$H^{\rm Schw}$, Eq. (16)	Schwarzschild Hamiltonian for a nonspinning test mass
$H^{\rm pm}_{\rm eff}$, Eq. (17)	effective Hamiltonian for nonspinning binaries, reduces to $H^{\rm Schw}$ when $\nu \to 0$
$H^{\rm align}_{\rm eff}$, Eq. (26)	effective Hamiltonian for spin-aligned binaries, reduces to $H^{\rm pm}_{\rm eff}$ in the zero-spin limit, reduces to $H^{\rm Kerr,eq}$ when $\nu \to 0$
$H^{\rm prec}_{\rm eff}$, Eq. (35)	effective Hamiltonian for spin-precessing binaries, reduces to $H^{\rm align}_{\rm eff}$ for aligned spins, reduces to $H^{\rm Kerr}$ when $\nu \to 0$
$H^{\rm simp}_{\rm eff}$, Eq. (48)	effective Hamiltonian for precessing-spin binaries, reduces to $H^{\rm align}_{\rm eff}$ for aligned spins, agrees with $H^{\rm prec}_{\rm eff}$ to $\mathcal{O}(S^3)$ when PN expanded for circular orbits

A. Kerr Hamiltonian

The Kerr Hamiltonian can be written in a 3-vector notation as (see Refs. [15, 17] for details)

$$\tilde{H}^{\rm Kerr} = \frac{2r}{\Lambda}\tilde{\boldsymbol{L}}\cdot\boldsymbol{a}_+ + \left[A^{\rm Kerr}\left(1 + B^{\rm Kerr}_p \boldsymbol{p}^2 + B^{\rm Kerr}_{np}(\boldsymbol{n}\cdot\boldsymbol{p})^2 + B^{\rm Kerr}_{npa}(\boldsymbol{n}\times\boldsymbol{p}\cdot\boldsymbol{a}_+)^2\right)\right]^{1/2}, \tag{11}$$

where the first term only contains odd-in-spin contributions, while the square root is the even-in-spin part, with

$$A^{\rm Kerr} = \frac{\Delta\Sigma}{\Lambda}, \qquad B^{\rm Kerr}_p = \frac{r^2}{\Sigma}, \qquad B^{\rm Kerr}_{np} = \frac{r^2}{\Sigma}\left[\frac{\Delta}{r^2} - 1\right], \qquad B^{\rm Kerr}_{npa} = -\frac{r^2}{\Sigma\Lambda}(\Sigma + 2r), \tag{12}$$

and

$$\Sigma = r^2 + (\boldsymbol{n}\cdot\boldsymbol{a}_+)^2, \qquad \Delta = r^2 - 2r + a_+^2, \qquad \Lambda = (r^2 + a_+^2)^2 - \Delta a_+^2 + \Delta(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2. \tag{13}$$

For equatorial orbits, the Kerr Hamiltonian reduces to

$$\tilde{H}^{\rm Kerr,eq} = \frac{2p_\phi a_+}{r^3 + a_+^2(r+2)} + \left[A^{\rm Kerr,eq}\left(1 + p^2 + B^{\rm Kerr,eq}_{np}p_r^2 + B^{\rm Kerr,eq}_{npa}\frac{p_\phi^2 a_+^2}{r^2}\right)\right]^{1/2}, \tag{14}$$

where

$$A^{\rm Kerr,eq} = \frac{1 - 2u + a_+^2u^2}{1 + a_+^2u^2(2u+1)}, \qquad B^{\rm Kerr,eq}_{np} = a_+^2u^2 - 2u, \qquad B^{\rm Kerr,eq}_{npa} = -\frac{r+2}{a_+^2(r+2) + r^3}. \tag{15}$$

with $u \equiv 1/r$.

In the zero-spin limit, we obtain the Schwarzschild Hamiltonian

$$\tilde{H}^{\rm Schw} = \sqrt{(1-2u)\left[1 + (1-2u)\,p_r^2 + p_\phi^2u^2\right]}. \tag{16}$$

B. Effective Hamiltonian for nonspinning binaries

The effective Hamiltonian in the zero-spin (point-mass) limit can be written as[1]

$$\tilde{H}_{\rm eff}^{\rm pm} = \sqrt{A_{\rm pm}\left[1 + A_{\rm pm}\bar{D}_{\rm pm}p_r^2 + p_\phi^2 u^2 + Q_{\rm pm}\right]}\,, \tag{17}$$

where $Q_{\rm pm}(r,p_r)$ is at least quartic in p_r. In the test-mass limit, we have

$$A_{\rm pm}(r) \xrightarrow{\nu=0} 1-2u, \qquad \bar{D}_{\rm pm}(r) \xrightarrow{\nu=0} 1, \qquad Q_{\rm pm}(r,p_r) \xrightarrow{\nu=0} 0, \tag{18}$$

and the effective Hamiltonian reduces to the Schwarzschild Hamiltonian (16). For the potentials A, $\bar{D}$, and Q, we use the results of Ref. [6] (see Table IV there), which are missing two quadratic-in-ν coefficients in A and $\bar{D}$ at 5PN.

The 5PN Taylor-expanded potential A is given by

$$\begin{aligned} A_{\rm pm}^{\rm Tay}(u) &= 1 - 2u + 2\nu u^3 + \nu\left(\frac{94}{3} - \frac{41\pi^2}{32}\right)u^4 + \left[\nu\left(\frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128\gamma_E}{5} + \frac{256\ln 2}{5}\right)\right. \\ &\quad \left. + \left(\frac{41\pi^2}{32} - \frac{221}{6}\right)\nu^2 + \frac{64}{5}\nu\ln u\right]u^5 + \left[\nu a_6 + \left(-\frac{144\nu^2}{5} - \frac{7004\nu}{105}\right)\ln u\right]u^6, \end{aligned} \tag{19}$$

where γ_E is the Euler gamma constant, and we replaced the coefficient of u^6 in $A(r)$, except for the log part, by the parameter a_6, which is calibrated to quasi-circular NR simulations. Note that we pull out a factor of ν from a_6 compared to the definition in Ref. [6]. Then, we perform a (1,5) Padé resummation of $A_{\rm pm}^{\rm Tay}(u)$, while treating $\ln u$ as a constant, i.e., we use

$$A_{\rm pm} = P_5^1[A_{\rm pm}^{\rm Tay}(u)]. \tag{20}$$

The 5PN potential $\bar{D}_{\rm pm}$ reads

$$\begin{aligned} \bar{D}_{\rm pm}^{\rm Tay}(u) &= 1 + 6\nu u^2 + \left(52\nu - 6\nu^2\right)u^3 + \left[\nu\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184\gamma_E}{15} - \frac{6496\ln 2}{15} + \frac{2916\ln 3}{5}\right)\right. \\ &\quad \left. + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592\nu}{15}\ln u\right]u^4 + \left(-\frac{3392\nu^2}{15} - \frac{1420\nu}{7}\right)u^5\ln u \\ &\quad + \left[\nu\left(\frac{294464}{175} - \frac{2840\gamma_E}{7} - \frac{63707\pi^2}{512} + \frac{120648\ln 2}{35} - \frac{19683\ln 3}{7}\right) + \left(\frac{1069}{3} - \frac{205\pi^2}{16}\right)\nu^3\right. \\ &\quad \left. + \left(d_5^{\nu^2} - \frac{6784\gamma_E}{15} + \frac{67736}{105} + \frac{58320\ln 3}{7} - \frac{326656\ln 2}{21}\right)\nu^2\right]u^5, \end{aligned} \tag{21}$$

where we set the remaining unknown coefficient $d_5^{\nu^2}$ to zero, but it can be calibrated in the future to NR results for eccentric orbits. We perform a (2,3) Padé resummation of $\bar{D}_{\rm pm}^{\rm Tay}(u)$

$$\bar{D}_{\rm pm} = P_3^2[\bar{D}_{\rm pm}^{\rm Tay}(u)]. \tag{22}$$

The 5.5PN contributions to A and $\bar{D}$ are known [6]; however, since we Padé resum these potentials, we find it more convenient to stop at 5PN.

For the Q potential, we use the full 5.5PN expansion, which is also expanded in eccentricity to $\mathcal{O}(p_r^8)$, and it reads

$$\begin{aligned} Q_{\rm pm} = p_r^4\Bigg\{&2(4-3\nu)\nu u^2 + u^3\left[10\nu^3 - 83\nu^2 + \nu\left(-\frac{5308}{15} + \frac{496256\ln 2}{45} - \frac{33048\ln 3}{5}\right)\right] + u^4\left[\left(640 - \frac{615\pi^2}{32}\right)\nu^3\right. \\ &+ \nu^2\left(\frac{31633\pi^2}{512} - \frac{1184\gamma_E}{5} + \frac{150683}{105} + \frac{33693536\ln 2}{105} - \frac{6396489\ln 3}{70} - \frac{9765625\ln 5}{126}\right) \\ &+ \nu\left(\frac{1295219}{350} - \frac{93031\pi^2}{1536} + \frac{10856\gamma_E}{105} - \frac{40979464}{315}\ln 2 + \frac{14203593\ln 3}{280} + \frac{9765625\ln 5}{504}\right) \end{aligned}$$

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[1] The Hamiltonian can be obtained by writing the effective metric in the form $ds_{\rm eff}^2 = -A dt^2 + dr^2/(A\bar{D}) + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$, with the mass-shell condition $1 + g_{\rm eff}^{\mu\nu}p_\mu p_\nu + Q = 0$, which can be solved for $p_0 = H_{\rm eff}$, yielding the Hamiltonian in Eq. (17).

$$
\begin{aligned}
&+\left(\frac{5428\nu}{105}-\frac{592\nu^2}{5}\right)\ln u\Big]+\frac{88703\pi\nu u^{9/2}}{1890}\Big\}\\
&+p_r^6\Big\{u^2\left[6\nu^3-\frac{27\nu^2}{5}+\nu\left(-\frac{827}{3}-\frac{2358912}{25}\ln 2+\frac{1399437\ln 3}{50}+\frac{390625\ln 5}{18}\right)\right]\\
&\quad+u^3\Big[-14\nu^4+116\nu^3+\nu^2\left(\frac{159089}{75}-\frac{4998308864\ln 2}{1575}+\frac{26171875\ln 5}{18}-\frac{45409167\ln 3}{350}\right)\\
&\quad+\nu\left(\frac{2613083}{1050}+\frac{6875745536\ln 2}{4725}-\frac{23132628\ln 3}{175}-\frac{101687500\ln 5}{189}\right)\Big]-\frac{2723471\pi\nu u^{7/2}}{756000}\Big\}\\
&+p_r^8\Big\{u\nu\left(-\frac{35772}{175}+\frac{21668992\ln 2}{45}+\frac{6591861\ln 3}{350}-\frac{27734375\ln 5}{126}\right)+u^2\Big[-6\nu^4+\frac{24\nu^3}{7}\\
&\quad+\nu^2\left(\frac{870976}{525}+\frac{703189497728\ln 2}{33075}+\frac{332067403089\ln 3}{39200}-\frac{13841287201\ln 7}{4320}-\frac{468490234375\ln 5}{42336}\right)\\
&\quad+\nu\left(\frac{5790381}{2450}-\frac{16175693888\ln 2}{1575}+\frac{875090984375\ln 5}{169344}+\frac{13841287201\ln 7}{17280}-\frac{393786545409\ln 3}{156800}\right)\Big]\\
&\quad+\frac{5994461\pi\nu u^{5/2}}{12700800}\Big\}. \qquad (23)
\end{aligned}
$$

The calibration parameter a_6 is a function of ν; to determine its value in the limit $\nu \to 0$, we use the gravitational-self-force results of Refs. [25, 26] for the frequency shift of the innermost stable circular orbit (ISCO), which is

$$M\Omega^{\rm 1SF}_{\rm ISCO} = 6^{-3/2}(1 + C_\Omega/q), \qquad C_\Omega = 1.25101539 \pm 4\times 10^{-8}. \qquad (24)$$

The ISCO can be computed from the Hamiltonian by solving $\partial H/\partial r = 0 = \partial^2 H/\partial r^2$ for r and p_ϕ with $p_r = 0$. The value of a_6 that gives best agreement with $\Omega^{\rm 1SF}_{\rm ISCO}$ is

$$a_6|_{\nu\to 0} \simeq 39.1. \qquad (25)$$

C. Effective Hamiltonian for aligned spins

For aligned spins, the effective Hamiltonian reduces to the equatorial Kerr Hamiltonian (14), and to include higher PN information, we use the following ansatz:

$$
\begin{aligned}
\tilde{H}^{\rm align}_{\rm eff} &= \frac{1}{r^3 + a_+^2(r+2)}\left[p_\phi(g_{a_+}a_+ + g_{a_-}\delta a_-) + {\rm SO}_{\rm calib} + G_{a^3}\right]\\
&+\left[A\left(1 + p^2 + B_{np}p_r^2 + B^{\rm Kerr,eq}_{npa}\frac{p_\phi^2 a_+^2}{r^2} + Q\right)\right]^{1/2}, \qquad (26)
\end{aligned}
$$

where g_{a_+} and g_{a_-} include the SO corrections, $\rm SO_{calib}$ is an NR calibration term, and G_{a^3} contains $\rm S^3$ corrections. The nonspinning and SS contributions are included in A, B_{np} and Q, while the $\rm S^4$ corrections are added in A. The potential $B^{\rm Kerr,eq}_{npa}$ is kept the same as in the Kerr Hamiltonian.

1. *Spin-orbit and cubic-in-spin contributions*

In some papers, the gyro-gravitomagnetic factors g_{a_+} and g_{a_-} in the SO part of the Hamiltonian[2] were chosen to be in a gauge such that they are functions of $1/r$ and p_r^2 only [10, 11]. However, other papers [21, 22] have made

[2] The SO part of EOB Hamiltonians is usually expressed in terms of $S \equiv S_1 + S_2$ and $S_* \equiv S_1 m_2/m_1 + S_2 m_1/m_2$, using $H_{\rm SO} \propto (g_S S + g_{S_*} S_*)\, p_\phi / r^3$. The relation between the gyro-gravitomagnetic factors in this case and our definition in Eq. (26) is that

$$g_{a_+} = \frac{1}{2}(g_S + g_{S_*}), \qquad g_{a_-} = \frac{1}{2}(g_S - g_{S_*}). \qquad (27)$$

different choices. For SEOBNRv5, we find better results when using a gauge in which g_{a_+} and g_{a_-} depend on $1/r$ and $\tilde{L}^2/r^2$, but not on p_r^2, such that

$$g_{a_+} = \frac{7}{4} + \left[\frac{\tilde{L}^2}{r^2}\left(-\frac{45\nu}{32} - \frac{15}{32}\right) + \frac{1}{r}\left(\frac{23\nu}{32} - \frac{3}{32}\right)\right] + \left[\frac{\tilde{L}^4}{r^4}\left(\frac{345\nu^2}{256} + \frac{75\nu}{128} + \frac{105}{256}\right) + \frac{\tilde{L}^2}{r^3}\left(-\frac{1591\nu^2}{768} - \frac{267\nu}{128} + \frac{59}{256}\right) + \frac{1}{r^2}\left(\frac{109\nu^2}{192} - \frac{177\nu}{32} - \frac{5}{64}\right)\right], \tag{28a}$$

$$g_{a_-} = \frac{1}{4} + \left[\frac{\tilde{L}^2}{r^2}\left(\frac{15}{32} - \frac{9\nu}{32}\right) + \frac{1}{r}\left(\frac{11\nu}{32} + \frac{3}{32}\right)\right] + \left[\frac{\tilde{L}^4}{r^4}\left(\frac{75\nu^2}{256} - \frac{45\nu}{128} - \frac{105}{256}\right) + \frac{\tilde{L}^2}{r^3}\left(-\frac{613\nu^2}{768} - \frac{35\nu}{128} - \frac{59}{256}\right) + \frac{1}{r^2}\left(\frac{103\nu^2}{192} - \frac{\nu}{32} + \frac{5}{64}\right)\right], \tag{28b}$$

where the square brackets collect different PN orders. These PN expressions were obtained by canonically transforming the 3.5PN results of, e.g., Ref. [27].

The 4.5PN SO coupling was derived in Refs. [18, 28–30], and can be included in the effective Hamiltonian.[3] However, we found that using a calibration term at 5.5PN had a small effect on the dynamics, and thus only included the 3.5PN information with a 4.5PN calibration term of the form

$$\mathrm{SO}_{\mathrm{calib}} = d_{\mathrm{SO}} \frac{\nu}{r^3} p_\phi a_+. \tag{30}$$

For the cubic-in-spin term G_{a^3}, we obtain

$$G_{a^3} = \frac{p_\phi}{r^2}\left[-\frac{a_+^3}{4} + \frac{\delta}{4} a_- a_+^2 + C_+^{a^3} + \frac{3}{8} C_+^{a^2}\left(\delta a_- + 3a_+\right) - \frac{3}{2} a_- C_-^{a^2}\right], \tag{31}$$

where only the first two terms contribute for BHs. The coefficients $C_\pm^{\cdots}$ are defined in Eq. (5).

2. *Spin-spin and quartic-in-spin contributions*

We include SS and S^4 PN information in the effective Hamiltonian (26) through the following ansatz (cf. Eq. (15)):

$$\begin{aligned} A &= \frac{a_+^2 u^2 + A_{\mathrm{pm}} + A_{\mathrm{SS}} + A_{S^4}}{1 + a_+^2 u^2 (2u + 1)}, \\ B_{np} &= -1 + a_+^2 u^2 + A_{\mathrm{pm}} \bar{D}_{\mathrm{pm}} + B_{np}^{\mathrm{SS}}, \\ Q &= Q_{\mathrm{pm}} + Q_{\mathrm{SS}}, \end{aligned} \tag{32}$$

where the nonspinning contributions A_{pm}, $\bar{D}_{\mathrm{pm}}$ and Q_{pm} are given by Eqs. (20), (22) and (23), respectively.

The SS contributions are given by

$$A_{\mathrm{SS}} = -\frac{C_+^{a^2}}{r^3} + \frac{1}{r^4}\left[\frac{9a_+^2}{8} - \frac{5}{4}\delta a_- a_+ + a_-^2\left(\frac{\nu}{2} + \frac{1}{8}\right) - \delta C_-^{a^2} - C_+^{a^2}\right] + \frac{1}{r^5}\left[a_+^2\left(-\frac{175\nu}{64} - \frac{225}{64}\right) + \delta a_- a_+\left(\frac{117}{32} - \frac{39\nu}{16}\right) + a_-^2\left(\frac{21\nu^2}{16} - \frac{81\nu}{64} - \frac{9}{64}\right) - \frac{51}{28}\delta C_-^{a^2} + \left(\frac{207\nu}{28} - \frac{51}{28}\right) C_+^{a^2}\right], \tag{33a}$$

---

[3] The 4.5PN gyros are given by Eq. (5.6) of Ref. [18] in terms of $1/r$ and p_r^2. Using $\tilde{L}^2/r^2$ instead of p_r^2 yields the following 4.5PN terms

$$g_{a_+}^{4.5\mathrm{PN}} = g_{a_+}^{3.5\mathrm{PN}} + \left[\frac{\tilde{L}^6}{r^6}\left(-\frac{5425\nu^3}{4096} - \frac{1785\nu^2}{2048} - \frac{1715\nu}{4096} - \frac{1575}{4096}\right) + \frac{\tilde{L}^4}{r^5}\left(\frac{75187\nu^3}{20480} + \frac{37603\nu^2}{10240} + \frac{3717\nu}{4096} - \frac{1023}{4096}\right) + \frac{\tilde{L}^2}{r^4}\left(-\frac{15093\nu^3}{5120} + \frac{80189\nu^2}{7680} - \frac{13059\nu}{1024} + \frac{209}{1024}\right) + \frac{1}{r^3}\left(\frac{1079\nu^3}{2048} - \frac{24131\nu^2}{3072} + \frac{\left(23376\pi^2 - 525331\right)\nu}{18432} - \frac{175}{2048}\right)\right], \tag{29a}$$

$$g_{a_-}^{4.5\mathrm{PN}} = g_{a_+}^{3.5\mathrm{PN}} + \left[\frac{\tilde{L}^6}{r^6}\left(-\frac{1225\nu^3}{4096} + \frac{525\nu^2}{2048} + \frac{1785\nu}{4096} + \frac{1575}{4096}\right) + \frac{\tilde{L}^4}{r^5}\left(\frac{26491\nu^3}{20480} + \frac{4801\nu^2}{10240} - \frac{4843\nu}{20480} + \frac{1023}{4096}\right) + \frac{\tilde{L}^2}{r^4}\left(-\frac{9549\nu^3}{5120} - \frac{1777\nu^2}{7680} - \frac{28883\nu}{5120} - \frac{209}{1024}\right) + \frac{1}{r^3}\left(\frac{1823\nu^3}{2048} - \frac{1025\nu^2}{3072} - \frac{5\left(10043 + 48\pi^2\right)\nu}{18432} + \frac{175}{2048}\right)\right]. \tag{29b}$$

$$B_{np}^{\rm SS} = \frac{1}{r^3}\left[a_+^2\left(3\nu+\frac{45}{16}\right)-\frac{21}{8}\delta a_- a_+ + a_-^2\left(\frac{3\nu}{4}-\frac{3}{16}\right)+(3\nu-3)C_+^{a^2}\right]+\frac{1}{r^4}\left[a_+^2\left(-\frac{1171\nu}{64}-\frac{861}{64}\right)\right.$$
$$\left.+\delta a_- a_+\left(\frac{13\nu}{16}+\frac{449}{32}\right)+a_-^2\left(\frac{\nu^2}{16}+\frac{115\nu}{64}-\frac{37}{64}\right)+\left(6\nu-\frac{19}{4}\right)\delta C_-^{a^2}+\left(\frac{111\nu}{4}-\frac{3}{4}\right)C_+^{a^2}\right], \tag{33b}$$
$$Q_{\rm SS} = \frac{p_r^4}{r^3}\left[a_+^2\left(-5\nu^2+\frac{165\nu}{32}+\frac{25}{32}\right)+\delta a_- a_+\left(\frac{45\nu}{8}-\frac{5}{16}\right)+a_-^2\left(-\frac{15\nu^2}{8}+\frac{75\nu}{32}-\frac{15}{32}\right)+\left(\frac{35\nu}{4}-5\nu^2\right)C_+^{a^2}\right], \tag{33c}$$

where we recall that the constants $C_\pm^{a^2}$ are zero for BHs.

The quartic-in-spin contribution in A is given by

$$A_{S^4} = \frac{1}{r^5}\left[-\frac{3}{4}C_+^{a^4}-\frac{3}{2}a_+C_+^{a^3}+\frac{3}{2}a_-C_-^{a^3}+\left(-\frac{9}{8}a_-^2-\frac{5a_+^2}{8}\right)C_+^{a^2}-\frac{9}{8}(C_+^{a^2})^2+\frac{9}{8}(C_-^{a^2})^2+\frac{9}{4}a_-a_+C_-^{a^2}\right], \tag{34}$$

which vanishes for BHs since the Kerr Hamiltonian with the mapping (9) automatically reproduces them [23].

D. Effective Hamiltonian for precessing spins

For precessing spins, the effective Hamiltonian reduces to the Kerr Hamiltonian (11), and to include higher PN information, we write the following ansatz:

$$\tilde H_{\rm eff}^{\rm prec} = \frac{r}{\Lambda}\left[\tilde{\boldsymbol L}\cdot\left(g_{a_+}\boldsymbol a_+ + g_{a_-}\delta\boldsymbol a_-\right)+{\rm SO}_{\rm calib}+G_{a^3}\right]$$
$$+\left[A\left(1+B_p\boldsymbol p^2+B_{np}(\boldsymbol n\cdot\boldsymbol p)^2+B_{npa}^{\rm Kerr}(\boldsymbol n\times\boldsymbol p\cdot\boldsymbol a_+)^2+Q\right)\right]^{1/2}, \tag{35}$$

where g_{a_+} and g_{a_-} include the SO corrections, $\mathrm{SO}_{\rm calib}$ is an NR calibration term, and G_{a^3} contains S^3 corrections. The nonspinning and SS contributions are included in A, B_p, B_{np} and Q, while the S^4 corrections are added in A. The potential $B_{npa}^{\rm Kerr}$ is kept the same as in the Kerr Hamiltonian.

1. *Spin-orbit and cubic-in-spin contributions*

The gyro-gravitomagnetic factors g_{a_+} and g_{a_-} are given by Eq. (28), since they are exactly the same as in the aligned-spin case. The calibration term is also the same, except for adding a dot product

$${\rm SO}_{\rm calib} = d_{\rm SO}\frac{\nu}{r^3}\tilde{\boldsymbol L}\cdot\boldsymbol a_+. \tag{36}$$

For the cubic-in-spin term G_{a^3}, we obtain

$$G_{a^3} = \tilde{\boldsymbol L}\cdot\boldsymbol a_+\Bigg\{\frac{\tilde L^2}{r^3}\left[\frac{\delta}{2}(\boldsymbol n\cdot\boldsymbol a_-)(\boldsymbol n\cdot\boldsymbol a_+)-\frac{(\boldsymbol n\cdot\boldsymbol a_+)^2}{4}\right]+\frac{p_r^2}{r}\left[\frac{5}{4}(\boldsymbol n\cdot\boldsymbol a_+)^2-\delta\frac{3}{2}(\boldsymbol n\cdot\boldsymbol a_-)(\boldsymbol n\cdot\boldsymbol a_+)\right]$$
$$+\frac{1}{r^2}\Bigg[-\frac{a_+^2}{4}+(\boldsymbol n\cdot\boldsymbol a_+)^2+\delta\frac{5}{24}(\boldsymbol a_+\cdot\boldsymbol a_-)-\delta\frac{5}{3}(\boldsymbol n\cdot\boldsymbol a_-)(\boldsymbol n\cdot\boldsymbol a_+)$$
$$+\frac{1}{8}a_1^2\left(3\delta\widetilde C_{1ES^2}+4\widetilde C_{1BS^3}\right)+\frac{1}{8}a_2^2\left(4\widetilde C_{2BS^3}-3\widetilde C_{2ES^2}\delta\right)-\frac{3}{8}(\boldsymbol a_1\cdot\boldsymbol a_2)\left(\widetilde C_{1ES^2}(\delta-3)-\widetilde C_{2ES^2}(\delta+3)\right)$$
$$+(\boldsymbol n\cdot\boldsymbol a_1)^2\left(-\frac{3}{8}\widetilde C_{1ES^2}(2\delta+3)-\frac{5\widetilde C_{1BS^3}}{2}\right)+\frac{3}{4}(\boldsymbol n\cdot\boldsymbol a_1)(\boldsymbol n\cdot\boldsymbol a_2)\left(\widetilde C_{1ES^2}(2\delta-7)-\widetilde C_{2ES^2}(2\delta+7)\right)$$
$$+\frac{1}{8}(\boldsymbol n\cdot\boldsymbol a_2)^2\left(\widetilde C_{2ES^2}(6\delta-9)-20\widetilde C_{2BS^3}\right)\Bigg]\Bigg\}$$
$$+\tilde{\boldsymbol L}\cdot\boldsymbol a_-\Bigg\{\frac{p_r^2\delta(\boldsymbol n\cdot\boldsymbol a_+)^2}{4r}-\frac{\tilde L^2\delta(\boldsymbol n\cdot\boldsymbol a_+)^2}{4r^3}+\frac{1}{r^2}\Bigg[\frac{\delta a_+^2}{24}+\frac{2}{3}\delta(\boldsymbol n\cdot\boldsymbol a_+)^2$$

$$
+\frac{1}{8}a_1^2\left(4\widetilde{C}_{1BS^3}-3\widetilde{C}_{1ES^2}\right)+\frac{1}{8}a_2^2\left(3\widetilde{C}_{2ES^2}-4\widetilde{C}_{2BS^3}\right)-\frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(\delta-3)+\widetilde{C}_{2ES^2}(\delta+3)\right)
+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\widetilde{C}_{1ES^2}(\delta-6)-\frac{5\widetilde{C}_{1BS^3}}{2}\right)+\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(2\delta-7)+\widetilde{C}_{2ES^2}(2\delta+7)\right)
+(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{5\widetilde{C}_{2BS^3}}{2}-\frac{3}{8}\widetilde{C}_{2ES^2}(\delta+6)\right)\Bigg]\Bigg\}. \tag{37}
$$

2. Spin-spin and quartic-in-spin contributions

We include SS and S^4 PN information in the effective Hamiltonian (35) through the following ansatz for the potentials (cf. Eq. (12)):

$$
\begin{aligned}
A &= \frac{\left(a_+^2u^2+A_{\rm pm}+A_{\rm SS}^{\rm align}+A_{S^4}^{\rm align}\right)\left(1+(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^2+A_{\rm SS}^{\rm prec}+A_{S^4}^{\rm prec}\right)}{1+a_+^2u^2+2a_+^2u^3+(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^2-2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^3+a_+^2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^4},\\
B_p &= \left[1+(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^2+B_{p,\rm SS}^{\rm prec}\right]^{-1},\\
B_{np} &= \frac{-1+a_+^2u^2+A_{\rm pm}\bar{D}_{\rm pm}+B_{np,\rm SS}^{\rm align}+B_{np,\rm SS}^{\rm prec}}{1+(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2u^2},\\
Q &= Q_{\rm pm}+Q_{\rm SS}^{\rm align}+Q_{\rm SS}^{\rm prec},
\end{aligned} \tag{38}
$$

where the nonspinning contributions $A_{\rm pm}$, $\bar{D}_{\rm pm}$ and $Q_{\rm pm}$ are given by Eqs. (20), (22) and (23), respectively.

The spin-spin contributions to the potentials are given by

$$
A_{\rm SS}^{\rm align} = -\frac{C_+^{a^2}}{r^3}+\frac{1}{r^4}\left[\frac{9a_+^2}{8}-\frac{5}{4}\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_++a_-^2\left(\frac{\nu}{2}+\frac{1}{8}\right)-\delta C_-^{a^2}-C_+^{a^2}\right]+\frac{1}{r^5}\left[a_+^2\left(-\frac{175\nu}{64}-\frac{225}{64}\right)
+\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{117}{32}-\frac{39\nu}{16}\right)+a_-^2\left(\frac{21\nu^2}{16}-\frac{81\nu}{64}-\frac{9}{64}\right)-\frac{51}{28}\delta C_-^{a^2}+\left(\frac{207\nu}{28}-\frac{51}{28}\right)C_+^{a^2}\right], \tag{39a}
$$

$$
A_{\rm SS}^{\rm prec} = \frac{3C_+^{n\cdot a^2}}{r^3}+\frac{1}{r^4}\left[\frac{33}{8}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{\nu}{2}-\frac{3}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(\frac{7\nu}{4}-\frac{15}{4}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+3\delta C_-^{n\cdot a^2}+(3\nu+6)C_+^{n\cdot a^2}\right]
+\frac{1}{r^5}\left[\delta(17\nu+8)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{41\nu^2}{8}+\frac{551\nu}{32}-\frac{219}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(-\frac{11\nu^2}{8}+\frac{1771\nu}{96}-\frac{293}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2
+\delta\left(\frac{81\nu}{16}+\frac{1245}{224}\right)C_-^{n\cdot a^2}+\left(-\frac{13\nu^2}{16}+\frac{515\nu}{56}+\frac{3555}{224}\right)C_+^{n\cdot a^2}\right], \tag{39b}
$$

$$
B_{p,\rm SS}^{\rm prec} = \frac{1}{r^3}\left[-\frac{3}{4}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(\frac{3\nu}{4}-\frac{3}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(\frac{7\nu}{4}+\frac{15}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+(3\nu-3)C_+^{n\cdot a^2}\right]
+\frac{1}{r^4}\left[\delta\left(\frac{49\nu}{4}+\frac{43}{8}\right)\boldsymbol{n}\cdot\boldsymbol{a}_-\boldsymbol{n}\cdot\boldsymbol{a}_++\left(-\frac{19\nu^2}{8}+\frac{545\nu}{32}-\frac{219}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(-\frac{11\nu^2}{8}+\frac{805\nu}{96}-\frac{125}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2
+\delta\left(\frac{81\nu}{16}-\frac{189}{32}\right)C_-^{n\cdot a^2}+\left(-\frac{13\nu^2}{16}+\frac{203\nu}{8}-\frac{51}{32}\right)C_+^{n\cdot a^2}\right], \tag{39c}
$$

$$
B_{np,\rm SS}^{\rm align} = \frac{1}{r^3}\left[a_+^2\left(3\nu+\frac{45}{16}\right)-\frac{21}{8}\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_++a_-^2\left(\frac{3\nu}{4}-\frac{3}{16}\right)+(3\nu-3)C_+^{a^2}\right]+\frac{1}{r^4}\left[a_+^2\left(-\frac{1171\nu}{64}-\frac{861}{64}\right)
+\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{13\nu}{16}+\frac{449}{32}\right)+a_-^2\left(\frac{\nu^2}{16}+\frac{115\nu}{64}-\frac{37}{64}\right)+\left(6\nu-\frac{19}{4}\right)\delta C_-^{a^2}+\left(\frac{111\nu}{4}-\frac{3}{4}\right)C_+^{a^2}\right], \tag{39d}
$$

$$
B_{np,\rm SS}^{\rm prec} = \frac{1}{r^3}\left[\frac{45}{8}\delta(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{15\nu}{4}-\frac{45}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2\right]+\frac{1}{r^4}\left[\delta\left(\frac{129\nu}{4}-\frac{17}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)
+\left(-\frac{33\nu^2}{4}+\frac{981\nu}{16}-\frac{165}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(-\frac{11\nu^2}{2}+\frac{1901\nu}{48}+\frac{199}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\delta\left(\frac{9\nu}{4}-\frac{75}{8}\right)C_-^{n\cdot a^2}
+\left(-\frac{13\nu^2}{4}+\frac{37\nu}{4}+\frac{39}{8}\right)C_+^{n\cdot a^2}\right], \tag{39e}
$$

$$Q_{\rm SS}^{\rm align} = \frac{p_r^4}{r^3}\left[a_+^2\left(-5\nu^2+\frac{165\nu}{32}+\frac{25}{32}\right)+\delta\boldsymbol{a}_-\cdot\boldsymbol{a}_+\left(\frac{45\nu}{8}-\frac{5}{16}\right)+a_-^2\left(-\frac{15\nu^2}{8}+\frac{75\nu}{32}-\frac{15}{32}\right)+\left(\frac{35\nu}{4}-5\nu^2\right)C_+^{a^2}\right], \tag{39f}$$

$$\begin{aligned} Q_{\rm SS}^{\rm prec} = \frac{p_r^4}{r^3}\Bigg[&\delta\left(\frac{35}{16}-\frac{273\nu}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(\frac{105\nu^2}{8}-\frac{525\nu}{32}+\frac{105}{32}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 \\ &+\left(\frac{119\nu^2}{4}-\frac{2849\nu}{96}-\frac{175}{32}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\left(35\nu^2-\frac{245\nu}{4}\right)C_+^{n\cdot a^2}\Bigg] \\ &+\frac{p_r^3}{r^3}\Bigg[\delta\left(\frac{69\nu}{8}-\frac{5}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)+\left(-\frac{59\nu^2}{4}+\frac{341\nu}{24}+\frac{25}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_+) \\ &+\delta\left(\frac{69\nu}{8}-\frac{5}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-)+\left(-\frac{15\nu^2}{2}+\frac{75\nu}{8}-\frac{15}{8}\right)(\boldsymbol{p}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_-) \\ &+\left(35\nu-20\nu^2\right)\left(\widetilde{C}_{1ES^2}\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{p}\cdot\boldsymbol{a}_1)}{M^2}+\widetilde{C}_{2ES^2}\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{p}\cdot\boldsymbol{a}_2)}{M^2}\right)\Bigg]. \end{aligned} \tag{39g}$$

The quartic-in-spin contributions in A are given by

$$\begin{aligned} A_{S^4}^{\rm align} = \frac{1}{r^5}\Bigg\{&-\frac{1}{4}a_1^2a_2^2\left[\widetilde{C}_{1ES^2}(3\widetilde{C}_{2ES^2}+2)+2\widetilde{C}_{2ES^2}\right]-\frac{3}{2}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)^2\left[\widetilde{C}_{1ES^2}\widetilde{C}_{2ES^2}+\widetilde{C}_{1ES^2}+\widetilde{C}_{2ES^2}\right] \\ &+a_1^2(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(\widetilde{C}_{1ES^2}-3\widetilde{C}_{1BS^3})+\frac{1}{4}a_1^4(2\widetilde{C}_{1ES^2}-3\widetilde{C}_{1ES^4})+1\leftrightarrow 2\Bigg\}, \end{aligned} \tag{40a}$$

$$\begin{aligned} A_{S^4}^{\rm prec} = \frac{1}{r^5}\Bigg\{&(\boldsymbol{n}\cdot\boldsymbol{a}_1)^4\left(\frac{21\widetilde{C}_{1ES^2}}{2}-\frac{35\widetilde{C}_{1ES^4}}{4}\right)+(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{n}\cdot\boldsymbol{a}_1)^3(21\widetilde{C}_{1ES^2}-35\widetilde{C}_{1BS^3}) \\ &+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\Bigg[(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(-\frac{105}{4}\widetilde{C}_{1ES^2}\widetilde{C}_{2ES^2}-21\widetilde{C}_{1ES^2}-21\widetilde{C}_{2ES^2}\right)+(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)(15\widetilde{C}_{1BS^3}-12\widetilde{C}_{1ES^2}) \\ &\quad+a_1^2\left(\frac{15\widetilde{C}_{1ES^4}}{2}-9\widetilde{C}_{1ES^2}\right)+\frac{15}{4}a_2^2\widetilde{C}_{1ES^2}\widetilde{C}_{2ES^2}+\frac{3a_2^2\widetilde{C}_{1ES^2}}{2}+3a_2^2\widetilde{C}_{2ES^2}\Bigg] \\ &+(\boldsymbol{n}\cdot\boldsymbol{a}_1)a_1^2(\boldsymbol{n}\cdot\boldsymbol{a}_2)(15\widetilde{C}_{1BS^3}-6\widetilde{C}_{1ES^2})+a_1^2(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{15\widetilde{C}_{1ES^2}\widetilde{C}_{2ES^2}}{4}+3\widetilde{C}_{1ES^2}+\frac{3\widetilde{C}_{2ES^2}}{2}\right) \\ &+(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\left(15\widetilde{C}_{1ES^2}\widetilde{C}_{2ES^2}+\frac{27\widetilde{C}_{1ES^2}}{2}+\frac{27\widetilde{C}_{2ES^2}}{2}\right)+1\leftrightarrow 2\Bigg\}, \end{aligned} \tag{40b}$$

which are zero for BHs since the Kerr Hamiltonian with the mapping (9) reproduces them [23].

E. Hamiltonian in tortoise coordinates

The tortoise-coordinate r_* is defined by [31, 32]

$$\frac{dr_*}{dr} = \frac{1}{\xi(r)}, \qquad \xi(r) \equiv A_{\rm pm}(r)\sqrt{\bar{D}_{\rm pm}(r)}. \tag{41}$$

The conjugate momentum to r_* is p_{r_*}, which is given by the relation

$$p_{r_*} = p_r\xi(r). \tag{42}$$

The nonspinning effective Hamiltonian (17) in terms of p_{r_*} takes the simpler form

$$\tilde{H}_{\rm eff}^{\rm pm} = \sqrt{p_{r_*}^2 + A(r)\left[1+p_\phi^2u^2+Q(r,p_{r_*})\right]}, \tag{43}$$

where we obtain $Q(r,p_{r_*})$ from Eq. (23) by converting p_r to p_{r_*} using Eq. (42), then PN expand to 5.5PN order.

For both aligned and precessing spins, a convenient choice for ξ is

$$\xi(r) = \frac{\bar{D}_{\rm pm}^{1/2}\left(A_{\rm pm} + a_+^2 u^2\right)}{1 + a_+^2 u^2}, \tag{44}$$

which is similar to what was used for ξ in SEOBNRv4 [32, 33] except for the different resummation and PN orders in $A_{\rm pm}$ and $\bar{D}_{\rm pm}$. In the $\nu \to 0$ limit, ξ reduces to the Kerr value $(dr/dr_*)_{\rm Kerr} = (r^2 - 2r + a_+^2)/(r^2 + a_+^2)$. The PN expansion of $\xi(r)$ is given by

$$\xi(r) \simeq 1 - \frac{2}{r} + \frac{3\nu}{r^2} + \frac{2a^2}{r^3} + \dots \tag{45}$$

Instead of replacing p_r by p_{r_*}/ξ everywhere in the Hamiltonian, which would complicate it, we expand $Q_{\rm pm}(r, p_r)$ from Eq. (23) to 5.5PN order to obtain $Q_{\rm pm}(r, p_{r_*})$. We do not include spin contributions in that expansion since the a^2 term in Eq. (45) is at 2PN order, and thus contributes to $Q_{\rm pm}$ at 5PN, while the SS PN corrections are included up to 4PN.

The equations of motion (EOMs) for aligned spins, in terms of p_{r_*}, are given by Eqs. (10) of Ref. [34], which read

$$\dot{r} = \xi \left.\frac{\partial H}{\partial p_{r_*}}\right|_r, \qquad \dot{\phi} = \frac{\partial H}{\partial p_\phi}, \qquad \dot{p}_{r_*} = -\xi \left.\frac{\partial H}{\partial r}\right|_{p_{r_*}} + \frac{p_{r_*}}{p_\phi}\mathcal{F}_\phi, \qquad \dot{p}_\phi = \mathcal{F}_\phi. \tag{46}$$

F. Comparison with other models

Table II summarizes the main differences of the SEOBNRv5 Hamiltonian compared to that of SEOBNRv4 [21, 22, 35, 36] and TEOBResumS [14, 16, 37]. The SEOBNRv5 Hamiltonian contains the same spin PN information as the Hamiltonians derived in Ref. [17], which extended the results of Ref. [15] to higher orders; however, we use different resummations/factorizations from those employed in the above references.

TABLE II. Summary of the main differences of the SEOBNRv5 Hamiltonian, which builds on the results of Refs. [15, 17], compared to that of SEOBNRv4 and TEOBResumS.

	SEOBNRv5	SEOBNRv4 [21, 22, 35, 36]	TEOBResumS [14, 16, 37]
nonspinning part	4PN with partial 5PN in $A_{\rm pm}$ and $\bar{D}_{\rm pm}$, 5.5PN in $Q_{\rm pm}$	4PN in $A_{\rm pm}$, 3PN in $\bar{D}_{\rm pm}$ and $Q_{\rm pm}$	4PN with partial 5PN in $A_{\rm pm}$, 3PN in $\bar{D}_{\rm pm}$ and $Q_{\rm pm}$
$A_{\rm pm}$ resummation	(1,5) Padé	horizon factorization and log resummation	(1,5) Padé
$\bar{D}_{\rm pm}$ resummation	(2,3) Padé	log	Taylor expanded ($D_{\rm pm} \equiv 1/\bar{D}_{\rm pm}$ is inverse-Taylor resummed)
Hamiltonian in the $\nu \to 0$ limit	reduces to Kerr Hamiltonian for a *test mass* in a generic orbit	reduces to Kerr Hamiltonian for a *test spin*, to linear order in spin, in a generic orbit	the A potential reduces to Kerr, but not the full Hamiltonian
spin-orbit part	3.5PN, in $(r, \tilde{L}^2)$ gauge, Taylor expanded	3.5PN, added in the spin map	3.5PN, in (r, p_r^2) gauge, inverse-Taylor resummed
higher-order spin information	NNLO SS (4PN), LO S^3 (3.5PN), LO S^4 (4PN)	LO SS (2PN)	NLO SS (3PN) for circular orbits
precessing-spin Hamiltonian	yes	yes	no
spin-multipole constants included	yes	no	yes (to NNLO SS, for aligned spins and circular orbits)

II. PN-EOB DYNAMICS FOR SPIN-PRECESSING BINARIES

In Sec. I D, we derived an effective Hamiltonian for precessing spins that reduces to the Kerr Hamiltonian for generic orbits. The EOMs from that Hamiltonian read

$$\dot{\boldsymbol{r}} = \frac{\partial \tilde{H}_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial \tilde{H}_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{r}} + \boldsymbol{\mathcal{F}}, \qquad \dot{\boldsymbol{S}}_{\rm i} = \frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{S}_{\rm i}} \times \boldsymbol{S}_{\rm i} + \dot{\boldsymbol{S}}_{\rm i}^{\rm RR}, \tag{47}$$

where $\boldsymbol{\mathcal{F}}$ is the radiation-reaction (RR) force, and $\dot{\boldsymbol{S}}_{\rm i}^{\rm RR}$ is the RR contribution to the spin-evolution equations, which starts at $\mathcal{O}(v^{11}S^2)$ [38, 39] and is thus neglected to the order we consider here.

These equations are computationally expensive to evolve; therefore, we evolve the EOMs in the co-precessing frame [40–42], in which the z-axis remains perpendicular to the orbital plane, and the dynamics can be approximated by the aligned-spin case. We use a simpler precessing-spin Hamiltonian that reduces to $H_{\rm eff}^{\rm align}$ for aligned-spins and includes in-plane spin components for circular orbits. This Hamiltonian is used, after orbit-averaging the precessing terms, to evolve the EOMs for the dynamical variables r, p_r, ϕ and p_ϕ, while the evolution equations for the spin and angular momentum vectors are computed in a PN expansion, also in an orbit average for circular orbits. In the following subsections, we present the Hamiltonian, then derive the PN EOMs for precessing spins, in EOB coordinates, up to NNLO SO (3.5PN) and NNLO SS (4PN).

A. Simple precessing-spin Hamiltonian

The precessing-spin Hamiltonian presented in Sec. I D reduces to the exact Kerr Hamiltonian (11) for generic orbits. Here, we consider a simpler Hamiltonian that starts with an ansatz similar to the aligned-spin Hamiltonian in Eqs. (26) and (32), then complement it with precessing-spin corrections for circular orbits only, i.e., we do not include in-plane spin terms proportional to p_r^n.

We use the following ansatz for the effective Hamiltonian (cf. Eqs. (26) and (35))

$$\begin{aligned}\tilde{H}_{\rm eff}^{\rm simp} &= \frac{1}{r^3 + a_+^2(r+2)} \left[\tilde{\boldsymbol{L}} \cdot (g_{a_+}\boldsymbol{a}_+ + g_{a_-}\delta\boldsymbol{a}_-) + {\rm SO}_{\rm calib} + G_{a^3}\right] \\ &\quad + \left[A\left(1 + B_p\frac{\tilde{L}^2}{r^2} + (1+B_{np})p_r^2 + B_{npa}^{\rm Kerr,eq}\frac{(\tilde{\boldsymbol{L}}\cdot\boldsymbol{a}_+)^2}{r^2} + Q\right)\right]^{1/2},\end{aligned} \tag{48}$$

where the gyro-gravitomagnetic factors are the same as in Eqs. (28), and the SO calibration term is given by

$${\rm SO}_{\rm calib} = d_{\rm SO}\frac{\nu}{r^3}\tilde{\boldsymbol{L}}\cdot\boldsymbol{a}_+. \tag{49}$$

with the same value of $d_{\rm SO}$ as the aligned-spin model. The SS corrections are added such that (cf. Eq. (32))

$$\begin{aligned} A &= \frac{a_+^2u^2 + A_{\rm pm} + A_{\rm SS}^{\rm align} + A_{\rm SS}^{\rm prec}}{1 + a_+^2u^2(2u+1)}, \\ B_p &= 1 + B_{p,{\rm SS}}^{\rm prec}, \\ B_{np} &= -1 + a_+^2u^2 + A_{\rm pm}\bar{D}_{\rm pm} + B_{np,{\rm SS}}^{\rm align}, \\ Q &= Q_{\rm pm} + Q_{\rm SS}^{\rm align}, \end{aligned} \tag{50}$$

where the nonspinning contributions are given by Eqs. (19)–(23), the SS corrections $A_{\rm SS}^{\rm align}$, $B_{np,{\rm SS}}^{\rm align}$ and $Q_{\rm SS}^{\rm align}$ are the same as in Eqs. (39), except for replacing a_+a_- by $\boldsymbol{a}_+\cdot\boldsymbol{a}_-$. The purely in-plane SS contributions are given by

$$\begin{aligned} A_{\rm SS}^{\rm prec} &= \frac{2\left(\boldsymbol{n}\cdot\boldsymbol{a}_+\right)^2 + 3C_+^{n\cdot a^2}}{r^3} \\ &\quad + \frac{1}{r^4}\left[\frac{33}{8}\delta\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(-\frac{\nu}{2} - \frac{3}{8}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 + \left(\frac{7\nu}{4} - \frac{31}{4}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 + 3\delta C_-^{n\cdot a^2} + 3\nu C_+^{n\cdot a^2}\right] \\ &\quad + \frac{1}{r^5}\left[\delta\left(17\nu - \frac{1}{4}\right)\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_- + \left(-\frac{41\nu^2}{8} + \frac{583\nu}{32} - \frac{171}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 \right.\end{aligned}$$

$$+\left(\frac{187}{64}-\frac{11\nu^2}{8}+\frac{1435\nu}{96}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\delta\left(\frac{81\nu}{16}-\frac{99}{224}\right)C_-^{n\cdot a^2}+\left(\frac{867}{224}-\frac{13\nu^2}{16}+\frac{179\nu}{56}\right)C_+^{n\cdot a^2}\Bigg], \tag{51a}$$

$$B_{p,\mathrm{SS}}^{\mathrm{prec}}=-\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{r^2}+\frac{1}{r^3}\left[\frac{3}{4}\delta\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_-+\left(\frac{3}{16}-\frac{3\nu}{4}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(-\frac{7\nu}{4}-\frac{15}{16}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+(3-3\nu)C_+^{n\cdot a^2}\right]$$
$$+\frac{1}{r^4}\Bigg[\delta\left(-\frac{49\nu}{4}-\frac{43}{8}\right)\boldsymbol{n}\cdot\boldsymbol{a}_+\boldsymbol{n}\cdot\boldsymbol{a}_-+\left(\frac{19\nu^2}{8}-\frac{545\nu}{32}+\frac{219}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2+\left(\frac{11\nu^2}{8}-\frac{805\nu}{96}+\frac{125}{64}\right)(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2$$
$$+\delta\left(\frac{189}{32}-\frac{81\nu}{16}\right)C_-^{n\cdot a^2}+\left(\frac{13\nu^2}{16}-\frac{203\nu}{8}+\frac{51}{32}\right)C_+^{n\cdot a^2}\Bigg]. \tag{51b}$$

The cubic-in-spin term G_{a^3} reads

$$G_{a^3}=\tilde{\boldsymbol{L}}\cdot\boldsymbol{a}_+\Bigg\{\frac{\tilde{L}^2}{r^3}\left[\frac{\delta}{2}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)-\frac{(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4}\right]+\frac{1}{r^2}\Bigg[-\frac{a_+^2}{4}-\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\delta\frac{5}{24}(\boldsymbol{a}_+\cdot\boldsymbol{a}_-)-\delta\frac{5}{3}(\boldsymbol{n}\cdot\boldsymbol{a}_-)(\boldsymbol{n}\cdot\boldsymbol{a}_+)$$
$$+\frac{1}{8}a_1^2\left(3\delta\widetilde{C}_{1ES^2}+4\widetilde{C}_{1BS^3}\right)+\frac{1}{8}a_2^2\left(4\widetilde{C}_{2BS^3}-3\widetilde{C}_{2ES^2}\delta\right)-\frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(\delta-3)-\widetilde{C}_{2ES^2}(\delta+3)\right)$$
$$+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\widetilde{C}_{1ES^2}(2\delta+3)-\frac{5\widetilde{C}_{1BS^3}}{2}\right)+\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(2\delta-7)-\widetilde{C}_{2ES^2}(2\delta+7)\right)$$
$$+\frac{1}{8}(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\widetilde{C}_{2ES^2}(6\delta-9)-20\widetilde{C}_{2BS^3}\right)\Bigg]\Bigg\}$$
$$+\delta\tilde{\boldsymbol{L}}\cdot\boldsymbol{a}_-\Bigg\{-\frac{\tilde{L}^2(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2}{4r^3}+\frac{1}{r^2}\Bigg[\frac{a_+^2}{24}+\frac{5}{12}(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2+\frac{1}{8}a_1^2\left(4\widetilde{C}_{1BS^3}-3\widetilde{C}_{1ES^2}\right)+\frac{1}{8}a_2^2\left(3\widetilde{C}_{2ES^2}-4\widetilde{C}_{2BS^3}\right)$$
$$-\frac{3}{8}(\boldsymbol{a}_1\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(\delta-3)+\widetilde{C}_{2ES^2}(\delta+3)\right)+(\boldsymbol{n}\cdot\boldsymbol{a}_1)^2\left(-\frac{3}{8}\widetilde{C}_{1ES^2}(\delta-6)-\frac{5\widetilde{C}_{1BS^3}}{2}\right)$$
$$+\frac{3}{4}(\boldsymbol{n}\cdot\boldsymbol{a}_1)(\boldsymbol{n}\cdot\boldsymbol{a}_2)\left(\widetilde{C}_{1ES^2}(2\delta-7)+\widetilde{C}_{2ES^2}(2\delta+7)\right)+(\boldsymbol{n}\cdot\boldsymbol{a}_2)^2\left(\frac{5\widetilde{C}_{2BS^3}}{2}-\frac{3}{8}\widetilde{C}_{2ES^2}(\delta+6)\right)\Bigg]\Bigg\}. \tag{52}$$

We do not include S^4 corrections for simplicity.

This Hamiltonian reduces to $H_{\mathrm{eff}}^{\mathrm{align}}$ from Sec. I C for aligned spins. It also agrees with $H_{\mathrm{eff}}^{\mathrm{prec}}$ from Sec. I D when PN expanded to 4PN and to $\mathcal{O}(S^3)$ for circular orbits ($p_r\to 0$). We remark that partial precessing-spin EOB Hamiltonian $H_{\mathrm{EOB}}^{\mathrm{pprec}}$ takes into account all spin components through $(\boldsymbol{l}^2,\boldsymbol{a}_\pm\cdot\boldsymbol{l}_N,\boldsymbol{a}_\pm\cdot\boldsymbol{l},\boldsymbol{a}_+\cdot\boldsymbol{a}_-)$, including orbit-averaged in-plane spin components, instead of only $\boldsymbol{a}_\pm\cdot\boldsymbol{l}_N$.

B. Orbit averaging the precessing-spin contributions

To simplify the EOMs, we remove the explicit dependence of the Hamiltonian on the $\boldsymbol{n}\cdot\boldsymbol{a}_{\mathrm{i}}$ terms by taking their orbit average. Since the spin-precession timescale ($\sim v^{-5}$) is larger than the orbital timescale ($\sim v^{-3}$), orbit-averaging the in-plane spin contributions is expected to provide a good approximation for the dynamics.

We define the unit vectors $(\boldsymbol{l}_N,\boldsymbol{n},\boldsymbol{\lambda}_N)$ with components

$$\begin{aligned}\boldsymbol{l}_N&=(0,0,1),\\ \boldsymbol{n}&=(\cos\phi,\sin\phi,0),\\ \boldsymbol{\lambda}_N\equiv\boldsymbol{l}_N\times\boldsymbol{n}&=(-\sin\phi,\cos\phi,0),\end{aligned} \tag{53}$$

where ϕ is the orbital phase and we take $\boldsymbol{l}_N$ to be aligned with the z-axis. When neglecting RR, an orbit average yields

$$\begin{aligned}\langle n^i\rangle&=\frac{1}{2\pi}\int_0^\pi n^i\mathrm{d}\phi=0=\langle\lambda_N^i\rangle,\\ \langle n^in^j\rangle&=\langle\lambda_N^i\lambda_N^j\rangle=\frac{1}{2}\left(\delta^{ij}-l_N^il_N^j\right),\end{aligned} \tag{54}$$

which lead to the following relations for the spin dot products:

$$\begin{aligned}
\langle(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{n}\rangle &= \langle(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_{\rm i})\boldsymbol{\lambda}_N\rangle = \frac{1}{2}\left[\boldsymbol{S}_{\rm i} - (\boldsymbol{l}_N\cdot\boldsymbol{S}_{\rm i})\boldsymbol{l}_N\right],\\
\left\langle(\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})^2\right\rangle &= \left\langle(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_{\rm i})^2\right\rangle = \frac{1}{2}\left[S_{\rm i}^2 - (\boldsymbol{l}_N\cdot\boldsymbol{S}_{\rm i})^2\right],\\
\langle(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\rangle &= \langle(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\rangle = \frac{1}{2}\left[\boldsymbol{S}_1\cdot\boldsymbol{S}_2 - (\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\right].
\end{aligned} \tag{55}$$

The Hamiltonian from Sec. II A depends on $(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2$, $(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2$ and $(\boldsymbol{n}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-)$, which we approximate by their orbit averages, given by

$$\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{a}_+)^2 &\simeq \frac{1}{2}\left[a_+^2 - (\boldsymbol{l}_N\cdot\boldsymbol{a}_+)^2\right],\\
(\boldsymbol{n}\cdot\boldsymbol{a}_-)^2 &\simeq \frac{1}{2}\left[a_-^2 - (\boldsymbol{l}_N\cdot\boldsymbol{a}_-)^2\right],\\
(\boldsymbol{n}\cdot\boldsymbol{a}_+)(\boldsymbol{n}\cdot\boldsymbol{a}_-) &\simeq \frac{1}{2}\left[\boldsymbol{a}_+\cdot\boldsymbol{a}_- - (\boldsymbol{l}_N\cdot\boldsymbol{a}_+)(\boldsymbol{l}_N\cdot\boldsymbol{a}_-)\right].
\end{aligned} \tag{56}$$

C. Equations of motion

The "Newtonian" angular-momentum vector $\boldsymbol{L}_N$ is perpendicular to the orbital plane and is defined by

$$\boldsymbol{L}_N \equiv \mu\boldsymbol{R}\times\mathbf{v}, \tag{57}$$

where $\mathbf{v} \equiv \dot{\boldsymbol{R}} = {\rm d}\boldsymbol{R}/{\rm d}T$ is the velocity. Henceforth, we use dimensionful variables for clarity, and to facilitate comparisons with the literature.

We consider a co-precessing orthonormal triad $(\boldsymbol{l}_N, \boldsymbol{n}, \boldsymbol{\lambda}_N)$, so that $\boldsymbol{l}_N$ is in the direction of $\boldsymbol{L}_N$ and $\boldsymbol{\lambda}_N \equiv \boldsymbol{l}_N\times\boldsymbol{n}$. However, the Hamiltonian is expressed in terms of the canonical angular momentum $\boldsymbol{L}\equiv\boldsymbol{R}\times\boldsymbol{P}$. We denote the $\boldsymbol{L}$-based unit vectors by $(\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{\lambda})$, where $\boldsymbol{l}$ is the direction of $\boldsymbol{L}$ and $\boldsymbol{\lambda}\equiv\boldsymbol{l}\times\boldsymbol{n}$.

In the co-precessing frame, the dynamics can be approximated by the aligned-spin EOMs

$$\dot{R} = \frac{\partial H_{\rm EOB}^{\rm simp}}{\partial P_R}, \qquad \dot{\phi} = \frac{\partial H_{\rm EOB}^{\rm simp}}{\partial P_\phi}, \qquad \dot{P}_R = -\frac{\partial H_{\rm EOB}^{\rm simp}}{\partial R} + \mathcal{F}_R, \qquad \dot{P}_\phi = \mathcal{F}_\phi, \tag{58}$$

where we use the Hamiltonian from Sec. II A after replacing the $\boldsymbol{n}\cdot\boldsymbol{a}_{\rm i}$ terms by their orbit average using Eqs. (56). At each time step, we also evolve the PN equations for the spins and angular momentum, given by

$$\dot{\boldsymbol{S}}_{\rm i} = \boldsymbol{\Omega}_{S_{\rm i}}\times\boldsymbol{S}_{\rm i}, \qquad \boldsymbol{L} = \boldsymbol{L}(\boldsymbol{l}_N, v, \boldsymbol{S}_{\rm i}), \qquad \dot{\boldsymbol{l}}_N = \dot{\boldsymbol{l}}_N(\boldsymbol{l}_N, v, \boldsymbol{S}_{\rm i}), \tag{59}$$

where $v\equiv(M\Omega)^{1/3}$, and $\boldsymbol{\Omega}_{S_{\rm i}} \equiv \partial H_{\rm EOB}^{\rm PN}/\partial\boldsymbol{S}_{\rm i}$ is the spin-precession frequency, computed in a PN expansion from the EOB Hamiltonian. These equations are derived in the following subsections to NNLO SS in an orbit average for circular orbits.

The coupled Eqs. (58) and (59) can be solved simultaneously for the dynamical variables. Alternatively, Eqs. (59) can be solved independently of Eqs. (58) by adding to them the PN evolution equation for the orbital frequency, which can be written as

$$\dot{v} = \frac{\dot{E}}{{\rm d}E(v)/{\rm d}v}, \tag{60}$$

where $E(v)$ is the energy of the binary system and $\dot{E}$ is the rate of energy loss.

Since $\boldsymbol{l}_N$ is perpendicular to $\boldsymbol{R}$ and $\mathbf{v}$, we can write the velocity as

$$\mathbf{v} = \dot{R}\boldsymbol{n} + \Omega R\boldsymbol{\lambda}_N, \tag{61}$$

which can be considered as a definition for the orbital frequency Ω, implying that $\Omega = \boldsymbol{l}_N\cdot(\boldsymbol{n}\times\mathbf{v})/R$.

Circular orbits are defined by $P_R = 0$ and $\dot{P}_R = 0$. Hence, to obtain R and L for circular orbits as functions of Ω, we solve

$$\frac{\rm d}{{\rm d}T}(\boldsymbol{R}\cdot\boldsymbol{P}) = \boldsymbol{R}\cdot\dot{\boldsymbol{P}} + \dot{\boldsymbol{R}}\cdot\boldsymbol{P} = 0, \quad \text{and} \quad v^3 = M\boldsymbol{l}_N\cdot\frac{\boldsymbol{n}\times\mathbf{v}}{R}, \tag{62}$$

for $r(v, \boldsymbol{l}, \boldsymbol{n})$ and $L(v, \boldsymbol{l}, \boldsymbol{n})$, perturbatively in a PN expansion, after using the EOMs (47) without RR. This yields expressions in terms of $\boldsymbol{l}$, and in the following subsection, we obtain them in terms of $\boldsymbol{l}_N$.

D. Angular momentum vector

To obtain the angular momentum unit vector $\boldsymbol{l}$ in terms of $\boldsymbol{l}_N$, we first use the EOMs (47), and the definition of $\boldsymbol{L}_N$ from Eq. (57), to get $\boldsymbol{l}_N$ in a PN expansion, i.e.,

$$\boldsymbol{l}_N \equiv \frac{\boldsymbol{L}_N}{|\boldsymbol{L}_N|} = \frac{\mu}{|\boldsymbol{L}_N|}\boldsymbol{R}\times\frac{\partial H_{\rm EOB}^{\rm prec}}{\partial \boldsymbol{P}}. \tag{63}$$

Then, we specialize to circular orbits by setting $P_R = 0$ and replacing R and L by their solution from Eqs. (62). We also replace $\boldsymbol{\lambda}$ using

$$\begin{aligned}\boldsymbol{S}_{\rm i} &= (\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{n} + (\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{\lambda} + (\boldsymbol{l}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{l}\\ &= (\boldsymbol{n}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{n} + (\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_{\rm i})\boldsymbol{\lambda}_N + (\boldsymbol{l}_N\cdot\boldsymbol{S}_{\rm i})\boldsymbol{l}_N,\end{aligned} \tag{64}$$

which implies that

$$\begin{aligned}(\boldsymbol{\lambda}\cdot\boldsymbol{S}_{\rm i})^2 + (\boldsymbol{l}\cdot\boldsymbol{S}_{\rm i})^2 &= (\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_{\rm i})^2 + (\boldsymbol{l}_N\cdot\boldsymbol{S}_{\rm i})^2,\\ (\boldsymbol{\lambda}\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}\cdot\boldsymbol{S}_2) + (\boldsymbol{l}\cdot\boldsymbol{S}_1)(\boldsymbol{l}\cdot\boldsymbol{S}_2) &= (\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2) + (\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2).\end{aligned} \tag{65}$$

That way, the right-hand side of Eq. (63) only depends on $\boldsymbol{l}$, $\boldsymbol{l}_N$, $\boldsymbol{\lambda}_N$ and v.

To solve Eq. (63) for $\boldsymbol{l}(\boldsymbol{l}_N)$, we expand it in spin, using

$$\boldsymbol{l}(\boldsymbol{l}_N) \equiv \boldsymbol{l}_N + \boldsymbol{l}^{\rm SO}(\boldsymbol{l}_N) + \boldsymbol{l}^{\rm SS}(\boldsymbol{l}_N), \tag{66a}$$

where $\boldsymbol{l}$ is in the same direction as $\boldsymbol{l}_N$ for nonspinning binaries, while $\boldsymbol{l}^{\rm SO}$ and $\boldsymbol{l}^{\rm SS}$ are the SO and SS parts. Solving order by order in spin, we obtain

$$\begin{aligned}\boldsymbol{l}^{\rm SO} &= \frac{(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)\boldsymbol{\lambda}_N}{M\mu}\bigg\{-\frac{v^3}{2}(3X_2+\nu) + v^5\left[\frac{\nu^2}{8} - \frac{9\nu}{8} + \left(\frac{3\nu}{2}+\frac{9}{8}\right)X_2\right]\\ &\quad + v^7\left[\frac{\nu^3}{48} + \frac{9\nu^2}{4} - \frac{27\nu}{16} + \left(-\frac{\nu^2}{2} + \frac{15\nu}{4} + \frac{27}{16}\right)X_2\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{66b}$$

$$\begin{aligned}\boldsymbol{l}^{\rm SS} &= \frac{v^4}{M^2\mu^2}\boldsymbol{\lambda}_N\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(X_2-\nu) + \nu(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\right]\\ &\quad + \frac{v^6}{M^2\mu^2}\bigg\{\boldsymbol{\lambda}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)\left[\frac{\nu^2}{6} + \frac{5\nu}{2} + \left(-\frac{11\nu}{3} - \frac{5}{2}\right)X_2\right] + \boldsymbol{\lambda}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(-\frac{7\nu^2}{6} - 4\nu\right)\\ &\qquad + \boldsymbol{l}_N(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[-\frac{\nu^2}{8} + \frac{9\nu}{8} + \left(-\frac{3\nu}{4} - \frac{9}{8}\right)X_2\right] + \boldsymbol{l}_N(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(-\frac{\nu^2}{8} - \frac{3}{2}\nu\right)\bigg\}\\ &\quad + \frac{v^8}{M^2\mu^2}\bigg\{\boldsymbol{\lambda}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)\left[-\frac{211\nu^3}{144} - \frac{41\nu^2}{16} - \frac{9\nu}{16} + \left(-\frac{479\nu^2}{144} + 2\nu + \frac{9}{16}\right)X_2\right]\\ &\qquad + \boldsymbol{l}_N(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^3}{16} - \frac{45\nu^2}{16} - \frac{27\nu}{16} + \left(\frac{15\nu^2}{16} + \frac{9\nu}{8} + \frac{27}{16}\right)X_2\right]\\ &\qquad + \boldsymbol{\lambda}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left[-\frac{179\nu^3}{144} + \frac{9\nu^2}{32} - \frac{3\nu}{8} + \left(\frac{3\nu^2}{8} + \frac{3\nu}{2}\right)X_2\right]\\ &\qquad + \boldsymbol{l}_N(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left[\frac{\nu^3}{16} + \frac{69\nu^2}{32} + \frac{9\nu}{8}\right]\bigg\} + 1\leftrightarrow 2,\end{aligned} \tag{66c}$$

which is independent of the spin-quadrupole constants. Substituting $\boldsymbol{l}(\boldsymbol{l}_N)$ in the solution of Eqs. (62) yields $R(\boldsymbol{l}_N, \boldsymbol{n}, v)$ and $L(\boldsymbol{l}_N, \boldsymbol{n}, v)$, which are given in Appendix A.

Finally, we use these relations to obtain $\boldsymbol{L} = L\boldsymbol{l}$ and take its orbit average using Eqs. (55), leading to

$$\boldsymbol{L} \equiv \boldsymbol{L}^{S^0} + \boldsymbol{L}^{\rm SO} + \boldsymbol{L}^{S_1S_2} + \boldsymbol{L}^{S^2} + \boldsymbol{L}^{S^2\tilde{C}}, \tag{67a}$$

$$\begin{aligned}\boldsymbol{L}^{S^0} &= \mu M\boldsymbol{l}_N\bigg\{\frac{1}{v} + \left(\frac{\nu}{6} + \frac{3}{2}\right)v + \left(\frac{\nu^2}{24} - \frac{19\nu}{8} + \frac{27}{8}\right)v^3 + \left[\frac{7\nu^3}{1296} + \frac{31\nu^2}{24} + \left(\frac{41\pi^2}{24} - \frac{6889}{144}\right)\nu + \frac{135}{16}\right]v^5\\ &\quad + v^7\bigg[-\frac{55\nu^4}{31104} - \frac{215\nu^3}{1728} + \left(\frac{356035}{3456} - \frac{2255\pi^2}{576}\right)\nu^2 + \nu\left(\frac{98869}{5760} - \frac{128\gamma_E}{3} - \frac{6455\pi^2}{1536} - \frac{256}{3}\ln 2 - \frac{128\log v}{3}\right)\end{aligned}$$

$$+\frac{2835}{128}\bigg]\bigg\}, \tag{67b}$$

$$\begin{aligned}
\boldsymbol{L}^{\rm SO} &= v^2\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{l}_N\left(-\frac{7\nu}{12}-\frac{7X_2}{4}\right)+\boldsymbol{S}_1\left(-\frac{\nu}{4}-\frac{3X_2}{4}\right)\right]\\
&+v^4\left\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{l}_N\left(\frac{11\nu^2}{144}-\frac{55\nu}{16}+\left(\frac{55\nu}{24}-\frac{33}{16}\right)X_2\right)+\boldsymbol{S}_1\left(\frac{\nu^2}{48}-\frac{15\nu}{16}+\left(\frac{5\nu}{8}-\frac{9}{16}\right)X_2\right)\right\}\\
&+v^6\bigg\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{l}_N\left[\frac{5\nu^3}{96}+\frac{275\nu^2}{32}-\frac{405\nu}{32}+\left(-\frac{25\nu^2}{32}+\frac{195\nu}{8}-\frac{135}{32}\right)X_2\right]\\
&\quad+\boldsymbol{S}_1\left[\frac{\nu^3}{96}+\frac{55\nu^2}{32}-\frac{81\nu}{32}+\left(-\frac{5\nu^2}{32}+\frac{39\nu}{8}-\frac{27}{32}\right)X_2\right]\bigg\}+1\leftrightarrow 2,
\end{aligned} \tag{67c}$$

$$\begin{aligned}
\boldsymbol{L}^{S_1S_2} &= \frac{v^3}{M^2}\left[\boldsymbol{l}_N(2(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)-(\boldsymbol{S}_1\cdot\boldsymbol{S}_2))+\frac{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{S}_2}{2}+\frac{(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\boldsymbol{S}_1}{2}\right]\\
&+\frac{v^5}{M^2}\left\{\boldsymbol{l}_N\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{13\nu}{36}-\frac{7}{6}\right)+\frac{2\nu(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)}{3}\right]+\boldsymbol{S}_2(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left(\frac{5}{4}-\frac{7\nu}{24}\right)+\boldsymbol{S}_1(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{5}{4}-\frac{7\nu}{24}\right)\right\}\\
&+\frac{v^7}{M^2}\bigg\{\boldsymbol{l}_N\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(-\frac{361\nu^2}{432}+\frac{361\nu}{288}+\frac{15}{4}\right)+\left(-\frac{5\nu^2}{72}-\frac{245\nu}{24}-\frac{5}{4}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]\\
&\quad+\boldsymbol{S}_2(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left(-\frac{223}{288}\nu^2-\frac{349\nu}{64}+\frac{15}{8}\right)+\boldsymbol{S}_1(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(-\frac{223}{288}\nu^2-\frac{349\nu}{64}+\frac{15}{8}\right)\bigg\},
\end{aligned} \tag{67d}$$

$$\begin{aligned}
\boldsymbol{L}^{S^2} &= \frac{v^3}{M\mu}\left\{\boldsymbol{l}_N\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2(X_2-\nu)+S_1^2\left(\frac{\nu}{2}-\frac{X_2}{2}\right)\right]+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{S}_1\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right\}\\
&+\frac{v^5}{M\mu}\bigg\{\boldsymbol{l}_N\left\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{121\nu^2}{72}+\frac{35\nu}{8}+\left(\frac{11\nu}{2}-\frac{35}{8}\right)X_2\right]+S_1^2\left[-\frac{\nu^2}{3}-\nu+\left(1-\frac{5\nu}{3}\right)X_2\right]\right\}\\
&\quad+\boldsymbol{S}_1(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left[\frac{5\nu^2}{24}-\frac{11\nu}{8}+\left(\frac{11}{8}-\frac{\nu}{2}\right)X_2\right]\bigg\}\\
&+\frac{v^7}{M\mu}\bigg\{\boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[-\frac{505\nu^3}{864}+\frac{347\nu^2}{96}+\frac{111\nu}{32}+\left(-\frac{2833\nu^2}{288}+\frac{199\nu}{16}-\frac{111}{32}\right)X_2\right]\\
&\quad+\boldsymbol{l}_NS_1^2\left[\frac{5\nu^3}{144}+\frac{275\nu^2}{48}-\frac{15\nu}{16}+\left(\frac{295\nu^2}{144}-\frac{455\nu}{48}+\frac{15}{16}\right)X_2\right]\\
&\quad+\boldsymbol{S}_1(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left[-\frac{235\nu^3}{288}+\frac{563\nu^2}{96}-\frac{21\nu}{32}+\left(-\frac{113\nu^2}{32}-\frac{7\nu}{3}+\frac{21}{32}\right)X_2\right]\bigg\}+1\leftrightarrow 2,
\end{aligned} \tag{67e}$$

$$\begin{aligned}
\boldsymbol{L}^{S^2\tilde{C}} &= \frac{\widetilde{C}_{1ES^2}}{M\mu}\boldsymbol{l}_N\bigg\{v^3\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{3X_2}{2}-\frac{3\nu}{2}\right)+S_1^2\left(\frac{\nu}{2}-\frac{X_2}{2}\right)\right]\\
&+v^5\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\nu^2-3\nu+(3\nu+3)X_2\right)+S_1^2\left(-\frac{\nu^2}{3}+\nu+(-\nu-1)X_2\right)\right]\\
&+v^7\bigg[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{48}+\frac{1475\nu^2}{112}-\frac{135\nu}{16}+\left(-\frac{65\nu^2}{16}+\frac{55\nu}{16}+\frac{135}{16}\right)X_2\right)\\
&\quad+S_1^2\left(\frac{5\nu^3}{144}-\frac{1475\nu^2}{336}+\frac{45\nu}{16}+\left(\frac{65\nu^2}{48}-\frac{55\nu}{48}-\frac{45}{16}\right)X_2\right)\bigg]\bigg\}+1\leftrightarrow 2.
\end{aligned} \tag{67f}$$

For aligned spins, $L(v)$ is gauge invariant, and our result agrees with the literature, e.g., with Refs. [24, 43–45]. However, for precessing spins, $\boldsymbol{L}$ is gauge dependent, and our result disagrees with Refs. [43, 46], even at LO SO, because these references used the covariant (Tulczyjew-Dixon) spin-supplementary condition (SSC) [47, 48], while we use the canonical Newton-Wigner (NW) SSC [49, 50] since we are working in a Hamiltonian formalism [19, 20]. Appendix B shows how to transform between our result and that of Refs. [43, 46] at LO SO.

One consistency check of the above calculations is that using these expansions, we can explicitly check that $\mathbf{v}\cdot\boldsymbol{\lambda}_N = \Omega\, R(\boldsymbol{l}_N, \boldsymbol{n}, v)$, and $\mathbf{v}\cdot\boldsymbol{l}_N = 0 = \mathbf{v}\cdot\boldsymbol{n}$, as expected from Eq. (61).

E. Spin-evolution equations

We obtain the spin-precession frequency $\boldsymbol{\Omega}_{S_i}$ by differentiating the Hamiltonian with respect to the spin vector. Then, we take the circular-orbit limit by setting $P_R = 0$ and replacing R and L by Eqs. (A1) and (A2). Finally, averaging the spin components over an orbit using Eqs. (55) yields

$$\dot{\boldsymbol{S}}_1 = \boldsymbol{\Omega}_{S_1} \times \boldsymbol{S}_1, \tag{68a}$$

$$\begin{aligned}
\boldsymbol{\Omega}_{S_1} = \frac{\boldsymbol{l}_N}{M}\Big\{ & v^5\left(\frac{3X_2}{2}+\frac{\nu}{2}\right) + v^7\left[\left(\frac{9}{8}-\frac{5\nu}{4}\right)X_2 - \frac{\nu^2}{24}+\frac{15\nu}{8}\right] \\
& + v^9\left[\left(\frac{5\nu^2}{16}-\frac{39\nu}{4}+\frac{27}{16}\right)X_2 - \frac{\nu^3}{48}-\frac{55\nu^2}{16}+\frac{81\nu}{16}\right]\Big\} \\
& + \frac{v^6}{M^2\mu}\left\{\boldsymbol{l}_N\left[\boldsymbol{l}_N\cdot\boldsymbol{S}_1\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)-\frac{3\nu}{2}\boldsymbol{l}_N\cdot\boldsymbol{S}_2\right]+\frac{\nu}{2}\boldsymbol{S}_2\right\} \\
& + \frac{v^8}{M^2\mu}\left\{\boldsymbol{l}_N\left[\boldsymbol{l}_N\cdot\boldsymbol{S}_1\left(-\frac{17\nu^2}{12}-\frac{9\nu}{4}+\left(\frac{9}{4}-\frac{15\nu}{4}\right)X_2\right)+\boldsymbol{l}_N\cdot\boldsymbol{S}_2\left(\frac{\nu^2}{12}-\frac{\nu}{2}\right)\right]-\frac{\nu^2}{4}\boldsymbol{S}_2\right\} \\
& + \frac{v^{10}}{M^2\mu}\Big\{\boldsymbol{l}_N\Big[\boldsymbol{l}_N\cdot\boldsymbol{S}_1\left(\frac{121\nu^3}{144}-\frac{91\nu^2}{16}-\frac{27\nu}{16}+\left(\frac{385\nu^2}{48}-\frac{97\nu}{16}+\frac{27}{16}\right)X_2\right) \\
& \qquad + \boldsymbol{l}_N\cdot\boldsymbol{S}_2\left(\frac{103\nu^3}{144}+\frac{139\nu^2}{48}-\frac{9\nu}{4}\right)\Big]+\left(\frac{\nu^3}{48}+\frac{49\nu^2}{16}+\frac{3\nu}{8}\right)\boldsymbol{S}_2\Big\}, \\
& + \frac{\widetilde{C}_{1ES^2}}{M^2\mu}\boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\Big\{v^6\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[-\frac{3\nu^2}{4}+\frac{9\nu}{4}+\left(-\frac{9\nu}{4}-\frac{9}{4}\right)X_2\right] \\
& \qquad + v^{10}\left[\frac{\nu^3}{16}-\frac{885\nu^2}{112}+\frac{81\nu}{16}+\left(\frac{39\nu^2}{16}-\frac{33\nu}{16}-\frac{81}{16}\right)X_2\right]\Big\},
\end{aligned} \tag{68b}$$

and similarly $\dot{\boldsymbol{S}}_2 = \boldsymbol{\Omega}_{S_2} \times \boldsymbol{S}_2$, with $\boldsymbol{\Omega}_{S_2}$ given by Eq. (68b) after exchanging the two bodies' labels $1 \leftrightarrow 2$. The SO and LO SS parts of the spin-precession frequency agree with the orbit-averaged results given by Eqs. (1)-(5) of Refs. [46, 51], but the NLO and NNLO SS terms do not agree with Refs. [51, 52] because of the different gauge.

F. Evolution of the orbital frequency

The evolution equation for the orbital frequency is given by Eq. (60) in terms of the energy loss and binding energy. The circular-orbit binding energy can be obtained from the Hamiltonian (minus the rest mass) by setting $P_R = 0$, replacing R, L and $\boldsymbol{l}$ by Eqs. (A1), (A2) and (66), then taking the orbit average. This leads to

$$E(v) \equiv -\frac{\mu v^2}{2}\left(\bar{E}^{S^0}+\bar{E}^{\rm SO}+\bar{E}^{S_1S_2}+\bar{E}^{S^2}+\bar{E}^{S^2\tilde{C}}\right), \tag{69a}$$

$$\bar{E}^{S^0} = 1+\left(-\frac{\nu}{12}-\frac{3}{4}\right)v^2+\left(-\frac{\nu^2}{24}+\frac{19\nu}{8}-\frac{27}{8}\right)v^4+\left[-\frac{35\nu^3}{5184}-\frac{155\nu^2}{96}+\left(\frac{34445}{576}-\frac{205\pi^2}{96}\right)\nu-\frac{675}{64}\right]v^6, \tag{69b}$$

$$\begin{aligned}
\bar{E}^{\rm SO} = \frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{M\mu}\Big\{ & v^3\left(\frac{2\nu}{3}+2X_2\right)+v^5\left[-\frac{\nu^2}{9}+5\nu+\left(3-\frac{10\nu}{3}\right)X_2\right] \\
& + v^7\left[-\frac{\nu^3}{12}-\frac{55\nu^2}{4}+\frac{81\nu}{4}+\left(\frac{5\nu^2}{4}-39\nu+\frac{27}{4}\right)X_2\right]\Big\} + 1\leftrightarrow 2,
\end{aligned} \tag{69c}$$

$$\begin{aligned}
\bar{E}^{S_1S_2} = {} & \frac{v^4}{M^2\mu^2}\left[-3\nu(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)+\nu(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]+\frac{v^6}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{5\nu^2}{18}-\frac{5\nu}{3}\right)-\frac{5\nu^2}{6}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right] \\
& + \frac{v^8}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{721\nu^3}{216}+\frac{973\nu^2}{72}-\frac{21\nu}{2}\right)+\left(\frac{7\nu^3}{72}+\frac{343\nu^2}{24}+\frac{7\nu}{4}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right],
\end{aligned} \tag{69d}$$

$$\begin{aligned}
\bar{E}^{S^2} = {} & \frac{v^4}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+S_1^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right] \\
& + \frac{v^6}{M^2\mu^2}\Big\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[-\frac{85\nu^2}{36}-\frac{15\nu}{4}+\left(\frac{15}{4}-\frac{25\nu}{4}\right)X_2\right]+S_1^2\left[\frac{5\nu^2}{12}+\frac{5\nu}{4}+\left(\frac{25\nu}{12}-\frac{5}{4}\right)X_2\right]\Big\}
\end{aligned}$$

$$+\frac{v^8}{M^2\mu^2}\bigg\{S_1^2\left[-\frac{7\nu^3}{144}-\frac{385\nu^2}{48}+\frac{21\nu}{16}+\left(-\frac{413\nu^2}{144}+\frac{637\nu}{48}-\frac{21}{16}\right)X_2\right]$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{847\nu^3}{432}-\frac{637\nu^2}{48}-\frac{63\nu}{16}+\left(\frac{2695\nu^2}{144}-\frac{679\nu}{48}+\frac{63}{16}\right)X_2\right]\bigg\}+1\leftrightarrow 2, \tag{69e}$$

$$\bar{E}^{S^2\tilde{C}}=\frac{\widetilde{C}_{1ES^2}}{M^2\mu^2}\bigg\{v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{3\nu}{2}-\frac{3X_2}{2}\right)+S_1^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right]$$
$$+v^6\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^2}{4}+\frac{15\nu}{4}+\left(-\frac{15\nu}{4}-\frac{15}{4}\right)X_2\right)+S_1^2\left(\frac{5\nu^2}{12}-\frac{5\nu}{4}+\left(\frac{5\nu}{4}+\frac{5}{4}\right)X_2\right)\right]$$
$$+v^8\bigg[S_1^2\left(-\frac{7\nu^3}{144}+\frac{295\nu^2}{48}-\frac{63\nu}{16}+\left(-\frac{91\nu^2}{48}+\frac{77\nu}{48}+\frac{63}{16}\right)X_2\right)$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{7\nu^3}{48}-\frac{295\nu^2}{16}+\frac{189\nu}{16}+\left(\frac{91\nu^2}{16}-\frac{77\nu}{16}-\frac{189}{16}\right)X_2\right)\bigg]\bigg\}+1\leftrightarrow 2. \tag{69f}$$

Note that we did not include the 4PN nonspinning contribution in the binding energy to keep it at the same order as the energy flux. The nonspinning and SO parts agree with Eqs. (233) and (415) of Ref. [53], while the SS part agrees in the aligned-spin limit with, e.g., Refs. [24, 44].

The NNLO SO contribution to the energy flux was derived in Ref. [54], while the NNLO SS (4PN beyond the LO) contribution was derived in Ref. [45], though the SS tail contribution at 3.5PN is known for aligned spins only. The result in Ref. [45] is expressed in terms of gauge-dependent quantities. Therefore, we used their EOMs to obtain the circular-orbit energy flux as a function of v, and orbit-averaged the in-plane spin components, leading to

$$\dot{E}\equiv-\frac{32\nu^2v^{10}}{5}\left(\dot{\bar{E}}^{S^0}+\dot{\bar{E}}^{\rm SO}+\dot{\bar{E}}^{S_1S_2}+\dot{\bar{E}}^{S^2}+\dot{\bar{E}}^{S^2\tilde{C}}\right), \tag{70a}$$

$$\dot{\bar{E}}^{S^0}=1+v^2\left(-\frac{35\nu}{12}-\frac{1247}{336}\right)+4\pi v^3+v^4\left(\frac{65\nu^2}{18}+\frac{9271\nu}{504}-\frac{44711}{9072}\right)+\pi v^5\left(-\frac{583\nu}{24}-\frac{8191}{672}\right)$$
$$+v^6\left[-\frac{775\nu^3}{324}-\frac{94403\nu^2}{3024}-\frac{134543\nu}{7776}+\pi^2\left(\frac{41\nu}{48}+\frac{16}{3}\right)-\frac{1712\ln v}{105}-\frac{1712\gamma_E}{105}+\frac{6643739519}{69854400}-\frac{3424\ln 2}{105}\right]$$
$$+\pi v^7\left(\frac{193385\nu^2}{3024}+\frac{214745\nu}{1728}-\frac{16285}{504}\right), \tag{70b}$$

$$\dot{\bar{E}}^{\rm SO}=\frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{M\mu}\bigg\{v^3\left(-\frac{3\nu}{2}-\frac{5X_2}{4}\right)+v^5\left(\frac{157\nu^2}{18}-\frac{23\nu}{8}+\left(\frac{43\nu}{4}-\frac{13}{16}\right)X_2\right)+\pi v^6\left(-\frac{17\nu}{3}-\frac{31X_2}{6}\right)$$
$$+v^7\left[-\frac{1117\nu^3}{54}+\frac{625\nu^2}{189}+\frac{180955\nu}{13608}+\left(-\frac{1501\nu^2}{36}+\frac{1849\nu}{126}+\frac{9535}{336}\right)X_2\right]$$
$$+\pi v^8\left[\frac{21241\nu^2}{336}-\frac{10069\nu}{672}+\left(\frac{130583\nu}{2016}-\frac{7163}{672}\right)X_2\right]\bigg\}+1\leftrightarrow 2, \tag{70c}$$

$$\dot{\bar{E}}^{S_1S_2}=\frac{\nu v^4}{M^2\mu^2}\left[\frac{289}{48}(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)-\frac{103}{48}(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]$$
$$+\frac{\nu v^6}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(-\frac{2023\nu}{72}-\frac{5647}{168}\right)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(\frac{821\nu}{72}+\frac{2123}{84}\right)\right]$$
$$+\frac{\nu v^8}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{2161\nu^2}{48}+\frac{60241\nu}{252}+\frac{107771}{1512}\right)+(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\left(-\frac{4405\nu^2}{144}-\frac{194687\nu}{1008}-\frac{895429}{9072}\right)\right]$$
$$+\frac{63\pi\nu v^7}{4M^2\mu^2}(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2), \tag{70d}$$

$$\dot{\bar{E}}^{S^2}=\frac{v^4}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{287X_2}{96}-\frac{287\nu}{96}\right)+S_1^2\left(\frac{89\nu}{96}-\frac{89X_2}{96}\right)\right]$$
$$+\frac{v^6}{M^2\mu^2}\left\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{2621\nu^2}{144}+\frac{1255\nu}{56}-\left(\frac{461\nu}{72}+\frac{1255}{56}\right)X_2\right]+S_1^2\left[\left(\frac{185\nu}{72}+\frac{801}{56}\right)X_2-\frac{727\nu^2}{144}-\frac{801\nu}{56}\right]\right\}$$
$$+\frac{v^8}{M^2\mu^2}\bigg\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[-\frac{5615\nu^3}{96}-\frac{62031\nu^2}{448}-\frac{250813\nu}{6048}+\left(-\frac{11903\nu^2}{288}+\frac{202963\nu}{1344}+\frac{250813}{6048}\right)X_2\right]$$

$$+ S_1^2\left[\frac{3371\nu^3}{288}+\frac{406253\nu^2}{4032}+\frac{963901\nu}{18144}+\left(\frac{439\nu^2}{96}-\frac{389723\nu}{4032}-\frac{963901}{18144}\right)X_2\right]\Bigg\}$$
$$+\frac{\pi v^7}{M^2\mu^2}(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{65X_2}{8}-\frac{65\nu}{8}\right)+1\leftrightarrow 2, \tag{70e}$$
$$\dot{E}^{S^2\tilde{C}}=\frac{\widetilde{C}_{1ES^2}}{M^2\mu^2}\Bigg\{v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2(3X_2-3\nu)+S_1^2(\nu-X_2)\right]$$
$$+v^6\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{129\nu^2}{8}+\frac{837\nu}{112}+\left(-\frac{135\nu}{16}-\frac{837}{112}\right)X_2\right)+S_1^2\left(-\frac{43\nu^2}{8}-\frac{279\nu}{112}+\left(\frac{45\nu}{16}+\frac{279}{112}\right)X_2\right)\right]$$
$$+v^8\Bigg[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{81\nu^3}{2}-\frac{41191\nu^2}{672}+\frac{74911\nu}{3024}+\left(-\frac{209\nu^2}{48}+\frac{46801\nu}{672}-\frac{74911}{3024}\right)X_2\right)$$
$$+S_1^2\left(\frac{27\nu^3}{2}+\frac{41191\nu^2}{2016}-\frac{74911\nu}{9072}+\left(\frac{209\nu^2}{144}-\frac{46801\nu}{2016}+\frac{74911}{9072}\right)X_2\right)\Bigg]$$
$$+\pi v^7(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2(8X_2-8\nu)\Bigg\}+1\leftrightarrow 2, \tag{70f}$$

where we expressed the aligned-spin SS tail part ($\mathcal{O}(v^7)$ beyond the LO) in terms of $\boldsymbol{l}_N\cdot\boldsymbol{S}_\mathrm{i}$ as an approximation for the precessing case which would also depend on S_1^2 and $\boldsymbol{S}_1\cdot\boldsymbol{S}_2$.

Inserting E and $\dot{E}$ in Eq. (60) and PN expanding yields

$$\dot{v}\equiv\frac{32\nu v^9}{5M}\left(\dot{\bar{v}}^{S^0}+\dot{\bar{v}}^{\mathrm{SO}}+\dot{\bar{v}}^{S_1S_2}+\dot{\bar{v}}^{S^2}+\dot{\bar{v}}^{S^2\tilde{C}}\right), \tag{71a}$$
$$\dot{\bar{v}}^{S^0}=1+\left(-\frac{11\nu}{4}-\frac{743}{336}\right)v^2+4\pi v^3+\left(\frac{59\nu^2}{18}+\frac{13661\nu}{2016}+\frac{34103}{18144}\right)v^4+\pi\left(-\frac{189\nu}{8}-\frac{4159}{672}\right)v^5$$
$$+v^6\left[\frac{541\nu^2}{896}-\frac{5605\nu^3}{2592}-\frac{56198689\nu}{217728}+\pi^2\left(\frac{451\nu}{48}+\frac{16}{3}\right)-\frac{1712\ln v}{105}-\frac{1712\gamma_E}{105}+\frac{16447322263}{139708800}-\frac{3424\ln 2}{105}\right]$$
$$+\pi\left(\frac{91495\nu^2}{1512}+\frac{358675\nu}{6048}-\frac{4415}{4032}\right)v^7, \tag{71b}$$
$$\dot{\bar{v}}^{\mathrm{SO}}=\frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{M\mu}\Bigg\{v^3\left(-\frac{19\nu}{6}-\frac{25X_2}{4}\right)+v^5\left[\frac{79\nu^2}{6}-\frac{21611\nu}{1008}+\left(\frac{281\nu}{8}-\frac{809}{84}\right)X_2\right]+\pi v^6\left(-\frac{37\nu}{3}-\frac{151X_2}{6}\right)$$
$$+v^7\left[-\frac{10819\nu^3}{432}+\frac{40289\nu^2}{288}-\frac{1932041\nu}{18144}+\left(-\frac{2903\nu^2}{32}+\frac{257023\nu}{1008}-\frac{1195759}{18144}\right)X_2\right]$$
$$+\pi v^8\left[\frac{34303\nu^2}{336}-\frac{46957\nu}{504}+\left(\frac{50483\nu}{224}-\frac{1665}{28}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \tag{71c}$$
$$\dot{\bar{v}}^{S_1S_2}=\frac{\nu v^4}{M^2\mu^2}\left[\frac{721(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)}{48}-\frac{247(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)}{48}\right]$$
$$+\frac{\nu v^6}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{14433}{224}-\frac{11779\nu}{288}\right)+\left(\frac{6373\nu}{288}+\frac{16255}{672}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]$$
$$+\frac{\nu\pi v^7}{M^2\mu^2}\left[\frac{207(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)}{4}-12(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\right]+\frac{\nu v^8}{M^2\mu^2}\Bigg[\left(-\frac{162541\nu^2}{3456}-\frac{195697\nu}{896}-\frac{9355721}{72576}\right)(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{33163\nu^2}{3456}-\frac{10150387\nu}{24192}+\frac{21001565}{24192}\right)\Bigg], \tag{71d}$$
$$\dot{\bar{v}}^{S^2}=\frac{v^4}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{719X_2}{96}-\frac{719\nu}{96}\right)+S_1^2\left(\frac{233\nu}{96}-\frac{233X_2}{96}\right)\right]$$
$$+\frac{v^6}{M^2\mu^2}\Bigg\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{25373\nu^2}{576}+\frac{2185\nu}{448}+\left(\frac{19423\nu}{576}-\frac{2185}{448}\right)X_2\right]$$
$$+S_1^2\left[-\frac{6011\nu^2}{576}-\frac{8503\nu}{448}+\left(\frac{8503}{448}-\frac{1177\nu}{576}\right)X_2\right]\Bigg\}$$
$$+\frac{\pi v^7}{M^2\mu^2}\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{209X_2}{8}-\frac{209\nu}{8}\right)+S_1^2(6\nu-6X_2)\right]$$

$$+\frac{v^8}{M^2\mu^2}\Bigg\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\left(\frac{11888267}{48384}-\frac{2392243\nu^2}{6912}+\frac{4063301\nu}{16128}\right)X_2-\frac{869429\nu^3}{6912}+\frac{14283281\nu^2}{48384}-\frac{11888267\nu}{48384}\right]$$
$$+S_1^2\left[\frac{138323\nu^3}{6912}+\frac{711521\nu^2}{5376}+\frac{8207303\nu}{145152}+\left(\frac{250693\nu^2}{6912}-\frac{812353\nu}{5376}-\frac{8207303}{145152}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \quad (71e)$$

$$\dot{v}^{S^2\tilde{C}}=\frac{\widetilde{C}_{1ES^2}}{M^2\mu^2}\Bigg\{v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{15X_2}{2}-\frac{15\nu}{2}\right)+S_1^2\left(\frac{5\nu}{2}-\frac{5X_2}{2}\right)\right]$$
$$+v^6\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{129\nu^2}{4}-\frac{1977\nu}{224}+\left(\frac{1977}{224}-\frac{73\nu}{16}\right)X_2\right)+S_1^2\left(-\frac{43\nu^2}{4}+\frac{659\nu}{224}+\left(\frac{73\nu}{48}-\frac{659}{224}\right)X_2\right)\right]$$
$$+v^8\Bigg[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{1567\nu^3}{24}+\frac{29329\nu^2}{224}-\frac{597271\nu}{6048}+\left(-\frac{5675\nu^2}{96}-\frac{1517\nu}{168}+\frac{597271}{6048}\right)X_2\right)$$
$$+S_1^2\left(\frac{1567\nu^3}{72}-\frac{29329\nu^2}{672}+\frac{597271\nu}{18144}+\left(\frac{5675\nu^2}{288}+\frac{1517\nu}{504}-\frac{597271}{18144}\right)X_2\right)\Bigg]$$
$$+\pi v^7\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2(26X_2-26\nu)+S_1^2(6\nu-6X_2)\right]\Bigg\}+1\leftrightarrow 2. \quad (71f)$$

The NNLO SO and LO SS parts of $\dot{v}$ agree with, e.g., Eq. (A1) of Ref. [55].

G. Evolution of the angular momentum vector

To derive the PN expansion for $\dot{\boldsymbol{l}}_N$, we start from the equation for the total angular momentum $\boldsymbol{J}=\boldsymbol{L}+\boldsymbol{S}_1+\boldsymbol{S}_2$. We start by neglecting RR, and in the following subsection compute the RR contribution. Setting $\dot{\boldsymbol{J}}\simeq 0$ yields

$$\dot{\boldsymbol{L}}+\dot{\boldsymbol{S}}_1+\dot{\boldsymbol{S}}_2=0, \quad (72)$$

where $\dot{\boldsymbol{S}}_{\rm i}$ is given by Eq. (68), while $\dot{\boldsymbol{L}}$ can be computed by taking the time derivative of Eq. (67).

Solving Eq. (72) for $\dot{\boldsymbol{l}}_N$ yields[4]

$$\dot{\boldsymbol{l}}_N\equiv\dot{\boldsymbol{l}}_N^{\rm SO}+\dot{\boldsymbol{l}}_N^{S_1S_2}+\dot{\boldsymbol{l}}_N^{S^2}+\dot{\boldsymbol{l}}_N^{S^2\tilde{C}}, \quad (73a)$$

$$\dot{\boldsymbol{l}}_N^{\rm SO}=\frac{\boldsymbol{l}_N\times\boldsymbol{S}_1}{M^2\mu}\Bigg\{v^6\left(-\frac{\nu}{2}-\frac{3X_2}{2}\right)+v^8\left[\frac{\nu^2}{4}-\frac{9\nu}{4}+\left(\frac{9\nu}{4}+\frac{9}{4}\right)X_2\right]$$
$$+v^{10}\left[-\frac{\nu^3}{48}+\frac{81\nu^2}{16}-\frac{27\nu}{16}+\left(-\frac{21\nu^2}{16}+\frac{63\nu}{16}+\frac{27}{16}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \quad (73b)$$

$$\dot{\boldsymbol{l}}_N^{S_1S_2}=\frac{\nu}{M^3\mu^2}\Bigg\{\frac{3}{2}v^7\left[(\boldsymbol{l}_N\times\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)+(\boldsymbol{l}_N\times\boldsymbol{S}_2)(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\right]$$
$$+v^9\Bigg[\left(\frac{5X_2}{4}-\frac{5}{8}\right)\boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)+(\boldsymbol{l}_N\times\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(-\frac{5\nu}{4}-\frac{15X_2}{8}-\frac{21}{4}\right)$$
$$+(\boldsymbol{l}_N\times\boldsymbol{S}_2)(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left(-\frac{5\nu}{4}+\frac{15X_2}{8}-\frac{57}{8}\right)+\left(\frac{3}{8}-\frac{3X_2}{4}\right)(\boldsymbol{S}_1\times\boldsymbol{S}_2)\Bigg]$$
$$+v^{11}\Bigg\{(\boldsymbol{l}_N\times\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left[-\frac{\nu^2}{6}+\frac{25\nu}{4}+\left(\frac{71\nu}{32}+\frac{9}{32}\right)X_2+\frac{15}{16}\right]$$
$$+(\boldsymbol{l}_N\times\boldsymbol{S}_2)(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left[-\frac{\nu^2}{6}+\frac{271\nu}{32}+\left(-\frac{71\nu}{32}-\frac{9}{32}\right)X_2+\frac{39}{32}\right]$$
$$+\boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1\times\boldsymbol{S}_2)\left[\frac{89\nu}{96}+\left(\frac{9}{16}-\frac{89\nu}{48}\right)X_2-\frac{9}{32}\right]$$
$$+(\boldsymbol{S}_1\times\boldsymbol{S}_2)\left[-\frac{9\nu}{16}+\left(\frac{9\nu}{8}+\frac{9}{16}\right)X_2-\frac{9}{32}\right]\Bigg\}\Bigg\}, \quad (73c)$$

---

[4] To solve Eq. (72), we split $\dot{\boldsymbol{l}}_N$ and $\dot{\boldsymbol{S}}_{\rm i}$ into SO and SS contributions, such that $\dot{\boldsymbol{l}}_N\equiv\dot{\boldsymbol{l}}_N^{\rm SO}+\dot{\boldsymbol{l}}_N^{\rm SS}$ and $\dot{\boldsymbol{S}}_{\rm i}\equiv\dot{\boldsymbol{S}}_{\rm i}^{\rm SO}+\dot{\boldsymbol{S}}_{\rm i}^{\rm SS}$, then solved order by order in spin for $\dot{\boldsymbol{l}}_N^{\rm SO}$ and $\dot{\boldsymbol{l}}_N^{\rm SS}$. When performing this calculation, several simplifications can be done: $\dot{\boldsymbol{S}}_{\rm i}$ is perpendicular to $\boldsymbol{S}_{\rm i}$, leading to $\boldsymbol{S}_1\cdot\dot{\boldsymbol{S}}_1=0=\boldsymbol{S}_2\cdot\dot{\boldsymbol{S}}_2$, and since $\dot{\boldsymbol{S}}_{\rm i}^{\rm SO}$ is perpendicular to $\boldsymbol{l}_N$, we get $\boldsymbol{l}_N\cdot\dot{\boldsymbol{S}}_{\rm i}^{\rm SO}=0$.

$$\dot{\boldsymbol{l}}_N^{S^2} = \frac{(\boldsymbol{l}_N \times \boldsymbol{S}_1)(\boldsymbol{l}_N \cdot \boldsymbol{S}_1)}{M^3\mu^2}\left\{v^7\left(\frac{3X_2}{2} - \frac{3\nu}{2}\right) + v^9\left[2\nu^2 + 9\nu + (3\nu - 9)X_2\right] + v^{11}\left[-\frac{23\nu^3}{16} - \frac{157\nu^2}{16} - \frac{93\nu}{16} + \left(-\frac{439\nu^2}{48} + 4\nu + \frac{93}{16}\right)X_2\right]\right\} + 1 \leftrightarrow 2, \tag{73d}$$

$$\dot{\boldsymbol{l}}_N^{S^2\tilde{C}} = \frac{\widetilde{C}_{1ES^2}}{M^3\mu^2}(\boldsymbol{l}_N \times \boldsymbol{S}_1)(\boldsymbol{l}_N \cdot \boldsymbol{S}_1)\left\{v^7\left(\frac{3X_2}{2} - \frac{3\nu}{2}\right) + v^9\left[\frac{11\nu^2}{8} + \frac{9\nu}{8} + \left(\frac{11\nu}{4} - \frac{9}{8}\right)X_2\right] + v^{11}\left[-\frac{43\nu^3}{96} + \frac{1077\nu^2}{224} + \frac{27\nu}{32} + \left(-\frac{479\nu^2}{96} + \frac{3\nu}{2} - \frac{27}{32}\right)X_2\right]\right\}, \tag{73e}$$

which agrees up to NLO SO, i.e. to $\mathcal{O}(v^8)$, with Eq. (4c) of Ref. [46] if one uses the coefficients of $\boldsymbol{L}(\boldsymbol{l}_N)$ from Eq. (67), instead of those in Ref. [46] because of the different SSC. Note that $\dot{\boldsymbol{l}}_N$ has a component parallel to $\boldsymbol{l}_N$, which enters at NLO and NNLO S_1S_2, and is given by

$$\dot{\boldsymbol{l}}_N \cdot \boldsymbol{l}_N = \frac{\nu}{M^3\mu^2}(\boldsymbol{l}_N \cdot \boldsymbol{S}_1 \times \boldsymbol{S}_2)\left\{v^9\left(\frac{X_2}{2} - \frac{1}{4}\right) + v^{11}\left[\frac{35\nu}{96} + \left(\frac{9}{8} - \frac{35\nu}{48}\right)X_2 - \frac{9}{16}\right]\right\}. \tag{74}$$

H. Radiation-reaction contribution to $\dot{\boldsymbol{l}}_N$

When computing $\dot{\boldsymbol{l}}_N$, RR enters when taking the time derivative of $\boldsymbol{L}$ through $\dot{v}$, which is given by Eq. (71), and from the nonzero $\dot{\boldsymbol{J}}$, which is given by

$$\begin{aligned}\dot{\boldsymbol{J}} &= \dot{\boldsymbol{L}} + \dot{\boldsymbol{S}}_1 + \dot{\boldsymbol{S}}_2 \\ &= \boldsymbol{R} \times \boldsymbol{\mathcal{F}} + \dot{\boldsymbol{S}}_1^{\rm RR} + \dot{\boldsymbol{S}}_2^{\rm RR},\end{aligned} \tag{75}$$

where we used the EOMs (47) to relate $\dot{\boldsymbol{J}}$ to the RR force $\boldsymbol{\mathcal{F}}$. Since we are working to NNLO SS, i.e. to $\mathcal{O}(v^{10})$ in $\dot{\boldsymbol{L}}$ and $\dot{\boldsymbol{S}}_{\rm i}$, we only need $\dot{\boldsymbol{J}}$ to $\mathcal{O}(v^3)$ beyond the LO, which is $\mathcal{O}(v^7)$, and we can neglect the RR contribution to $\dot{\boldsymbol{S}}_{\rm i}$ because it starts at $\mathcal{O}(v^{11}S^2)$ [38, 39].

The RR force $\boldsymbol{\mathcal{F}}$ for circular orbits in SEOBNR waveform models is chosen to be in a gauge such that [36, 56][5]

$$\boldsymbol{\mathcal{F}} = \frac{\dot{E}}{\Omega L}\boldsymbol{P}. \tag{76}$$

Using the energy loss from Eq. (70) and expanding to LO SO for circular orbits, we get

$$\boldsymbol{\mathcal{F}} = -\frac{32}{5}\nu^2 v^9 \boldsymbol{\lambda}\left\{1 + v^2\left(-\frac{13\nu}{4} - \frac{1247}{336}\right) + 4\pi v^3 + \frac{v^3}{M\mu}\left[\boldsymbol{l}\cdot\boldsymbol{S}_1\left(-\frac{4\nu}{3} - \frac{3X_2}{4}\right) + \boldsymbol{l}\cdot\boldsymbol{S}_2\left(-\frac{4\nu}{3} - \frac{3X_1}{4}\right)\right]\right\}, \tag{77}$$

where we did not write the $\boldsymbol{n}$ component of $\boldsymbol{\mathcal{F}}$ since it does not contribute to $\dot{\boldsymbol{J}}$ and is proportional to P_R. Then, from Eq. (75), and using Eq. (66) to replace $\boldsymbol{l} = \boldsymbol{n} \times \boldsymbol{\lambda}$ by $\boldsymbol{l}_N$, we obtain

$$\dot{\boldsymbol{J}} = -\frac{32}{5}M\nu^2 v^7\left\{\boldsymbol{l}_N\left[1 + v^2\left(-\frac{35\nu}{12} - \frac{1247}{336}\right) + 4\pi v^3\right] + \frac{v^3}{M\mu}\left[\boldsymbol{l}_N\left(\frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{4}(-5\nu - 2X_2) + \frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_2}{4}(-5\nu - 2X_1)\right) + \boldsymbol{S}_1\left(-\frac{\nu}{4} - \frac{3X_2}{4}\right) + \boldsymbol{S}_2\left(-\frac{\nu}{4} - \frac{3X_1}{4}\right)\right]\right\}. \tag{78}$$

Following similar steps as in the previous subsection, except for including $\dot{\boldsymbol{J}}$ and $\dot{v}$, we obtain the following RR contribution to $\dot{\boldsymbol{l}}_N$:

$$\dot{\boldsymbol{l}}_N^{\rm RR} = -\frac{64}{5}\frac{v^8}{M}\left\{\nu\boldsymbol{l}_N\left[1 + v^2\left(-\frac{37\nu}{12} - \frac{1751}{336}\right) + 4\pi v^3\right] + \frac{v^3}{M\mu}\left[\boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\left(\frac{9\nu^2}{8} - \frac{19\nu}{12} + 5X_2\nu - \frac{25}{8}X_2\right) + \boldsymbol{l}_N(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{9\nu^2}{8} - \frac{19\nu}{12} + 5X_1\nu - \frac{25}{8}X_1\right) + \boldsymbol{S}_1\left(\frac{\nu^2}{8} + \frac{3\nu X_2}{8}\right) + \boldsymbol{S}_2\left(\frac{\nu^2}{8} + \frac{3\nu X_1}{8}\right)\right]\right\}. \tag{79}$$

We do not include this RR contribution in the `SEOBNRv5PHM` waveform model, but we checked that it has a negligible effect on the dynamics.

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[5] This relation was derived in Ref. [56] for precessing spins, and is given at LO SO by Eq. (3.27) there, which includes an extra term depending on $(\boldsymbol{P}\cdot\boldsymbol{S}_{\rm i})\boldsymbol{L}$ that averages to zero over an orbit.

III. 3.5PN SPIN CONTRIBUTIONS TO THE FACTORIZED MODES

[This section is mostly extracted from Ref. [57], and it contains the expressions for the 3.5PN factorized modes for aligned spins in circular orbits. The nonspinning part is only written to the order needed to produce the spin contributions. We mark in color the new terms that were included, or corrected, in `SEOBNRv5` compared to `SEOBNRv4`.]

The waveform can be written in a factorized, resummed form as [31, 58–60]

$$h^{\mathrm{F}}_{\ell m} = h^{(N,\epsilon_p)}_{\ell m} \hat{S}^{(\epsilon_p)}_{\mathrm{eff}} T_{\ell m} e^{i\delta_{\ell m}} f_{\ell m} \,, \tag{80}$$

where ϵ_p is the parity of $\ell+m$: $\epsilon_p = 0$ if $\ell+m$ is even, and $\epsilon_p = 1$ if $\ell+m$ is odd. The first term $h^{(N,\epsilon_p)}_{\ell m}$ is the leading (Newtonian) order waveform, which is known for any (ℓ, m), and its explicit expression is given in, e.g., Eq. (3) of Ref. [59].

The (dimensionless) effective source term $\hat{S}_{\mathrm{eff}}$ is given by either the effective energy E_{eff} or the orbital angular momentum p_ϕ, both expressed as functions of $v \equiv (M\omega)^{1/3} = \sqrt{x}$, such that

$$\hat{S}_{\mathrm{eff}} = \begin{cases} \frac{E_{\mathrm{eff}}(v)}{\mu}, & \ell+m \text{ even} \\ v\,\frac{p_\phi(v)}{\mu M}, & \ell+m \text{ odd} \end{cases} , \tag{81}$$

where E_{eff} is related to the total energy E via the EOB energy map $E = M\sqrt{1+2\nu\left(E_{\mathrm{eff}}/\mu - 1\right)}$. The factor $T_{\ell m}$ resums the infinite number of "leading logarithms" entering the tail effects [61], and is given by

$$T_{\ell m} = \frac{\Gamma\left(\ell+1-2i\hat{k}\right)}{\Gamma(\ell+1)} e^{\pi\hat{k}} e^{2i\hat{k}\ln(2m\omega r_0)}, \tag{82}$$

where $\Gamma(...)$ is the Euler gamma function, $\hat{k} \equiv m\omega E$, ω is the orbital frequency, and the constant r_0 takes the value $2M/\sqrt{e}$ to give agreement with waveforms computed in the test-body limit [59].

The remaining part of the factorized modes is expressed as an amplitude $f_{\ell m}$ and a phase $\delta_{\ell m}$, which are computed such that the expansion of $h^{\mathrm{F}}_{\ell m}$ agrees with the PN-expanded modes. To improve the agreement with numerical-relativity waveforms, $f_{\ell m}$ is further resummed as [59, 60] $f_{\ell m} = (\rho_{\ell m})^\ell$ to reduce the magnitude of the 1PN non-spinning coefficient, which grows linearly with ℓ. For spinning binaries, the non-spinning and spin contributions are separated for the odd m modes, such that

$$f_{\ell m} = \begin{cases} \rho^\ell_{\ell m}, & m \text{ even} \\ (\rho^{\mathrm{NS}}_{\ell m})^\ell + f^{\mathrm{S}}_{\ell m}, & m \text{ odd} \end{cases} , \tag{83}$$

where $\rho^{\mathrm{NS}}_{\ell m}$ is the non-spinning part of $\rho_{\ell m}$, while $f^{\mathrm{S}}_{\ell m}$ is the spin part of $f_{\ell m}$.

To simplify the expressions for the factorized modes, and to be consistent with the notation used in the literature, we introduce the dimensionless symmetric and antisymmetric spin parameters

$$\chi_S \equiv \frac{1}{2}\left(\chi_1 + \chi_2\right) , \tag{84a}$$

$$\chi_A \equiv \frac{1}{2}\left(\chi_1 - \chi_2\right) , \tag{84b}$$

and define the following combinations of the spin-multipole constants and spins:

$$\tilde{\kappa}_S \equiv \frac{1}{2}\left[\chi_1^2(\kappa_1 - 1) + \chi_2^2(\kappa_2 - 1)\right] , \tag{85a}$$

$$\tilde{\kappa}_A \equiv \frac{1}{2}\left[\chi_1^2(\kappa_1 - 1) - \chi_2^2(\kappa_2 - 1)\right] , \tag{85b}$$

$$\tilde{\lambda}_S \equiv \frac{1}{2}\left[\chi_1^3(\lambda_1 - 1) + \chi_2^3(\lambda_2 - 1)\right] , \tag{85c}$$

$$\tilde{\lambda}_A \equiv \frac{1}{2}\left[\chi_1^3(\lambda_1 - 1) - \chi_2^3(\lambda_2 - 1)\right] , \tag{85d}$$

which equal zero for black holes.

For the $(2,2)$ mode, we obtain

$$\begin{aligned}
\rho_{22} = {} & 1 + v^2\left(\tfrac{55}{84}\nu - \tfrac{43}{42}\right) + v^3\left[\left(\tfrac{2}{3}\nu - \tfrac{2}{3}\right)\chi_S - \tfrac{2}{3}\delta\chi_A\right] + v^4\Big[\tfrac{19583}{42336}\nu^2 - \tfrac{33025}{21168}\nu - \tfrac{20555}{10584} \\
& + \left(\tfrac{1}{2} - 2\nu\right)\chi_A^2 + \delta\chi_A\chi_S + \tfrac{1}{2}\chi_S^2 + \tfrac{1}{2}\delta\kappa_A + \kappa_S\left(\tfrac{1}{2} - \nu\right)\Big] \\
& + v^5\left[\delta\left(-\tfrac{19}{42}\nu - \tfrac{34}{21}\right)\chi_A + \left(\tfrac{209}{126}\nu^2 + \tfrac{49}{18}\nu - \tfrac{34}{21}\right)\chi_S\right] \\
& + v^6\Big[\delta\left(\tfrac{89}{126} - \tfrac{781}{252}\nu\right)\chi_A\chi_S + \left(-\tfrac{27}{14}\nu^2 - \tfrac{457}{504}\nu + \tfrac{89}{252}\right)\chi_A^2 + \left(\tfrac{10}{9}\nu^2 - \tfrac{1817}{504}\nu + \tfrac{89}{252}\right)\chi_S^2 \\
& + \left(\tfrac{67}{84} - \tfrac{139}{168}\nu\right)\delta\tilde{\kappa}_A + \left(-\tfrac{27}{28}\nu^2 - \tfrac{407}{168}\nu + \tfrac{67}{84}\right)\tilde{\kappa}_S\Big] \\
& + v^7\Big[\delta\left(\tfrac{97865}{63504}\nu^2 + \tfrac{50140}{3969}\nu + \tfrac{18733}{15876}\right)\chi_A + \left(\tfrac{50803}{63504}\nu^3 - \tfrac{245717}{63504}\nu^2 + \tfrac{74749}{5292}\nu + \tfrac{18733}{15876}\right)\chi_S \\
& + \delta\chi_A^3\left(\tfrac{1}{3} - \tfrac{4}{3}\nu\right) + \delta(2\nu+1)\chi_A\chi_S^2 + \left(-4\nu^2 - 3\nu + 1\right)\chi_A^2\chi_S + \left(\nu + \tfrac{1}{3}\right)\chi_S^3 \\
& + \tilde{\kappa}_S\left[\left(\tfrac{1}{3}\nu + \tfrac{4}{3}\right)\delta\chi_A + \left(-2\nu^2 - \tfrac{14}{3}\nu + \tfrac{4}{3}\right)\chi_S\right] + \tilde{\kappa}_A\left[\left(\tfrac{4}{3} - \tfrac{7}{3}\nu\right)\chi_A + \left(\tfrac{4}{3} - 2\nu\right)\delta\chi_S\right] \\
& + (\nu - 1)\delta\tilde{\lambda}_A + (3\nu - 1)\tilde{\lambda}_S\Big], \qquad (86a)
\end{aligned}$$

$$\delta_{22} = \tfrac{7}{3}\omega E + (\omega E)^2\left[\left(\tfrac{8}{3}\nu - \tfrac{4}{3}\right)\chi_S - \tfrac{4}{3}\delta\chi_A\right], \qquad (86b)$$

where the energy E in $\delta_{\ell m}$ is replaced by the Hamiltonian. The coefficient $19/42$ of $\mathcal{O}(v^5\delta\chi_A\nu)$ corrects a typo in the `SEOBNRv4` code.

For the $(2,1)$ mode, we obtain

$$\rho_{21}^{\rm NS} = 1 + \left(\tfrac{23}{84}\nu - \tfrac{59}{56}\right)v^2 + \left(\tfrac{617}{4704}\nu^2 - \tfrac{10993}{14112}\nu - \tfrac{47009}{56448}\right)v^4, \qquad (87a)$$

$$\begin{aligned}
f_{21}^{\rm S} = {} & v\left(-\tfrac{3}{2}\tfrac{\chi_A}{\delta} - \tfrac{3}{2}\chi_S\right) + v^3\Big[\left(\tfrac{131}{84}\nu + \tfrac{61}{12}\right)\tfrac{\chi_A}{\delta} + \left(\tfrac{79}{84}\nu + \tfrac{61}{12}\right)\chi_S\Big] \\
& + v^4\left[(-2\nu - 3)\chi_A^2 + \left(\tfrac{21}{2}\nu - 6\right)\frac{\chi_A\chi_S}{\delta} + \left(\tfrac{1}{2}\nu - 3\right)\chi_S^2 + \left(-\nu - \tfrac{1}{2}\right)\tilde{\kappa}_S - \frac{\tilde{\kappa}_A}{2\delta}\right] \\
& + v^5\Big[\left(-\tfrac{703}{112}\nu^2 + \tfrac{8797}{1008}\nu - \tfrac{81}{16}\right)\frac{\chi_A}{\delta} + \left(\tfrac{613}{1008}\nu^2 + \tfrac{1709}{1008}\nu - \tfrac{81}{16}\right)\chi_S \\
& + \left(\tfrac{3}{4} - 3\nu\right)\frac{\chi_A^3}{\delta} + \left(\tfrac{9}{4} - 6\nu\right)\frac{\chi_A\chi_S^2}{\delta} + \left(\tfrac{9}{4} - 3\nu\right)\chi_A^2\chi_S + \tfrac{3}{4}\chi_S^3 \\
& + \tfrac{3}{4}\tilde{\kappa}_A\chi_A + \left(\tfrac{3}{4} - 3\nu\right)\frac{\tilde{\kappa}_A\chi_S}{\delta} + \left(\tfrac{3}{4} - \tfrac{3}{2}\nu\right)\frac{\tilde{\kappa}_S\chi_A}{\delta} + \left(\tfrac{3}{4} - \tfrac{3}{2}\nu\right)\tilde{\kappa}_S\chi_S\Big] \\
& + v^6\Big[\left(\tfrac{5}{7}\nu^2 - \tfrac{9287}{1008}\nu + \tfrac{4163}{252}\right)\chi_A^2 + \left(\tfrac{139}{72}\nu^2 - \tfrac{2633}{1008}\nu + \tfrac{4163}{252}\right)\chi_S^2 \\
& + \left(\tfrac{9487}{504}\nu^2 - \tfrac{1636}{21}\nu + \tfrac{4163}{126}\right)\frac{\chi_A\chi_S}{\delta} + \left(\tfrac{473}{84}\nu^2 + \tfrac{8}{21}\nu + \tfrac{1}{21}\right)\frac{\tilde{\kappa}_A}{\delta} \\
& + \left(\tfrac{5}{14}\nu^2 + \tfrac{10}{21}\nu + \tfrac{1}{21}\right)\tilde{\kappa}_S\Big] \\
& + i(\omega E)^2\left[\left(-\tfrac{69}{140}\nu - \tfrac{17}{35}\right)\frac{\chi_A}{\delta} + \left(-\tfrac{41}{28}\nu - \tfrac{17}{35}\right)\chi_S\right], \qquad (87b)
\end{aligned}$$

$$\delta_{21} = \tfrac{2}{3}\omega E - \tfrac{25}{2}\nu v^5. \qquad (87c)$$

We note that the $\mathcal{O}(v^6\chi^2\nu^2)$ terms in the (2,1) mode disagree with those used in the `SEOBNRv4HM` model [62].[6] Another discrepancy with literature [34, 60] is in the $\mathcal{O}(\nu v^5)$ nonspinning part of δ_{21}, whose coefficient had the value $-493/42$ instead of $-25/2$, due to an error in the (2,1) mode in Ref. [63], which was later corrected in an erratum.

For the odd m modes, the functions $f_{\ell m}$ and $\delta_{\ell m}$ depend on $1/\delta$, which diverges for equal masses. In the implementation for equal masses, we pull the factor δ from the leading order into $f_{\ell m}$ to cancel $1/\delta$. To avoid the divergence in $\delta_{\ell m}$, we moved the spin part from $\delta_{\ell m}$ directly to $f_{\ell m}$ after multiplying it by i, as was done for the $(3,3)$ mode in `SEOBNRv4HM` [62].

The (3,2) and (4,3) modes are given by

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[6] The difference, for black holes, is given by

$$f_{21}^{\rm S} - f_{21}^{\rm S,(\texttt{SEOBNRv4HM})} = v^6\nu^2\left(\tfrac{165}{112}\chi_A^2 + \tfrac{87}{56}\frac{\chi_A\chi_S}{\delta} + \tfrac{165}{112}\chi_S^2\right). \qquad (88)$$

$$\rho_{32} = 1 + v\frac{4\nu\chi_S}{3(1-3\nu)} + v^2\left[\frac{-\frac{32}{27}\nu^2+\frac{223}{54}\nu-\frac{164}{135}}{1-3\nu} - \frac{16\nu^2\chi_S^2}{9(1-3\nu)^2}\right]$$
$$+ v^3\left[\left(\tfrac{13}{9}\nu+\tfrac{2}{9}\right)\frac{\delta\chi_A}{1-3\nu} + \left(\tfrac{607}{81}\nu^3+\tfrac{503}{81}\nu^2-\tfrac{1478}{405}\nu+\tfrac{2}{9}\right)\frac{\chi_S}{(1-3\nu)^2} + \frac{320\nu^3\chi_S^3}{81(1-3\nu)^3}\right]$$
$$+ v^4\left[\frac{\frac{77141}{40095}\nu^4-\frac{508474}{40095}\nu^3-\frac{945121}{320760}\nu^2+\frac{1610009\nu}{320760}-\frac{180566}{200475}}{(1-3\nu)^2} + \left(4\nu^2-3\nu+\tfrac{1}{3}\right)\frac{\chi_A^2}{1-3\nu}\right.$$
$$+ \left(-\tfrac{50}{27}\nu^2-\tfrac{88}{27}\nu+\tfrac{2}{3}\right)\frac{\delta\chi_A\chi_S}{(1-3\nu)^2} + \left(-\tfrac{2452}{243}\nu^4-\tfrac{1997}{243}\nu^3+\tfrac{1435}{243}\nu^2-\tfrac{43}{27}\nu+\tfrac{1}{3}\right)\frac{\chi_S^2}{(1-3\nu)^3}$$
$$\left.+ \left(\tfrac{1}{3}-\tfrac{1}{3}\nu\right)\frac{\delta\tilde{\kappa}_A}{1-3\nu} + \left(2\nu^2-\nu+\tfrac{1}{3}\right)\frac{\tilde{\kappa}_S}{1-3\nu}\right]$$
$$+ v^5\left[\left(-\tfrac{1184225}{96228}\nu^5-\tfrac{40204523}{962280}\nu^4+\tfrac{101706029}{962280}\nu^3-\tfrac{14103833}{192456}\nu^2+\tfrac{20471053}{962280}\nu-\tfrac{2788}{1215}\right)\frac{\chi_S}{(1-3\nu)^3}\right.$$
$$+ \left(\tfrac{889673}{106920}\nu^3-\tfrac{75737}{5346}\nu^2+\tfrac{376177}{35640}\nu-\tfrac{2788}{1215}\right)\frac{\delta\chi_A}{(1-3\nu)^2} + \left(\tfrac{608}{81}\nu^3+\tfrac{736}{81}\nu^2-\tfrac{16}{9}\nu\right)\frac{\delta\chi_A\chi_S^2}{(1-3\nu)^3}$$
$$+ \left(\tfrac{96176}{2187}\nu^5+\tfrac{43528}{2187}\nu^4-\tfrac{40232}{2187}\nu^3+\tfrac{376}{81}\nu^2-\tfrac{8\nu}{9}\right)\frac{\chi_S^3}{(1-3\nu)^4}$$
$$+ \left(-\tfrac{32}{3}\nu^3+8\nu^2-\tfrac{8}{9}\nu\right)\frac{\chi_A^2\chi_S}{(1-3\nu)^2} + \left(\tfrac{8}{9}\delta\nu^2-\tfrac{8}{9}\delta\nu\right)\frac{\chi_S\tilde{\kappa}_A}{(1-3\nu)^2}$$
$$\left.+ \left(-\tfrac{16}{3}\nu^3+\tfrac{8}{3}\nu^2-\tfrac{8}{9}\nu\right)\frac{\chi_S\tilde{\kappa}_S}{(1-3\nu)^2}\right], \tag{89a}$$

$$\delta_{32} = \left(\tfrac{11}{5}\nu+\tfrac{2}{3}\right)\frac{\omega E}{1-3\nu} + v^4\left[\frac{4\delta\chi_A}{1-3\nu} + \left(\tfrac{36}{5}\nu^2-20\nu+4\right)\frac{\chi_S}{(1-3\nu)^2}\right]$$
$$+ v^5\left[\left(-\tfrac{144}{5}\nu^3+80\nu^2-16\nu\right)\frac{\chi_S^2}{(1-3\nu)^3} - \frac{16\delta\nu\chi_A\chi_S}{(1-3\nu)^2}\right], \tag{89b}$$

$$\rho_{43}^{\rm NS} = 1 + \frac{v^2}{1-2\nu}\left(-\tfrac{10}{11}\nu^2+\tfrac{547}{176}\nu-\tfrac{111}{88}\right), \tag{90a}$$

$$f_{43}^{\rm S} = \frac{v}{1-2\nu}\left(\tfrac{5}{2}\nu\chi_S-\tfrac{5}{2}\nu\frac{\chi_A}{\delta}\right) + \frac{v^3}{1-2\nu}\left[\left(\tfrac{887}{44}\nu-\tfrac{3143}{132}\nu^2\right)\frac{\chi_A}{\delta} + \left(-\tfrac{529}{132}\nu^2-\tfrac{667}{44}\nu\right)\chi_S\right]$$
$$+ \frac{v^4}{1-2\nu}\left[\left(12\nu^2-\tfrac{37}{3}\nu+\tfrac{3}{2}\right)\chi_A^2 + \left(\tfrac{137}{6}\nu^2-18\nu+3\right)\frac{\chi_A\chi_S}{\delta} + \left(\tfrac{35}{6}\nu^2+\tfrac{1}{3}\nu+\tfrac{3}{2}\right)\chi_S^2\right.$$
$$\left.+ \left(6\nu^2-9\nu+\tfrac{3}{2}\right)\frac{\tilde{\kappa}_A}{\delta} + \left(6\nu^2-6\nu+\tfrac{3}{2}\right)\tilde{\kappa}_S\right]$$
$$+ i\frac{v^4}{(1-2\nu)^2}\left[\left(\tfrac{17999}{324}\nu^2-\tfrac{2605}{108}\nu+\tfrac{11}{4}\right)\frac{\chi_A}{\delta} + \left(\tfrac{2569}{324}\nu^2-\tfrac{2011}{108}\nu+\tfrac{11}{4}\right)\chi_S\right], \tag{90b}$$

$$\delta_{43} = \left(\tfrac{4961}{810}\nu+\tfrac{3}{5}\right)\frac{\omega E}{1-2\nu}, \tag{90c}$$

Since Ref. [57] was published when the model was close to being finalized, we only added the terms we considered most important, and we will add all new terms in a future update of the model.

Appendix A: Angular momentum and separation for circular orbits

In this Appendix, we write $R(\boldsymbol{l}_N, \boldsymbol{n}, v)$ and $L(\boldsymbol{l}_N, \boldsymbol{n}, v)$ for circular orbits and precessing spins to NNLO SS, which are obtained by solving Eqs. (62) and replacing $\boldsymbol{l}$ by $\boldsymbol{l}(\boldsymbol{l}_N)$ from Eq. (66).

For $L(\boldsymbol{l}_N, \boldsymbol{n}, v)$, we get

$$L \equiv \frac{M\mu}{v}\left(\bar{L}^{S^0} + \bar{L}^{\rm SO} + \bar{L}^{S_1S_2} + \bar{L}^{S^2} + \bar{L}^{S^2\tilde{C}}\right), \tag{A1a}$$

$$\bar{L}^{S^0} = 1 + v^2\left(\frac{\nu}{6}+\frac{3}{2}\right) + v^4\left(\frac{\nu^2}{24}-\frac{19\nu}{8}+\frac{27}{8}\right) + v^6\left[\frac{7\nu^3}{1296}+\frac{31\nu^2}{24}+\left(\frac{41\pi^2}{24}-\frac{6889}{144}\right)\nu+\frac{135}{16}\right]$$
$$+ v^8\Bigg[\frac{2835}{128}+\nu\left(\frac{98869}{5760}-\frac{128\gamma_E}{3}-\frac{6455\pi^2}{1536}-\frac{256\ln 2}{3}-\frac{128\ln v}{3}\right)+\left(\frac{356035}{3456}-\frac{2255\pi^2}{576}\right)\nu^2$$
$$-\frac{215\nu^3}{1728}-\frac{55\nu^4}{31104}\Bigg], \tag{A1b}$$
$$\bar{L}^{\rm SO} = \frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{M\mu}\Bigg\{v^3\left(-\frac{5\nu}{6}-\frac{5X_2}{2}\right)+v^5\left[\frac{7\nu^2}{72}-\frac{35\nu}{8}+\left(\frac{35\nu}{12}-\frac{21}{8}\right)X_2\right]$$
$$+v^7\left[\frac{\nu^3}{16}+\frac{165\nu^2}{16}-\frac{243\nu}{16}+\left(-\frac{15\nu^2}{16}+\frac{117\nu}{4}-\frac{81}{16}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \tag{A1c}$$
$$\bar{L}^{S_1S_2} = \frac{\nu}{M^2\mu^2}\Bigg\{2v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)-(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\right]$$
$$+v^6\Bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(\frac{16}{3}-\frac{2\nu}{3}\right)+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{4\nu}{9}+\frac{4}{3}\right)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(\frac{9\nu}{4}-\frac{7}{3}\right)\Bigg]$$
$$+v^8\Bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(-\frac{205\nu^2}{72}-\frac{5315\nu}{144}+\frac{15}{4}\right)+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(-\frac{265\nu^2}{108}-\frac{715\nu}{36}+\frac{25}{4}\right)$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(\frac{21\nu^2}{8}+\frac{235\nu}{18}-4\right)\Bigg]\Bigg\}, \tag{A1d}$$
$$\bar{L}^{S^2} = \frac{v^4}{M^2\mu^2}(\nu-X_2)\left[(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2-(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\right]+\frac{v^6}{M^2\mu^2}\Bigg\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{5\nu^2}{3}-\frac{23\nu}{3}+\left(\frac{23}{3}-8\nu\right)X_2\right]$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{14\nu^2}{9}+2\nu+\left(\frac{10\nu}{3}-2\right)X_2\right]+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[\frac{9\nu^2}{8}+\frac{109\nu}{24}+\left(\frac{65\nu}{12}-\frac{109}{24}\right)X_2\right]\Bigg\}$$
$$+\frac{v^8}{M^2\mu^2}\Bigg\{(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[-\frac{295\nu^3}{216}+\frac{365\nu^2}{24}+\frac{15\nu}{8}+\left(-\frac{815\nu^2}{72}+\frac{5\nu}{8}-\frac{15}{8}\right)X_2\right]$$
$$+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{185\nu^3}{144}+\frac{2275\nu^2}{144}-\frac{95\nu}{16}+\left(\frac{175\nu^2}{48}-\frac{985\nu}{36}+\frac{95}{16}\right)X_2\right]$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[\frac{21\nu^3}{16}-\frac{55\nu^2}{36}+\frac{65\nu}{16}+\left(-\frac{13\nu^2}{36}+\frac{1237\nu}{144}-\frac{65}{16}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \tag{A1e}$$
$$\bar{L}^{S^2\tilde{C}} = \frac{\widetilde{C}_{1ES^2}}{M^2\mu^2}\Bigg\{v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2(X_2-\nu)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2(2\nu-2X_2)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2(X_2-\nu)\right]$$
$$+v^6\Bigg[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{2\nu^2}{3}-2\nu+(2\nu+2)X_2\right)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(-\frac{4\nu^2}{3}+4\nu+(-4\nu-4)X_2\right)$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left(\frac{2\nu^2}{3}-2\nu+(2\nu+2)X_2\right)\Bigg]$$
$$+v^8\Bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(\frac{5\nu^3}{36}-\frac{1475\nu^2}{84}+\frac{45\nu}{4}+\left(\frac{65\nu^2}{12}-\frac{55\nu}{12}-\frac{45}{4}\right)X_2\right)$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{72}+\frac{1475\nu^2}{168}-\frac{45\nu}{8}+\left(-\frac{65\nu^2}{24}+\frac{55\nu}{24}+\frac{45}{8}\right)X_2\right)$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left(-\frac{5\nu^3}{72}+\frac{1475\nu^2}{168}-\frac{45\nu}{8}+\left(-\frac{65\nu^2}{24}+\frac{55\nu}{24}+\frac{45}{8}\right)X_2\right)\Bigg]\Bigg\}+1\leftrightarrow 2, \tag{A1f}$$

which agrees for aligned spins with Eq. (8.24) of Ref. [44] and Eq. (5.2) of Ref. [24].

For $R(\boldsymbol{l}_N,\boldsymbol{n},v)$, we get

$$R \equiv \frac{M}{v^2}\left(\bar{r}^{S^0}+\bar{r}^{\rm SO}+\bar{r}^{S_1S_2}+\bar{r}^{S^2}+\bar{r}^{S^2\tilde{C}}\right), \tag{A2a}$$
$$\bar{r}^{S^0} = 1+v^2\frac{\nu}{3}+v^4\left(\frac{\nu^2}{9}-\frac{5\nu}{4}\right)+v^6\left[\frac{2\nu^3}{81}+\frac{11\nu^2}{12}+\left(\frac{41\pi^2}{48}-\frac{1585}{72}\right)\nu\right]$$
$$+v^8\left[\left(\frac{544}{9}-\frac{451\pi^2}{192}\right)\nu^2+\nu\left(\frac{153211}{2880}-\frac{64\gamma_E}{3}-\frac{11375\pi^2}{3072}-\frac{128}{3}\ln 2-\frac{64\ln v}{3}\right)\right], \tag{A2b}$$

$$\bar{r}^{\rm SO} = \frac{\boldsymbol{l}_N\cdot\boldsymbol{S}_1}{M\mu}\Bigg\{v^3\left(-\frac{\nu}{6}-\frac{X_2}{2}\right)+v^5\left[-\frac{19\nu^2}{48}+\frac{3\nu}{16}+\left(\frac{\nu}{8}-\frac{3}{16}\right)X_2\right]$$
$$+v^7\left[-\frac{47\nu^3}{3456}-\frac{61\nu^2}{192}-\frac{5\nu}{128}+\left(\frac{907\nu^2}{1152}+\frac{139\nu}{32}+\frac{5}{128}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \tag{A2c}$$

$$\bar{r}^{S_1S_2} = \frac{\nu}{M^2\mu^2}\Bigg\{-2v^4(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)$$
$$+v^6\left[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(\frac{85}{24}-\frac{2\nu}{3}\right)+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{5\nu}{18}-\frac{8}{3}\right)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(\nu-\frac{8}{3}\right)\right]$$
$$+v^8\Bigg[(\boldsymbol{n}\cdot\boldsymbol{S}_1)(\boldsymbol{n}\cdot\boldsymbol{S}_2)\left(-\frac{59\nu^2}{48}-\frac{1115\nu}{96}-\frac{87}{32}\right)+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\left(\frac{5\nu^2}{12}+\frac{55\nu}{24}+\frac{3}{2}\right)$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_2)\left(\frac{35\nu^2}{24}+\frac{183\nu}{16}+\frac{15}{4}\right)\Bigg]\Bigg\}, \tag{A2d}$$

$$\bar{r}^{S^2} = \frac{v^4}{M^2\mu^2}(\nu-X_2)(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2+\frac{v^6}{M^2\mu^2}\Bigg\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^2}{12}-\frac{49\nu}{12}+\left(\frac{49}{12}-\frac{25\nu}{6}\right)X_2\right]$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{29\nu^2}{36}+\frac{11\nu}{4}+\left(\frac{11\nu}{6}-\frac{11}{4}\right)X_2\right]+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[\frac{\nu^2}{2}+\frac{17\nu}{6}+\left(\frac{7\nu}{3}-\frac{17}{6}\right)X_2\right]\Bigg\}$$
$$+\frac{v^8}{M^2\mu^2}\Bigg\{(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left[-\frac{101\nu^3}{32}+\frac{585\nu^2}{32}-\frac{67\nu}{32}+\left(-\frac{595\nu^2}{96}-\frac{163\nu}{8}+\frac{67}{32}\right)X_2\right]$$
$$+(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left[\frac{5\nu^3}{24}-\frac{47\nu^2}{24}-3\nu+\left(-\frac{13\nu^2}{8}-\frac{25\nu}{24}+3\right)X_2\right]$$
$$+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left[\frac{35\nu^3}{48}-\frac{105\nu^2}{16}-\frac{75\nu}{16}+\left(\frac{9\nu^2}{16}+\frac{15\nu}{8}+\frac{75}{16}\right)X_2\right]\Bigg\}+1\leftrightarrow 2, \tag{A2e}$$

$$\bar{r}^{S^2\tilde{C}} = \frac{\widetilde{C}_{1ES^2}}{M^2\mu^2}\Bigg\{v^4\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2(\nu-X_2)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left(\frac{X_2}{2}-\frac{\nu}{2}\right)\right]$$
$$+v^6\left[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}+\frac{5\nu X_2}{6}\right)+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}-\frac{3\nu}{2}+\left(\frac{3}{2}-\frac{19\nu}{6}\right)X_2\right)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left(\frac{\nu^2}{2}+\frac{5\nu X_2}{6}\right)\right]$$
$$+v^8\Bigg[(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)^2\left(\frac{325\nu^2}{84}+\left(-2\nu^2-\frac{5\nu}{6}\right)X_2\right)+(\boldsymbol{\lambda}_N\cdot\boldsymbol{S}_1)^2\left(\frac{325\nu^2}{84}+\left(-2\nu^2-\frac{5\nu}{6}\right)X_2\right)$$
$$+(\boldsymbol{n}\cdot\boldsymbol{S}_1)^2\left(-\frac{45\nu^3}{32}+\frac{2297\nu^2}{672}+\frac{69\nu}{32}+\left(\frac{11\nu^2}{32}-\frac{43\nu}{12}-\frac{69}{32}\right)X_2\right)\Bigg]\Bigg\}+1\leftrightarrow 2, \tag{A2f}$$

Appendix B: SSC transformation for the angular momentum

The transformation from the canonical NW SSC to the covariant SSC is given by the center-of-mass shift [64]

$$\mathbf{x}_{\rm i\,(NW)} \to \mathbf{x}_{\rm i\,(cov)} + \frac{1}{2m_{\rm i}}\mathbf{v}_{\rm i\,(cov)}\times\boldsymbol{S}_{\rm i}, \tag{B1}$$

where $\mathbf{x}_{\rm i}$ and $\mathbf{v}_{\rm i}\equiv\dot{\mathbf{x}}_{\rm i}$ are the position and velocity vectors of each body, with $\rm i = 1, 2$.

In the NW SSC, the vector $\boldsymbol{L}_{N\,(\rm NW)} = \mu\boldsymbol{R}_{(\rm NW)}\times\mathbf{v}_{(\rm NW)}$, where $\boldsymbol{R}\equiv\mathbf{x}_1-\mathbf{x}_2$ and $\mathbf{v}\equiv\mathbf{v}_1-\mathbf{v}_2$ are the relative position and velocity, respectively. Transforming to the covariant SSC leads to

$$\boldsymbol{L}_{N\,(\rm NW)} = \mu\boldsymbol{R}_{(\rm cov)}\times\mathbf{v}_{(\rm cov)} + \left\{\frac{X_2^2}{2}\left[\left(\frac{M}{R}+\mathbf{v}^2\right)\boldsymbol{S}_1-(\mathbf{v}\cdot\boldsymbol{S}_1)\mathbf{v}-\frac{M}{R}(\boldsymbol{n}\cdot\boldsymbol{S}_1)\boldsymbol{n}\right]+1\leftrightarrow 2\right\}, \tag{B2}$$

where all quantities on the right-hand side are in the covariant SSC, but we dropped the label to simplify the notation. Dividing $\boldsymbol{L}_{N\,(\rm NW)}$ by its magnitude, using $\boldsymbol{R}=R\boldsymbol{n}$ and $\mathbf{v}=R\Omega\boldsymbol{\lambda}_N=Rv^3\boldsymbol{\lambda}_N/M$ for circular orbits, and taking an orbit average yields

$$\boldsymbol{l}_{N\,(\rm NW)} = \boldsymbol{l}_{N\,(\rm cov)} + \frac{v^3}{2M\mu}\left\{X_2^2\left[\boldsymbol{S}_1-(\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{l}_N\right]+X_1^2\left[\boldsymbol{S}_2-(\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\boldsymbol{l}_N\right]\right\}. \tag{B3}$$

Our result for $\boldsymbol{L}(\boldsymbol{l}_N)$ in Eq. (67) is in the NW SSC, while Eq. (A5) of Ref. [46] is in the covariant SSC; the difference up to LO SO is given by

$$\boldsymbol{L}^{\text{Eq. (67)}}_{\text{(NW)}} - \boldsymbol{L}^{\text{Ref. [46]}}_{\text{(cov)}} = -\frac{v^2}{2}\left\{X_2^2\left[\boldsymbol{S}_1 - (\boldsymbol{l}_N\cdot\boldsymbol{S}_1)\boldsymbol{l}_N\right] + X_1^2\left[\boldsymbol{S}_2 - (\boldsymbol{l}_N\cdot\boldsymbol{S}_2)\boldsymbol{l}_N\right]\right\} + \mathcal{O}(v^3). \tag{B4}$$

The leading PN order of $\boldsymbol{L}(\boldsymbol{l}_N)$ in Eq. (67) is $\mu M \boldsymbol{l}_N/v$, and v is invariant under an SSC transformation; hence, the difference is due to the SSC transformation of $\boldsymbol{l}_N$, which is given by Eq. (B3). Indeed, we see from Eqs. (B4) and (B3) that accounting for that transformation cancels the difference between our result and that of Ref. [46].

Appendix C: PA dynamics

The equations of motion for aligned spins, in terms of p_{r_*}, are given by Eqs. (10) of Ref. [34], and read

$$\begin{aligned}\dot r &= \xi\,\frac{\partial H}{\partial p_{r_*}}\bigg|_r, \qquad \dot\phi = \frac{\partial H}{\partial p_\phi},\\ \dot p_{r_*} &= -\xi\,\frac{\partial H}{\partial r}\bigg|_{p_{r_*}} + \frac{p_{r_*}}{p_\phi}\mathcal{F}_\phi, \qquad \dot p_\phi = \mathcal{F}_\phi.\end{aligned} \tag{C1}$$

Explicitly, we have

$$\dot r = \frac{A}{2\nu\tilde H_{EOB}\tilde H_{\text{even}}}\left(\frac{2p_{r_*}}{\xi}(1+B_{np}) + \xi\frac{\partial Q}{\partial p_{r_*}}\right) \tag{C2}$$

$$\dot\phi = \frac{1}{\nu\tilde H_{EOB}}\left[p_\phi\frac{\partial\bar H_{\text{odd}}}{\partial p_\phi} + \bar H_{\text{odd}} + \frac{A}{\tilde H_{\text{even}}}\left(p_\phi u^2(1+B_{npa}a_+^2)\right)\right] \tag{C3}$$

$$\dot{p_{r_*}} = -\frac{\xi}{\nu\tilde H_{EOB}}\left(\frac{\partial\tilde H_{\text{even}}}{\partial r} + p_\phi\frac{\partial\bar H_{\text{odd}}}{\partial r}\right) + \frac{p_{r_*}}{p_\phi}\mathcal{F}_\phi \tag{C4}$$

$$\tag{C5}$$

where $\bar H_{\text{odd}} = \tilde H_{\text{odd}}/p_\phi$ and

$$\begin{aligned}\frac{\partial\tilde H_{\text{even}}}{\partial r} &= \frac{1}{2\tilde H_{\text{even}}}\left(K_0 p_\phi^2 + K_1\right),\\ K_0 &\equiv A'(u^2 + a_+^2u^2B_{npa}) + A\left(-2u^3(1+B_{npa}a_+^2) + a_+^2u^2\frac{\partial B_{npa}}{\partial r}\right),\\ K_1 &\equiv A'\left(1 + \frac{p_{r_*}^2}{\xi^2}(1+B_{np}) + Q\right)\\ &\quad + A\left(\frac{p_{r_*}^2}{\xi^2}\left[\frac{\partial B_{np}}{\partial r} - \frac{2}{\xi}\frac{d\xi}{dr}(1+B_{np})\right] + \frac{\partial Q}{\partial r}\right)\end{aligned} \tag{C6}$$

Quasi-circular initial conditions are taken from [56].

One can then integrate numerically Eq. (C1), to solve for the binary's dynamics.

In `SEOBNRv5` one can also employ the post-adiabatic (PA) approximation for the inspiral dynamics, which allows speeding up the evaluation of the model, especially for very long waveforms [65–67]. The implementation of the PA dynamics closely follows that of Ref. [67] to which we refer for most details. To obtain explicit algebraic equations for the momenta we follow the same procedure as described in Ref. [65, 68] which results in the following equations.

$$p_{r_*} = \frac{\xi}{2(1+B_{np})}\left[\mathcal{F}_\phi\left(\frac{dp_\phi}{dr}\right)^{-1}\frac{2\nu\tilde H_{EOB}\tilde H_{\text{even}}}{A} - \xi\frac{\partial Q}{\partial p_{r_*}}\right] \tag{C7}$$

$$K_0 p_\phi^2 + 2\tilde H_{\text{even}}\frac{\partial\bar H_{\text{odd}}}{\partial r}p_\phi + \tag{C8}$$

$$K_1 + \frac{2\nu \tilde{H}_{\rm even}\tilde{H}_{EOB}}{\xi}\left(\frac{dp_{r_*}}{dr}\frac{dr}{dt} - \frac{p_{r_*}}{p_\phi}\mathcal{F}_\phi\right) = 0 \tag{C9}$$

Here the only unknowns are the explicit p_{r_*} in the left-hand side of the first equation and the explicit p_ϕ^2, p_ϕ in the 2nd, and all the other instances of p_{r_*}, p_ϕ come from previous orders.

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