Measuring a Binary Black Hole Eccentrically Orbiting a Galactic Nucleus with Gravitational Waves: Interim Report 1

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1 Overview and Background

Since the first detection of gravitational waves (GWs) by LIGO in 2015, GW astronomy has cemented itself as a legitimate and advantageous method for observing binary systems of compact objects, the majority of which are binary black holes (BBHs) [10, 2, 1]. Currently, the ground detectors in LIGO/Virgo/KAGRA are most sensitive between 10 Hz to a few kHz, which describes the final moments before merger. In order for a GW detector to be sensitive to frequencies below 1 Hz, it must be built in space, where forming an interferometer with arm lengths much greater than LIGO's is feasible. LISA is a planned space based detector which would specialize in the millihertz band $\sim 10^{-3} - 10^{-1}$ Hz [12].

There are detector concepts which lie in the decihertz range – between LISA and ground-based detectors – such as B-DECIGO [7] and TianGO [8, 9], which may expand the range of astrophysical systems we can observe via GWs. These space-borne GW observatories could monitor inspiraling compact binary systems at times long before their orbital frequencies climb into the frequency bands of active ground-based detectors. Furthermore, since the instantaneous orbital decay timescale due to GW emission varies as $\tau_{GW} \propto f_{orb}^{-8/3}$ [11], a stellar-mass BBH system will remain in the frequency band of these low-frequency observatories much longer than in the frequency band of ground-based experiments. This will allow space-based observatories to monitor properties of a BBH system which may change over time scales on the order of years.

Consider a system consisting of a BBH orbiting around a nearby supermassive black hole (SMBH), which is sketched out in Fig. 1 along with the relative orientation of a hypothetical space-based GW observatory. When a BBH is in such a hierarchical triple system with the SMBH, the orbit of the BBH's center of mass about the SMBH (called the "outer orbit") produces a Doppler shift of the GW frequency [4]. Furthermore, when the semimajor axis of the outer orbit a_o is small enough, the angular momentum of the inner orbit (the BBH), $\boldsymbol{L}_{\rm i}$, will precess about the angular momentum of the outer orbit, $\boldsymbol{L}_{\rm o}$. This coupling between the orbit of the BBH around the SMBH and the inner angular momentum is called de Sitter precession. The inner orbit angular momentum $\boldsymbol{L}_{\rm i}$ evolves under de Sitter precession as

$$\frac{d\hat{\boldsymbol{L}}_{\rm i}}{dt} = \Omega_{\rm dS}\hat{\boldsymbol{L}}_{\rm o} \times \hat{\boldsymbol{L}}_{\rm i} \tag{1}$$

where

$$\Omega_{dS} = \Omega_0 \frac{M_3}{a_0(1-e_o^2)} \tag{2}$$

and $\Omega_0 = \sqrt{M_3/a_o^3}$[1]. For GWs emitted from the BBH with sufficiently low frequency, these perturbations due to the SMBH can be inferred from the waveform. Recently, Hang Yu and Yanbei Chen, with whom I will be working with this summer, studied how observations of the dS precession and Doppler shift in a hierarchical triple can allow inference of the mass of the SMBH, M_3, the semimajor axis of the outer orbit, a_o, and the eccentricity of the outer orbit, e_o, through the modulation of the detected GW waveform [13]. In particular, they find that if the precession period P_{dS} is comparable to the duration of observation by a space-borne GW observatory, then the dS precession of $\boldsymbol{L}_{\rm i}$ can be detected and combined with Doppler shift measurements to generate estimates of the triple system parameters to percent-level precision.

[1] We use geometric units in this paper where $G = c = 1$.

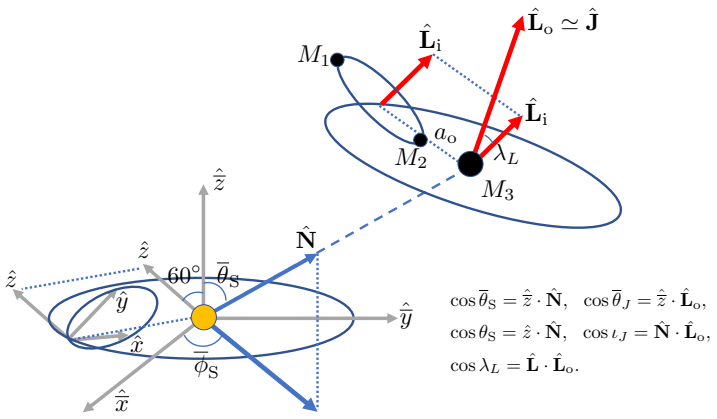

Figure 1: A cartoon of a BBH+SMBH triple system (top-right) along with the orbit of a space-based GW observatory around the Sun (bottom left). $\boldsymbol{L}_{\rm i}$ is the angular momentum of the two BHs in the binary, and $\boldsymbol{L}_{\rm o}$ is the orbital angular momentum of the BBH around the SMBH, with a_o being the semi-major axis of the outer orbit of the BBH around the SMBH. Figure from [13]. The effect of de Sitter precession is to rotate $\boldsymbol{L}_{\rm i}$ around $\boldsymbol{L}_{\rm o}$ while keeping λ_L constant.

However, the analysis of Ref. [13] assumes that the eccentricity of the outer orbit in the hierarchical triple is $e_o = 0$ – that is, the outer orbit is circular. The general case in which $e_0 > 0$ was not addressed in that study, and we expect a significant proportion of BBH+SMBH triples to have non-circular outer orbits. So, in order to develop a more complete understanding of this method to determine SMBH properties from general BBH+SMBH hierarchical triples, we must expand the methods of Ref. [13] to non-zero eccentricities.

2 Project and Methods

The central goal of my work will be to explore how the eccentricity of the outer orbit effects the uncertainty in parameter estimation of the BBH+SMBH triple system. It is expected that, for a given a_o, non-zero eccentricities will enhance the detectability of the de Sitter precession of L_i, as from Eq. 2, the dS precession frequency scales as $(1 - e_0)^{-2}$, and a more rapid precession frequency increases the likelihood that the modulation in the waveform due to the precession will be substantial over the duration of a mission. In order to quantify this change, I will ultimately produce figures similar to Figs. 4 and 5 of Ref. [13], which plot how the fractional error that space-borne GW observatories can achieve in estimating BBH+SMBH parameters varies with the true parameters of those systems.

The formation channels of BBH+SMBH triple systems, especially those which are dynamical in nature, are expected to produce a significant population of triples that fail to satisfy $e_o \approx 0$ [13]. Completing this extension of Yu and Chen's work could help with future studies of SMBHs at galactic centers. To determine the mass of an SMBH, astronomers primarily either examine the dynamics of nearby objects, like stars, as they orbit the black hole, or conduct "reverberation mapping" studies of the variation of emission line fluxes in the accretion disk of active galactic nuclei (AGNs) [3]. Yu and Chen claim that the parameter estimation method they outline can achieve precise results for a wider range of black hole masses and for triple systems at farther distances (up to 1 Gpc). Extending their analysis to non-circular outer orbits will be useful to account for more realistic scenarios. Furthermore, in Ref. [6], it is discussed how gravitational lensing of GWs by the SMBH combined with the dS precession of $\boldsymbol{L}_{\rm i}$ can further constrain the parameters of a triple system as estimated by a space-based GW observatory, even in the case of a circular outer orbit. One future extension would be to combine these lensing calculations with the results of my work to characterize the detectability of repeated lensing of GWs in an eccentric BBH+SMBH triple system.

In these kinds of theoretical studies, it is common to apply the Fisher matrix formalism in order to quantify the precision of parameter estimation [5]. The Fisher matrix Γ_{ab} is given by

$$\Gamma_{ab} \equiv \left(\frac{\partial \tilde{h}(f)}{\partial \theta_a} \middle| \frac{\partial \tilde{h}(f)}{\partial \theta_b} \right) , \tag{3}$$

where

$$\left(\tilde{g}|\tilde{h}\right) = 4\,\mathrm{Re}\int_0^\infty \frac{\tilde{g}^*(f)\tilde{h}(f)}{S_n(f)}df\,, \tag{4}$$

$\tilde{h}$ is the frequency-domain waveform, $S_n(f)$ is the PSD of the detector noise, and θ_a and θ_b are the various parameters of the system:

$\boldsymbol{\theta^a}$	**Definition**
$\log \mathcal{M}_z$	Detector Frame Chirp Mass: $\mu^{3/5}(m_1+m_2)^{2/5}$
q	Mass Ratio m_1/m_2
$\log D_L$	Luminosity Distance
t_c	Coalescence Time
ϕ_c	Coalescence Phase
$\bar{\theta}_S$,$\bar{\phi}_S$	Line of Sight of BBH+SMBH Triple
$\bar{\theta}_J$,$\bar{\phi}_J$	Orientation of Total Angular Momentum $\boldsymbol{J}$
λ_L	Angle Between $\boldsymbol{L}_i$ and $\boldsymbol{L}_o$
α_0	Initial Phase of $\boldsymbol{L}_i$ Around $\boldsymbol{L}_o$
$\log M_3$	SMBH Mass
$\log a_o$	Outer Orbit Semimajor Axis
$\phi^{(0)}$	Outer Orbit Initial Phase
e_o	Outer Orbit Eccentricity

Table 1: Relevant parameters in BBH+SMBH triple system for GW observed by detectors. Bars over angles indicate the Solar System coordinate frame.

For a network of GW detectors, the Fisher matrix is just the sum of the Fisher matrices from each detector on its own. The covariance matrix Σ_{ab}, which allows us to compute parameter uncertainties, is related to the Fisher matrix roughly as:

$$\Sigma_{ab} = [\Gamma^{-1}]_{ab} + \mathcal{O}(\rho^{-4}), \tag{5}$$

So as long as the SNR ρ is large, we can estimate the covariance matrix as the inverse of the Fisher matrix. The covariance matrix suggests that $(\Delta\theta_i)^2 = \Sigma_{ii}$, so in order to find the parameter estimation uncertainties, we need only examine the diagonal elements of the covariance matrix.

Ultimately, as long as one can describe the frequency-domain waveform for gravitational waves emitted from a BBH+SMBH triple, we can apply the Fisher matrix method to quantify the parameter estimation uncertainty. Enumerating such a waveform can be a difficult task, as this often requires forgoing an analytic solution for the equations of GR in favor of a post-Newtonian expansion. Fortunately, previous works have done so for some limiting cases, and some terms which contribute to the measured strain $\tilde{h}(f)$ are given in [13]. However, I will need to generalize these forms to account for non-zero eccentricities, which will also expand the size of the Fisher matrix by a few additional terms.

3 Current Progress

My work began by applying the Fisher information matrix method described above to a simple binary black hole system. In particular, I wrote code that placed three detectors on the surface of the Earth (namely, in the positions and orientations of Hanford, Livingston, and VIRGO) and calculated the strain measured in each detector from the gravitational waves emitted from a merging BBH. Then, using that code, I could calculate the derivative of the frequency-domain waveform with respect to its various relevant parameters, and thus systematically compute the Fisher matrix. After computing the inverse of the Fisher matrix, I could use the terms on the diagonal to estimate the uncertainty in parameter estimation by this three-detector network. This sequence allows for determination of the parameter estimation uncertainty for a given single set of BBH parameter values, so in order to understand how the estimation precision depends on those values, I needed to repeat this process over a range of possible values. For example, in Fig. 2 below, the

left-hand plot shows how the fractional error in the chirp mass $\mathcal{M}$ depends on the luminosity distance to the BBH and the chirp mass itself.

It is perhaps counterintuitive that even though the strain increases in magnitude with increasing chirp mass ($|\tilde{h}(f)| \sim \mathcal{M}^{5/6}$), our ability to measure the chirp mass declines. However, this can be explained by recalling that the Fisher matrix scales with the derivative of the frequency-domain waveform, essentially reflecting how sensitive the strain is to changing parameters. For chirp masses on the scale of tens of solar masses, the contribution of the chirp mass to the GW phase is actually the strongest contribution, for:

$$\frac{d\tilde{h}}{d(\log \mathcal{M})} = \mathcal{M}\frac{d\tilde{h}}{d\mathcal{M}} = \frac{5}{6}\tilde{h} + C_1\mathcal{M}^{-5/3}\tilde{h} + C_2\mathcal{M}^{-1}\tilde{h} \sim \mathcal{M}^{-5/6} \tag{6}$$

So, examining rough dimensional dependencies, we can see that

$$\Gamma_{\log\mathcal{M}\log\mathcal{M}} \sim \Big(\frac{d\tilde{h}}{d(\log\mathcal{M})}\Big)^2 \sim \mathcal{M}^{-5/3} \rightarrow \Sigma_{\log\mathcal{M}\log\mathcal{M}} \sim \mathcal{M}^{5/3} \rightarrow \boxed{\Delta(\log\mathcal{M}) \approx \frac{\Delta\mathcal{M}}{\mathcal{M}} \sim \mathcal{M}^{5/6}} \tag{7}$$

thus predicting that increasing chirp mass corresponds to increasing fractional uncertainty in measuring that parameter. On the right side of Fig. 2, the fractional uncertainty in measuring D_L is depicted. Now, $\frac{d\tilde{h}}{d\log D_L} = -\tilde{h}$, so in this case, a stronger signal will correspond to better parameter estimation precision. Indeed, this is what we see in the contour plot: the strain decreases with increasing D_L, so the fractional uncertainty in D_L grows with increasing D_L, and the contribution from both GW polarizations is weakest when the BBH is viewed edge-on, so the uncertainty is greatest when the the angular momentum of the BBH is perpendicular to our line of sight.

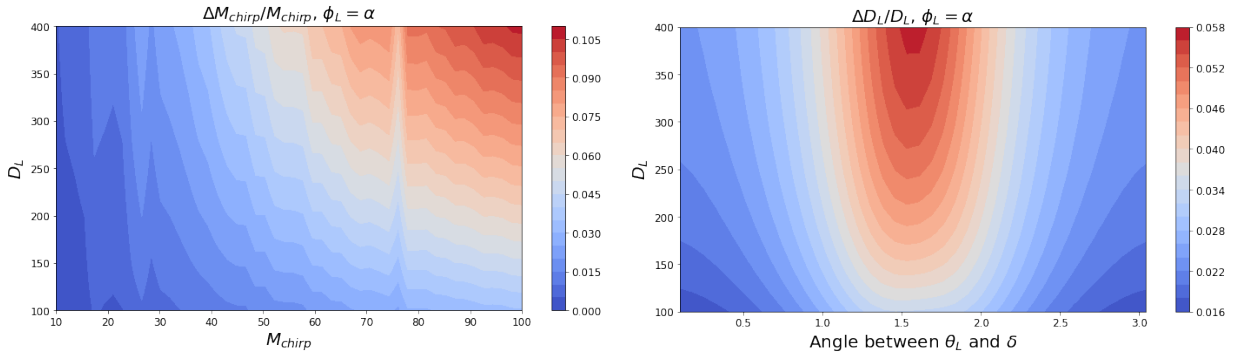

Figure 2: Left: Fractional error in $\mathcal{M}$ measurable by network of three ground-based GW observatories for varying $\mathcal{M}$ and D_L, assuming the azimuthal angles of the line of sight and angular momentum are equal. Right: Fractional error in D_L measurable by network of three ground-based GW observatories for varying $\theta_L - \delta$ and D_L, assuming the azimuthal angles of the line of sight and angular momentum are equal.

I have now moved onto studying triple systems (BBH+SMBH) with an L-shaped space-based detector, such as the proposed TianGO project. The geometry in this case is much more involved, as we must now account for the orientation of the BBH angular momentum relative to the outer orbit angular momentum as well as its precession due to the SMBH, in addition to all the parameters we needed to describe the BBH on its own. I am currently investigating whether Mathematica or Python are optimal for efficiently calculating the parameter estimation uncertainties both accurately and efficiently. I will then proceed to reproduce the results of [13], where the outer orbit of the BBH around the SMBH is circular, and after doing so, begin to implement changes to allow for eccentric outer orbits.

One unexpected difficulty I have faced in this work has been the limitations of computers in generating high-precision numbers. When computing the Fisher matrix, the various derivatives of $\tilde{h}$ can result in elements differing from one another in by more than ten orders of magnitude. With a machine precision that is only limited to a finite number of bits per number, the contributions of small numbers in computations (inverting Γ, for example) can be washed out, leading to unrealistic results (i.e., negative values on the diagonal of the covariance matrix). Fixing these kinds of problems is unfortunately not something that can be done easily, since they are intrinsically tied to the properties of standard computing hardware. The best

solution I have implemented thus far is to exclude certain parameters from the Fisher matrix which the waveform is highly sensitive to. These parameters are manifest as large entries in the diagonals of the Fisher matrix, corresponding to low uncertainties. When studying parameters with much smaller contributions to the Fisher matrix, we can effectively say that the parameters with large Fisher matrix terms are known very well, and then exclude them from the calculation so that when inverting the new Fisher matrix, the terms do not span as many orders of magnitude.

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