Understanding the Physical Degrees of Freedom in a Parameterized Test of General Relativity

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This paper provides a framework for understanding the physical degrees of freedom in a parameterized test of general relativity. In particular, we vary the post-Newtonian (PN) coefficients, the phenomenological coefficients, and the analytical black-hole perturbation theory waveform parameters, and observe how this would affect the waveform and hence the physical parameters. The physical parameters include the energy radiated and the rate of angular momentum. Although it is possible to map the dephasing coefficients to physical quantities, the inverse mapping of the physical quantities to the dephasing coefficients is unknown. Therefore, this paper presents a method of obtaining this inverse mapping using the Gaussian Mixture Model (GMM).

I. INTRODUCTION

The detection of gravitational waves (GWs) by the Advanced LIGO and Virgo [1–11] has opened new windows in observational astrophysics and cosmology. More specifically, it stands to test the limits of Einstein's theory of general relativity (GR). More recent work has been focused on testing GR in the strong-field/highly-relativistic regime. Such tests could potentially reconcile the deviations of GR with quantum field theory, through examining the higher-energy corrections to the Einstein-Hilbert action [12].

Of the many strong-field astrophysical events, this paper focuses on the coalescence of binary black holes (BBHs). This is because, firstly, the gravitational fields generated can be many orders of magnitude stronger than any other astrophysical event, as the BBHs' orbital separation can be smaller than the last stable orbit before merging. Secondly, BBH coalescence gives one of the cleanest signals for testing GR, as it is separated into three distinct phases: the inspiral, merger, and ringdown (IMR) phases [13].

A parametrized test is, simply put, a test where one measures the deviation of some parameters from their GR predictions. For parameterized tests of GR, the phenomenological models are most ideal, as they have a closed-form expression in the frequency domain and hence can be more computationally efficient. In particular, we focus on doing a parameterized test of GR on `IMRPhenomPv2` [14–17]. `IMRPhenomPv2` is a waveform model that approximates a signal of a precessing binary. It is used because it has good performance across the parameter space [15]. The purpose of the parameterized test is to understand the physical significance of varying the dephasing coefficients in the waveform and see whether such changes have deviations from GR.

Throughout the entirety of this paper, the geometric unit convention is adopted, where $c = G = 1$.

II. THEORY

A. Parameterized Test of GR

As mentioned in the Section I, a parameterized test of GR is to search for deviations of observations from the predictions of GR. To perform the parameterized tests, we introduce fractional deviations δp_i to the `IMRPhenomPv2` phase coefficients p_i [18], namely

$$p_i \to (1 + \delta p_i)p_i. \tag{1}$$

These fractional deviations are known as the dephasing coefficients. The phasing of `IMRPhenomPv2` consists of three regimes. The first of which is the inspiral regime which is parameterized by post-Newtonian (PN) coefficients [19] $\{\chi_0, \ldots, \chi_7\}$ and $\{\chi_{5l}, \chi_{6l}\}$. In this regime, there are also phenomenological parameters $\{\sigma_0, \ldots, \sigma_4\}$ that contribute to the high effective PN order. This corrects for non-adiabaticity in the late inspiral phase and for unknown high-order PN coefficients in the adiabatic regime. The second regime, is the intermediate regime, which is parameterized by the phenomenological coefficients $\{\beta_0, \ldots, \beta_3\}$. Finally, there is the merger-ringdown regime which is parameterized by a combination of the phenomenological coefficients and the analytical black-hole perturbation theory parameters $\{\alpha_0, \ldots, \alpha_5\}$ [13]. As one can see if $\delta p_i = 0$ this corresponds to a theory with no deviation with GR.

In FIG. 1, we have the phase evolution of an `IMRPhenomPv2` waveform with varied dephasing coefficients. A dephasing coefficient is chosen from each regime

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to illustrate how the dephasing coefficient changes the waveform and hence the physical parameters. More specifically, FIG. 1 illustrates how the dephasing coefficient changes the phase. In FIG. 2, we change the same dephasing coefficients as those in FIG. 1, however this time we plotted the strain data. This illustrates how the dephasing coefficient changes the shape of the GW waveform. Note that, the deviations of dephasing coefficients in FIG. 1 and FIG. 2 have been exaggerated to make the deviations more visible. Such high deviations have been ruled out with observations [1, 14, 20].

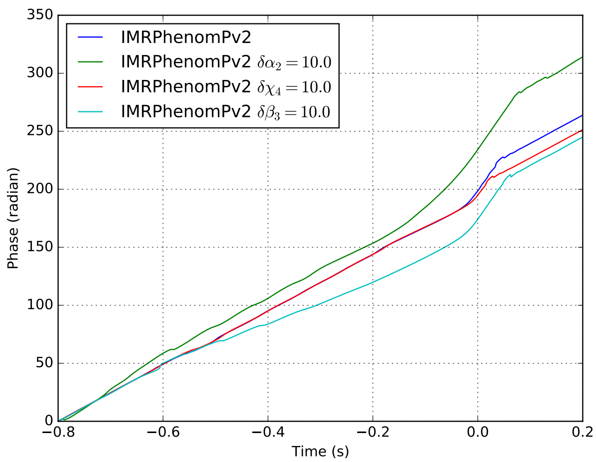

FIG. 1. Phase of GW versus time for `IMRPhenomPv2` with no modification (blue line), $\delta\alpha_2 = 10.0$ (orange line), $\delta\chi_4 = 10.0$ (green line), and $\delta\beta_3 = 10.0$ (red line).

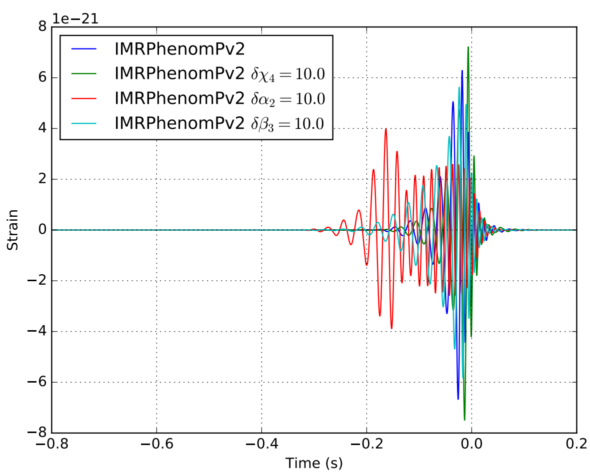

FIG. 2. Strain versus time for `IMRPhenomPv2` with no modification (blue line), $\delta\alpha_2 = 10.0$ (orange line), $\delta\chi_4 = 10.0$ (green line), and $\delta\beta_3 = 10.0$ (red line).

Throughout the entirety of this project, we will use the parameterized test of TIGER (Test Infrastructure for GEneral Relativity) [21, 22]. This infrastructure is preferred, as it is a theory-agnostic test of GR. This means that the infrastructure does not require an alternative theory of gravity to compare against. In addition to this, TIGER is dependent on the measurement of parameterizable deviations, like the aforementioned deviation in dephasing coefficients from a GR-consistent waveform model.

Let $\mathcal{H}_{\rm GR}$ be the hypothesis that some GW signal h is consistent with GR. To test how this hypothesis deviates from GR, we introduce another hypothesis $\mathcal{H}_{\rm MG}$ (MG stands for modified gravity) which is a hypothesis that the waveform model differs by one or more dephasing coefficients. Since $\mathcal{H}_{\rm GR}$ and $\mathcal{H}_{\rm MG}$ are mutually exclusive, and given some data d and information I, we can define the Bayes factor [13]

$$\mathcal{B} = \frac{p(d|\mathcal{H}_{\rm MG}, I)}{p(d|\mathcal{H}_{\rm GR}, I)}, \tag{2}$$

where $p(d|\mathcal{H}_{\rm GR}, I)$ and $p(d|\mathcal{H}_{\rm MG}, I)$ are the posterior probability densities of the data given hypotheses $\mathcal{H}_{\rm GR}$ and $\mathcal{H}_{\rm MG}$, respectively. If $\log \mathcal{B} > 0$, then the hypothesis $\mathcal{H}_{\rm MG}$ is favored, on the other hand if $\log \mathcal{B} < 0$ the hypothesis $\mathcal{H}_{\rm GR}$ is preferred [23]. Hence, we have a quantitative way of determining whether a waveform deviates from GR. This can be computed using some Bayesian inference software packages like `bilby` [24] or `LALInference` [25, 26].

B. Parameter Estimation

In Ref. [13], the Bayesian statistics framework is used to do parameter estimation. In such framework, the posterior distribution for some parameter λ is [13, 25, 26]

$$p(\lambda|\mathcal{H}_i, d, I) = \frac{p(\lambda|\mathcal{H}_i, I)p(d|\mathcal{H}_i, \lambda, I)}{p(d|I)}, \tag{3}$$

where $\mathcal{H}_i$ is the hypothesis that corresponds to a waveform model in which δp_i is a free parameter. In the equation above, d is the data, I is the background information, $p(\lambda|\mathcal{H}_i, I)$ is the prior probability density for the free parameters, and $p(d|\mathcal{H}_i, \lambda, I)$ is the probability of the data. $p(d|\mathcal{H}_i, \lambda, I)$ is defined as the likelihood function, which can be written as [13, 25, 26]

$$p(d|\mathcal{H}_i, \lambda, I) \propto e^{-\frac{1}{2}\langle d-h(\lambda)|d-h(\lambda)\rangle}, \tag{4}$$

where $h(\lambda)$ is the signal model and the inner product is defined as [13]

$$\langle a|b\rangle = 4\Re \int_{f_{\rm low}}^{f_{\rm high}} {\rm d}f \frac{a^*(f)b(f)}{S_n(f)}. \tag{5}$$

In Eq. (5), $f_{\rm high}$ is the high-frequency cutoff and $f_{\rm low}$ is the low-frequency cutoff. In the equation above, $S_n(f)$ is the power spectral density of noise. To obtain the posterior density for parameter δp_i, one has to marginalize over all parameters other than δp_i. These are also known as the nuisance parameters.

$$p(\delta p|\mathcal{H}_i, d, I) = \int \mathrm{d}\vec{\theta}\, p(\vec{\theta}, \delta p_i|\mathcal{H}_i, d, I), \tag{6}$$

where the integration is carried out over all nuisance parameters.

C. Rates of Energy and Angular Momentum of GWs

The physical parameters that we are interested in are the rates of energy and angular momentum. We can compute the energy and momentum using the Isaacson stress-energy tensor [27, 28]

$$t_{\mu\nu} = -\frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R^{(2)}\right\rangle, \tag{7}$$

where $R^{(2)}_{\mu\nu}$ is the Ricci tensor to quadratic order. $R^{(2)}_{\mu\nu}$ usually involves many terms quadratic in the metric perturbation, however we can drastically simplify this expression by performing integration by parts and using the transverse-traceless (TT) gauge condition

$$R^{(2)}_{\mu\nu} = -\frac{1}{4}\left\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\right\rangle. \tag{8}$$

Therefore the Isaacson stress-energy tensor can be written explicitly as

$$t_{\mu\nu} = \frac{1}{32\pi}\left\langle \partial_\mu h^{\mathrm{TT}}_{\alpha\beta}\partial_\nu h^{\alpha\beta}_{\mathrm{TT}}\right\rangle. \tag{9}$$

To compute the energy carried by a GW, we take the 00-component of the Isaacson stress-energy tensor and integrate over the volume V [27, 28]

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty}\frac{1}{16\pi}\int_S \mathrm{d}\Omega\, r^2\left\langle \dot{h}_+^2 + \dot{h}_\times^2\right\rangle, \tag{10}$$

where h_+ and $h_\times$ are the plus and cross polarizations of the GW, respectively. The overhead dot in Eq. (10) is the derivative with respect to coordinate time. Another useful expression is the energy spectrum, as it is much easier to integrate over all frequencies

$$\frac{\mathrm{d}E}{\mathrm{d}f} = \lim_{r\to\infty}\frac{\pi}{2}f^2\int_S \mathrm{d}\Omega\, r^2\left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2\right). \tag{11}$$

On the other hand, to compute the rate of angular momentum, we compute the linear momentum and take the cross product with the separation vector. The linear momentum is as follows [27, 28]

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = \lim_{r\to\infty}\frac{1}{32\pi}\int \mathrm{d}\Omega\, r^2\left\langle \dot{h}^{\mathrm{TT}}_{ij}\partial^k \dot{h}^{\mathrm{TT}}_{ij}\right\rangle. \tag{12}$$

Therefore, the total rate of change in angular momentum carried by the GWs can be written as

$$J^i = \frac{1}{2}\epsilon^{ijk}J_{kl}, \tag{13}$$

where J_{kl} is the conserved charge associated with rotation in the $kl-$plane. Using Noether's theorem, we find that the expression for rate of angular momentum is as follows [27, 28]

$$\begin{aligned}\frac{\mathrm{d}J_i}{\mathrm{d}t} = \lim_{r\to\infty}\frac{1}{32\pi}\int_S \mathrm{d}\Omega\, r^2\langle -\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{ab}x^k\partial^l h^{\mathrm{TT}}_{ab} \\ +2\epsilon^{ikl}\dot{h}^{\mathrm{TT}}_{al}h^{\mathrm{TT}}_{ak}\rangle.\end{aligned} \tag{14}$$

To carry out the integration, multipole expansion is performed on the rates of energy and angular momentum [28].

D. A Multipole Expansion of Energy

In Eq. (10), h_+ and $h_\times$ are dependent on t, the orbital phase ϕ, and the angle between the angular momentum $\vec{J}$ and line of sight $\hat{n}$, namely θ. To obtain an analytic expression for the integral over solid angle Ω, we separate h_+ and $h_\times$ into a time-dependent part and an angular part. This can be done using spin-weighted spherical harmonics ${}_sY_{\ell m}$. For outgoing GWs, we are concerned with the spin $s = -2$ [28],

$$h_+ - ih_\times = {}_{-2}Y_{22}(\theta,\phi)h_{2,2}(t) + {}_{-2}Y_{2-2}(\theta,\phi)h_{2,-2}(t), \tag{15}$$

where $h_{2,2}$ and $h_{2,-2}$ are time-dependent complex variables. For the waveform that we are concerned with, namely `IMRPhenomPv2`, the $(2,2)$- and $(2,-2)$-modes are the leading order terms. Therefore, Eq. (15) has been truncated to exclude higher-order multipoles. Taking the time derivative of Eq. (15) and multiplying this expression by its complex conjugate, we obtain the following expression

$$\begin{aligned}\dot{h}_+^2 + \dot{h}_\times^2 =& |{}_{-2}Y_{22}|^2|\dot{h}_{2,2}|^2 + {}_{-2}Y_{22}\;{}_{-2}Y^*_{2-2}\dot{h}_{2,2}\dot{h}^*_{2,-2} \\ &+ {}_{-2}Y^*_{22}\;{}_{-2}Y_{2-2}\dot{h}^*_{2,2}\dot{h}_{2,-2} + |{}_{-2}Y_{2-2}|^2|\dot{h}_{2,-2}|^2.\end{aligned} \tag{16}$$

With this, the task at hand is to calculate $h_{2,2}$ and $h_{2,-2}$. To do so, we first need to determine the spin-weighted spherical harmonics. Using the Wigner D matrix, ${}_{-2}Y_{22}$ and ${}_2Y_{22}$ in the θ and ϕ representation is as follows

$${}_{-2}Y_{22}(\theta,\phi) = \sqrt{\frac{5}{64\pi}}(1+\cos\theta)^2 e^{2i\phi}, \tag{17}$$

$${}_{-2}Y_{2-2}(\theta,\phi) = \sqrt{\frac{5}{64\pi}}(1-\cos\theta)^2 e^{-2i\phi}. \tag{18}$$

Notice that ${}_{-2}Y_{22}(0,0) = {}_{-2}Y_{2-2}(0,\pi) = \frac{1}{2}\sqrt{\frac{5}{\pi}}$ and ${}_{-2}Y_{2-2}(0,0) = {}_{-2}Y_{22}(0,\pi) = 0$. Therefore, to solve for $h_{2,2}$ and $h_{2,-2}$, we simply calculate $h_+ - ih_\times$ at $\theta = \phi = 0$, and $\phi = 0$, $\theta = \pi$. This can be done using `PyCBC` and `LALSimulation`. Doing so we find that

$$h_{2,2}(t) = \sqrt{\frac{4\pi}{5}}[h_+(t,0,0) - ih_\times(t,0,0)], \tag{19}$$

and

$$h_{2,-2}(t) = \sqrt{\frac{4\pi}{5}}[h_+(t,\pi,0) - ih_\times(t,\pi,0)]. \tag{20}$$

Since we are considering non-precessing binaries, θ contain no time dependence. Therefore, to integrate Eq. (15) over solid angle, we simply compute the following integrals,

$$\int_S \mathrm{d}\Omega\, |{}_{-2}Y_{22}|^2 = \int_S \mathrm{d}\Omega\, |{}_{-2}Y_{2-2}|^2 = 1, \tag{21}$$

and

$$\int_S \mathrm{d}\Omega\ {}_{-2}Y^*_{22}\ {}_{-2}Y_{2-2} = \int_S \mathrm{d}\Omega\ {}_{-2}Y_{22}\ {}_{-2}Y^*_{2-2} = \frac{1}{6}. \tag{22}$$

Using the results from Eq. (21) and Eq. (22), we can integrate Eq. (16) over solid angle to obtain the instantaneous power

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty} \frac{r^2}{16\pi} \Big\langle &|\dot{h}_{2,2}|^2 + |\dot{h}_{2,-2}|^2 \\ &+ \frac{1}{6}\left(\dot{h}^*_{2,2}\dot{h}_{2,-2} + \dot{h}_{2,2}\dot{h}^*_{2,-2}\right)\Big\rangle.\end{aligned} \tag{23}$$

III. RESULTS

A. Effects of Varying the Intrinsic Parameters

To calculate $\dot{h}_{2,2}$, $\dot{h}_{2,-2}$, $\dot{h}^*_{2,2}$, and $\dot{h}^*_{2,-2}$, we use numerical methods. More specifically, we use the central difference method to calculate the time derivatives of the arrays. The algorithm for the central difference method is as follows [29]

$$f'(x) \approx \frac{f(x+\frac{1}{2}\delta) - f(x-\frac{1}{2}\delta)}{2\delta}, \tag{24}$$

where f is the function for which the derivative is taken with respect to variable x, and δ is the step size. The central difference method is comparatively more accurate than the forward and backward difference methods. This is because, the forward and backward difference methods have truncation errors of order $\mathcal{O}(\delta)$, whereas the central difference method has a truncation error of order $\mathcal{O}(\delta^2)$ [29].

After computing $\dot{h}_{2,2}$, $\dot{h}_{2,-2}$, $\dot{h}^*_{2,2}$, and $\dot{h}^*_{2,-2}$, we have obtained the instantaneous power. To better depict the evolution of energy over time, we use the cumulative trapezoidal method to numerically calculate the integral of power over time, namely the cumulative energy. The algorithm for the cumulative trapezoidal rule is as follows [30]

$$\int_a^b f(x)dx \approx \sum_{k=1}^{N} \frac{f(x_{k-1}) + f(x_k)}{2}\Delta x_k, \tag{25}$$

where f is the function being integrated with respect to variable x over the interval $[a, b]$, $\{x_k\}$ is the partition of $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$, and $\Delta x_k = x_k + x_{k-1}$ is the kth sub interval [30]. The cumulative trapezoidal rule is used because the algorithm is much simpler to implement in code. There are indeed more accurate methods of obtaining the cumulative energy, however given a large number of data points, the cumulative trapezoidal rule yields sufficiently accurate results for calculating the integral. This can be seen in FIG. 3-5, as the energy versus time graph yields somewhat smooth functions. From here, we vary the intrinsic physical parameters (i.e. total mass M, mass ratio q, and spins $s1z$, $s2z$) and see how the energy evolves over time.

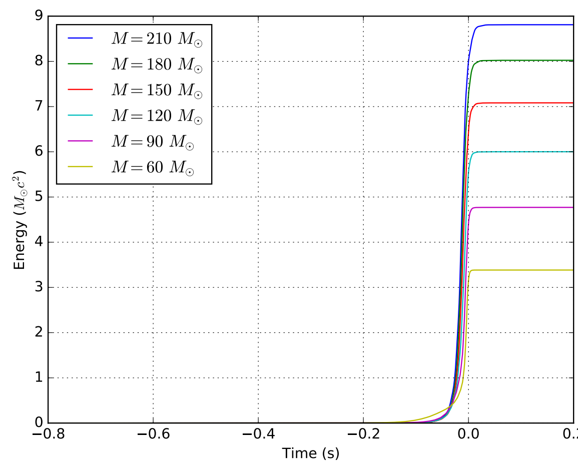

FIG. 3. The radiated energy of GW versus time in linear scale for `IMRPhenomPv2` with constant mass ratio $q = 1.00$ and varying total mass.

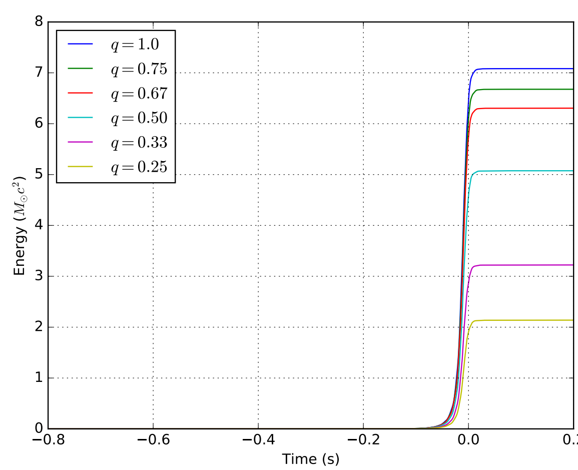

FIG. 4. The radiated energy of GW versus time in linear scale for `IMRPhenomPv2` with varying mass ratio and constant total mass of $M = 150\ M_\odot$.

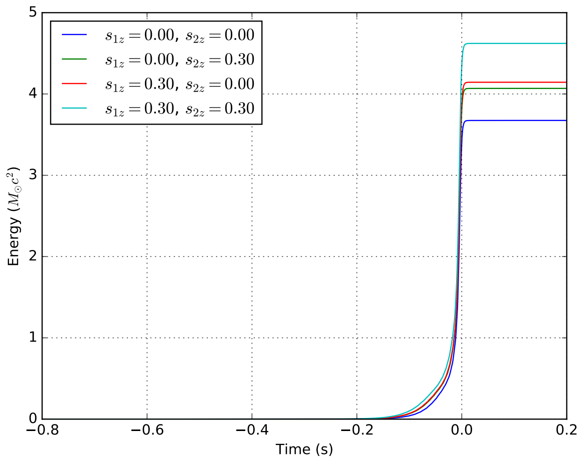

FIG. 5. The radiated energy of GW versus time in linear scale for `IMRPhenomPv2` with varying spin. Here, $m_1 = 35.0$ $M_\odot$ and $m_2 = 31.5$ $M_\odot$.

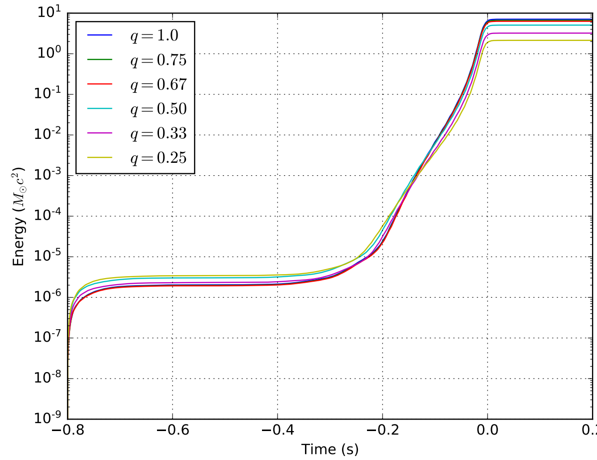

FIG. 7. The radiated energy of GW versus time for `IMRPhenomPv2` with varying mass ratio and constant total mass of $M = 150$ $M_\odot$ in logarithmic scale.

As seen from FIG. 3 and FIG. 4, the total mass M and the mass ratio q increases with the radiated energy of the GW. Similarly, from FIG. 5, it is apparent that the radiated energy of the GW increases with spin. From FIG. 3-5, it is apparent that a change in total mass M, mass ratio q, and spins s_{1z} and s_{2z}, has a major effect on the cumulative energy in the merger-ringdown phase. However, more interestingly, it seems that the variation of cumulative energy in the inspiral phase has little to no variation when plotting the cumulative energy versus time graph. To better illustrate the change in energy during the inspiral phase, we plot the y-axis of FIG. 3-5 in logarithmic scale.

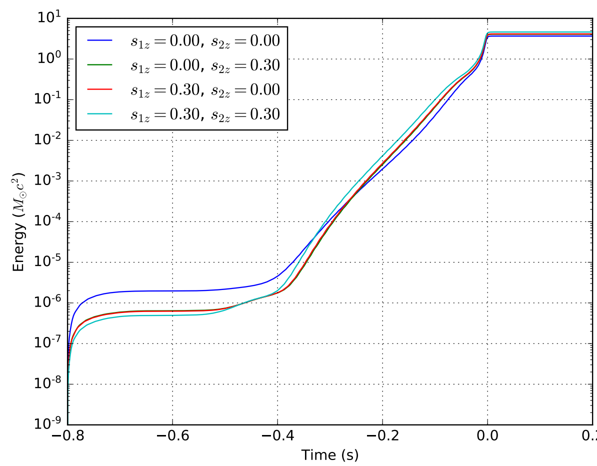

FIG. 8. The radiated energy of GW versus time for `IMRPhenomPv2` with varying spin in logarithmic scale. Here, $m_1 = 35.0$ $M_\odot$ and $m_2 = 31.5$ $M_\odot$.

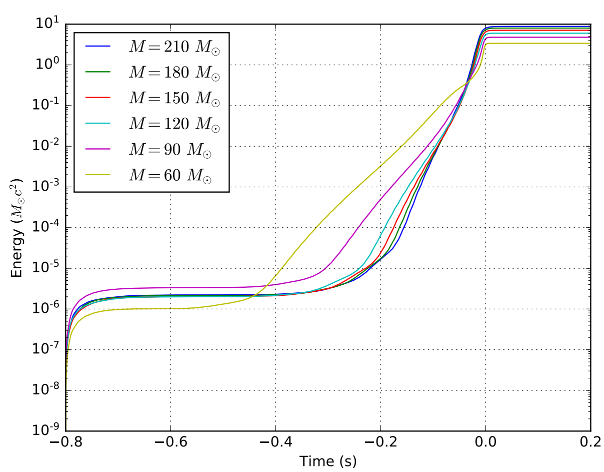

FIG. 6. The radiated energy of GW versus time for `IMRPhenomPv2` with constant mass ratio $q = 1.00$ and varying total mass in logarithmic scale.

From FIG. 6-8, we can see that during the inspiral phase there are small deviations in the total energy radiated by the GW. In FIG. 6, during times -0.38 s $\leq t \leq$ 0.05 s, we can see that the total energy radiated decreases with the total mass. This has an opposite correlation to that of the merger-ringdown phase, as indicated in FIG. 3, where the total mass increases with the GW energy. In FIG. 7, for times $t < -0.13$ s, we can see that the total energy radiated decreases with the mass ratio. Again, this has an opposite correlation to that of the merger-ringdown phase, as indicated in FIG. 4, where the mass ratio increases with the GW energy. In FIG. 8, we can see that at times $t < -0.35$ s, the total energy decreases with the spin. Once more, this has an opposite correlation to that of the merger-ringdown phase, as indicated in FIG. 5, where the spin increases with the GW energy.

IV. ANALYSIS AND DISCUSSION

For a more quantitative relation between the total mass, mass ratio, spin and energy. we plot the total energy versus the intrinsic parameters.

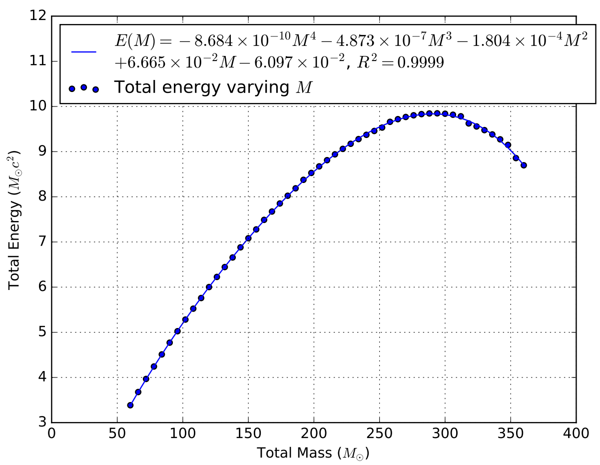

FIG. 9. Total energy of GW versus total mass with constant mass ratio and no spin. The dots represent the total energy obtained using the numerical calculations in Section III, and the line represent the line of best fit through polynomial regression.

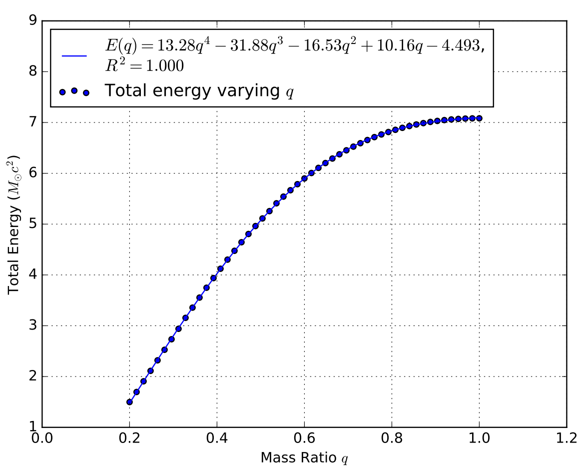

FIG. 10. Total energy of GW versus mass ratio with constant total mass and no spin in linear scale. The dots represent the total energy obtained using the numerical calculations in Section III, and the line represent the line of best fit through polynomial regression.

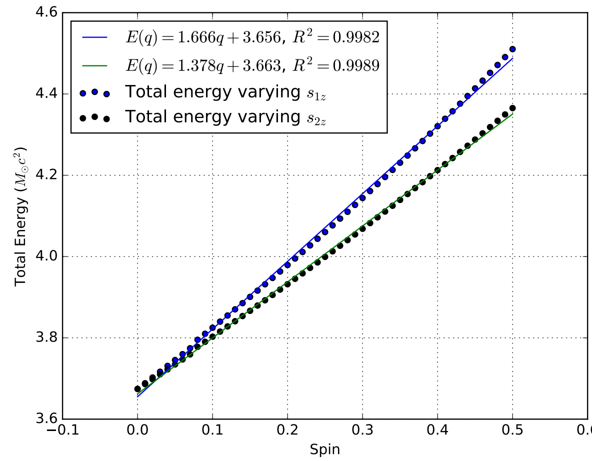

FIG. 11. Total energy of GW versus spin with $m_1 = 35.0$ $M_\odot$ and $m_2 = 31.5$ $M_\odot$. The dots represent the total energy obtained using the numerical calculations in Section III, and the line represent the line of best fit through polynomial regression.

As seen from FIG. 9, using polynomial regression, the total mass increases in a quartic manner with the total energy radiated. Similarly, from FIG. 10, we can see that the total energy radiated by the GW also increases quartically with the mass ratio. The quartic fit seems to be ideal, as $R^2 = 1.000$ for the polynomial regression in FIG. 9 and FIG. 10. The only underlying difference between the two are the coefficients in front of the polynomial. However, it is important to note that the use of polynomial regression in FIG. 9 and FIG. 10, are only valid for total mass in the range 30 $M_\odot \leq M \leq 330$ $M_\odot$, and mass ratio in the range $0.2 \leq q \leq 1.0$. This is because, as the total mass and mass ratio increases, we expect the gradient of total energy versus total mass/mass ratio to decrease and get closer to zero as $q \to \infty$ and $M \to \infty$. However, quartic expressions only have a zero gradient at a maxima or minima. Therefore, it is also correct to say that the quartic expressions in FIG. 9 and FIG. 10 are only the best fit lines for the given data. If one were to extrapolate, one would have to plot the new points and repeat the polynomial regression for the new data set. Unlike, FIG. 9 and FIG. 10, FIG. 11 shows that the total energy radiated by a GW increases linearly with the spin of the BHs. The linear fit seems to be ideal, as $R^2 = 0.9982$ and $R^2 = 0.9989$ for the linear regression of s_{1z} and s_{2z} data, respectively.

Apart from a quantitative analysis, it is important to verify whether our results make physical sense. To do so, we compare the total radiated energy with that of real GW signals. To clarify, the LIGO-Virgo-KAGRA (LVK) teams do not directly measure the energy, the calculations done by the LVK analysis team for real events are inferred using Eq .(10). Therefore, we are merely checking our calculations with the calculations done by the LVK analysis team. We can see from FIG. 3, the total energy radiated for a binary system with constant mass ratio $q = 1.0$, and total mass of $M = 60\ M_\odot$, is $\sim 3.5\ M_\odot c^2$. Comparing this to the results of GW150914, which has component masses of $m_1 = 35.6^{+4.8}_{-3.0}\ M_\odot$ and $m_2 = 30.6^{+3.0}_{-4.4}\ M_\odot$, mass ratio of $q = 0.85^{+0.17}_{-0.17}$, and a total radiated energy of $E = 3.1^{+0.4}_{-0.4}\ M_\odot c^2$ [1]. With this, given that the results in FIG. 3 yields a mass ratio, radiated energy, and total mass within the margin of error for the results of GW150914, it can be reasonably inferred that the results obtained in FIG. 3 are accurate. From FIG. 4, it is apparent that the total radiated energy of a binary system with mass ratio $q = 0.75$, and total mass 150 $M_\odot$, is $\sim 6.7\ M_\odot c^2$. This binary system is most similar to that of GW190521, where the component masses are $m_1 = 85^{+21}_{-14}\ M_\odot$ and $m_2 = 66^{+17}_{-18}\ M_\odot$, the mass ratio is $q = 0.78^{+0.36}_{-0.32}$, and the total radiated energy is $E = 7.6^{+2.2}_{-1.9}\ M_\odot c^2$ [10]. Again, one can infer that the results in FIG. 4 are accurate, as the mass ratio, total mass, and radiated energy are within the margin of error for the results of GW190521. For FIG. 5, we compare the results to that of an improved analysis on GW150914 with a fully spin-precessing waveform model [31]. In this analysis, the total mass, mass ratio, and radiated energies are the same as GW150914, but the spin associated with the primary and secondary masses are $s_{1z} = 0.26^{+0.52}_{-0.24}$ and $s_{2z} = 0.32^{+0.54}_{-0.29}$, respectively [31]. From FIG. 5, we can see that a binary system with $m_1 = 35.0\ M_\odot$, $m_2 = 31.5\ M_\odot$, $s_{1z} = 0.30$, and $s_{2z} = 0.30$, yields a total radiated energy of $\sim 4.6\ M_\odot c^2$. With this, it is apparent that the results from FIG. 5 agrees with the results obtained from Ref. [31], as the component masses, spins, and total energy are within the margin of error. Overall, the results seem to make physical sense, as they conform to the data gathered from observations.

V. FUTURE WORK

In the future, we plan to first do similar analyses by varying the dephasing coefficients and see how the energy evolves over time. In particular, we will focus on the dephasing coefficients in the inspiral phase, and then later extend this to the intermediate and merger-ringdown phase. After this, we vary the intrinsic parameters (such as total mass, mass ratio, and spin) and the dephasing coefficients to see how they affect the evolution of angular momentum. From there, we plan to use these physical quantities, namely energy, and angular momentum, along with the GMM to coherently constrain the dephasing parameters.

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VII. APPENDIX

A. Appendix A: The Frequency Dependence of `IMRPhenomPv2`

Phase of Coalescence	Dephasing Coefficient (δp_i)	Frequency (f) Dependence
Inspiral	$\delta\chi_0$	$f^{-\frac{5}{3}}$
Inspiral	$\delta\chi_1$	$f^{-\frac{4}{3}}$
Inspiral	$\delta\chi_2$	f^{-1}
Inspiral	$\delta\chi_3$	$f^{-\frac{2}{3}}$
Inspiral	$\delta\chi_4$	$f^{-\frac{1}{3}}$
Inspiral	$\delta\chi_{5l}$	$\ln(f)$
Inspiral	$\delta\chi_6$	$f^{\frac{1}{3}}$
Inspiral	$\delta\chi_{6l}$	$f^{\frac{1}{3}}\ln(f)$
Inspiral	$\delta\chi_7$	$f^{\frac{2}{3}}$
Intermediate	$\delta\beta_2$	$\ln(f)$
Intermediate	$\delta\beta_3$	f^{-3}
Merger-Ringdown	$\delta\alpha_2$	f^{-1}
Merger-Ringdown	$\delta\alpha_3$	$f^{\frac{3}{4}}$
Merger-Ringdown	$\delta\alpha_4$	$\arctan(af + b)$

TABLE I. The frequency dependence of `IMRPhenomPv2` dephasing coefficients used in parameterized tests of GR [1].

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