Effective Medium Description of Multilayer Coatings v.14

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The elastic properties of multilayer coatings determine the thermal noise imposed on beams reflected from interferometer mirrors. Those properties are not simple averages of the properties of the individual layers making up the coating, and the averaged properties that enter into modeling thermal noise and are different from those entering into measurement of the loss via the Q of mechanical resonators. It is the purpose of this document to establish a unified description of these elastic multilayers through the use of an effective medium theory for both isotropic (amorphous) and cubic (AlGaAs, AlGaP) layers that is applicable both for noise calculations and for analyzing loss measurements.

The effective medium approach is applicable when the variation of the elastic fields, other than that due to the layering itself, is slow compared to the thickness of the layer C_∞ s. By combining the constitutive relations (Hooke's Law) for the elastic stiffness in each layer with the continuity conditions for certain components of the stress and strain tensors, one can arrive at the connection between the stress and strain averaged over the layers, described in terms of an effective stiffness tensor computed from combinations of the stiffness tensors in each of the layers. Because the layering breaks the symmetry of the structure, the symmetry of this effective stiffness tensor will be lower than that of the layers themselves. In the two cases considered here: for isotropic layers the effective stiffness tensor has C_∞ symmetry, while for cubic layers the effective stiffness tensor has tetragonal symmetry.

This document is organized as follows: Section 1 establishes the effective stiffness tensor for isotropic and cubic layers. Section 2 uses the effective medium description to evaluate the stresses and strains in a thin multilayer coating in terms of the stresses and strains in the underlying substrate, and uses these elastic fields to compute the elastic energy density and power dissipated in the coating in terms of the substrate fields. Section 3 contains two applications of these results: computation of the thermal noise due to a mirror coating, and computation of the loss contributed by the coating in the measurement of the Q of a mechanical resonator. Section 4 is an appendix compiling relations between the various forms used to describe the mechanical properties of isotropic media, and giving results for integrals involved in the thermal noise calculation.

1. Formulating the effective medium

1.1. Symmetry of elastic multilayers

In an isotropic medium, the elastic properties are the same for stresses applied in any direction. This ceases to be true in a multilayer made up of isotropic media, because the layering makes the direction normal to the layers different from the in-plane directions, which remain essentially isotropic. Such a symmetry is known as C_∞. For fourth-rank tensors like the elastic stiffness, the implications of C_∞ symmetry are the same as for hexagonal point groups, with the layer-normal direction taking the role of the 6-fold (or **z**) axis. Hook's law in such a medium is expressed in terms of its stiffness tensor in the form

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{11} & c_{13} & & & \\ c_{13} & c_{13} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{66} \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \tag{1.1.1}$$

Here the second-rank stress (**T**) and strain (**S**) tensors are given in the Voight notation, where (1, 2, 3, 4, 5, 6) correspond to (*xx*, *yy*, *zz*, *yz*, *xz*, *xy*). Details can be found in B.A. Auld, *Acoustic Fields and Waves in Solids, vol. 1*, Wiley (1973), referred to hereafter as "Auld". One note on notation: the engineering (or Voight notation) strain $S_6 = 2\varepsilon_{xy} = \partial u_x / \partial y + \partial u_y / \partial x$ (see Auld Eq. 1.52 and page 56). This factor of 2 appears only for shear strains, not for compressions.

1.1.1. Isotropic layers

In an isotropic medium the tensor takes the same form as Eq. (1.1.1), but with

$$c_{13} = c_{12},\ c_{33} = c_{11},\ c_{66} = c_{44}$$

Note that there is an additional symmetry constraint on the elements of the C_∞ stiffness tensor, $c_{66} = (c_{11} - c_{12})/2$ [in isotropic media: $c_{44} = (c_{11} - c_{12})/2$], so that there are 5 independent elements in the C_∞ tensor, as compared to the 2 independent elements for an isotropic medium. Thus, describing the multilayer in terms of two quantities, like the Young's modulus and the Poisson ratio, or the bulk and shear moduli, is inappropriate.

1.1.2. Cubic layers

Note also that the form of the stiffness tensor for cubic media is the same as isotropic media, with the exception that the symmetry constraint $c_{44} = (c_{11} - c_{12})/2$ does not hold, so the stiffness tensor for layers of cubic materials will have the same form as Eq. (1.1.1), but again without the symmetry constraint $c_{66} = (c_{11} - c_{12})/2$. Thus, a material made of cubic-symmetry layers will have 6 independent elements (as does a tetragonal point group like 4mm). All of the results given here in terms of stiffness tensor elements, for example the form of the averaged effective-medium stiffness coefficients given in Eq. (1.2.10) will apply to cubic layers as well. Those results that rely explicitly on the isotropy of the individual layers, such as those involving Young's modulus and Poisson ratio (e.g. Eq. (3.1.11)) of course cannot be applied to cubic layers.

It is the purpose of this section to derive the connection between the properties of the layers making up the film, and those of an effective medium of the symmetry shown in Eq. (1.1.1). The individual layers for both isotropic and cubic layers will be described again by a stiffness tensor for computational convenience, though the elements of that tensor for isotropic media can all be given in terms of more conventional terminology like Young's modulus and Poisson ratio (see Appendix for those connections).

1.2. Effective medium analysis

The effective medium approach is applicable when the variation of the elastic fields, other than that due to the layering itself, is slow compared to the thickness of the layers. We can then apply a local averaging procedure to find an effective stiffness tensor, $[\overline{c}]$, that accurately approximates the response of the layered medium to applied stresses $\overline{T}$ and strains $\overline{S}$, also averaged over a period of the layer. We follow the approach of G. Backus, "Long-Wave Elastic Anisotropy Produced by Horizontal Layering," *Journal of Geophysical Research* **67**, pp. 4427-40 (1962), hereafter referred to as "Backus". The essence of the approach is to arrange the components of the constitutive relations in forms that do not contain products of quantities discontinuous at the interface between the layers. The required averaging is then conceptually straightforward, though in some case somewhat involved algebraically.

We assume a structure containing alternating layers of materials A and B, with thicknesses d_A and d_B and stiffness tensors c_A, and c_B. We define an averaging operator by

$$\overline{g} \equiv \langle g \rangle \equiv \frac{d_A}{d_A + d_B} g_A + \frac{d_B}{d_A + d_B} g_B \tag{1.2.1}$$

where g_A is any quantity or combination of quantities in layer A. The analysis begins by noting that the in-plane strains (S_1, S_2, S_6) and the surface normal stresses (T_3, T_4, T_5) are continuous across the layers.

1.2.1. Shears

Consider first the shears, since they are simpler than the compressions. According to the isotropic form of Eq. (1.1.1), in each layer we have

$$\begin{aligned} T_{4,J} &= c_{44,J} S_{4,J} \\ T_{5,J} &= c_{44,J} S_{5,J} \\ T_{6,J} &= c_{44,J} S_{6,J} \end{aligned} \tag{1.2.2}$$

where $J \in (A, B)$. In order to apply the averaging required for the effective medium approach, we need to arrange Eqs. (1.2.2) so that they contain no products of discontinuous quantities:

$$\begin{aligned} S_{4,J} &= c_{44,J}^{-1} T_{4,J} \\ S_{5,J} &= c_{44,J}^{-1} T_{5,J} \\ T_{6,J} &= c_{44,J} S_{6,J} \end{aligned} \tag{1.2.3}$$

In this form, the averaging function in Eq. (1.2.1) is straightforwardly applied:

$$\begin{aligned}\overline{S}_4 &\equiv \langle S_4 \rangle = \langle c_{44}^{-1} T_4 \rangle = \langle c_{44}^{-1} \rangle \overline{T}_4 \\ \overline{S}_5 &\equiv \langle S_5 \rangle = \langle c_{44}^{-1} T_5 \rangle = \langle c_{44}^{-1} \rangle \overline{T}_5 \\ \overline{T}_6 &\equiv \langle T_6 \rangle = \langle c_{44} S_6 \rangle = \langle c_{44} \rangle \overline{S}_6\end{aligned} \tag{1.2.4}$$

Comparing with Eq. (1.1.1) we see that

$$\begin{aligned}\overline{c}_{44} &= \overline{c}_{55} = \langle c_{44}^{-1} \rangle^{-1} \\ \overline{c}_{66} &= \langle c_{44} \rangle\end{aligned} \tag{1.2.5}$$

The different types of averaging involved can be thought of as analogous to averaging springs in series vs springs in parallel.

1.2.2. Compressions

The concept is the same for finding the effective stiffness components for the compressions, but a bit more work is involved due to the inherent coupling between them due to Poisson's ratio. Begin again with the relations in each layer given by the isotropic form of Eq. (1.1.1)

$$\begin{aligned}T_{1,\mathrm{J}} &= c_{11,\mathrm{J}}\overline{S}_1 + c_{12,\mathrm{J}}\overline{S}_2 + c_{12,\mathrm{J}}S_{3,\mathrm{J}} \\ T_{2,\mathrm{J}} &= c_{12,\mathrm{J}}\overline{S}_1 + c_{11,\mathrm{J}}\overline{S}_2 + c_{12,\mathrm{J}}S_{3,\mathrm{J}} \\ \overline{T}_3 &= c_{12,\mathrm{J}}\overline{S}_1 + c_{12,\mathrm{J}}\overline{S}_{2,} + c_{11,\mathrm{J}}S_{3,\mathrm{J}}\end{aligned} \tag{1.2.6}$$

where we already denote the continuous fields with overbars. With the third of Eqs. (1.2.6), we can arrange to have an "averageable" RHS for all three of Eqs. (1.2.6):

$$\begin{aligned}S_{3,\mathrm{J}} &= \left[c_{11,\mathrm{J}}^{-1}\overline{T}_3 - \frac{c_{12,\mathrm{J}}}{c_{11,\mathrm{J}}}\left(\overline{S}_1 + \overline{S}_2\right)\right] \\ T_{1,\mathrm{J}} &= \left(c_{11,\mathrm{J}} - \frac{c_{12,\mathrm{J}}^2}{c_{11,\mathrm{J}}}\right)\overline{S}_1 + \left(c_{12,\mathrm{J}} - \frac{c_{12,\mathrm{J}}^2}{c_{11,\mathrm{J}}}\right)\overline{S}_2 + \frac{c_{12,\mathrm{J}}}{c_{11,\mathrm{J}}}\overline{T}_3 \\ T_{2,\mathrm{J}} &= \left(c_{12,\mathrm{J}} - \frac{c_{12,\mathrm{J}}^2}{c_{11,\mathrm{J}}}\right)\overline{S}_1 + \left(c_{11,\mathrm{J}} - \frac{c_{12,\mathrm{J}}^2}{c_{11,\mathrm{J}}}\right)\overline{S}_2 + \frac{c_{12,\mathrm{J}}}{c_{11,\mathrm{J}}}\overline{T}_3\end{aligned} \tag{1.2.7}$$

Carrying out the averaging for the first of Eqs. (1.2.7)

$$\begin{aligned}\overline{S}_3 &= \langle c_{11}^{-1} \rangle \overline{T}_3 - \left\langle \frac{c_{12}}{c_{11}} \right\rangle \left(\overline{S}_1 + \overline{S}_2\right) \\ \Rightarrow \overline{T}_3 &= \langle c_{11}^{-1} \rangle^{-1} \overline{S}_3 + \langle c_{11}^{-1} \rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle \left(\overline{S}_1 + \overline{S}_2\right)\end{aligned} \tag{1.2.8}$$

Carrying out the averaging for the second two of Eqs. (1.2.8) and replacing $\overline{T}_3$ with the second of Eqs. (1.2.8) yields

$$\bar{T}_1 = \left[\left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \right] \bar{S}_1 + \left[\left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \right] \bar{S}_2 + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle \bar{S}_3$$

$$\bar{T}_2 = \left[\left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \right] \bar{S}_1 + \left[\left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \right] \bar{S}_2 + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle \bar{S}_3 \quad (1.2.9)$$

Comparing with Eq. (1.1.1), we have for the effective stiffness components

$$\begin{aligned}
\bar{c}_{11} &= \left\langle c_{11} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \\
\bar{c}_{12} &= \left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle + \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle^2 \\
\bar{c}_{13} &= \left\langle c_{11}^{-1} \right\rangle^{-1} \left\langle \frac{c_{12}}{c_{11}} \right\rangle \\
\bar{c}_{33} &= \left\langle c_{11}^{-1} \right\rangle^{-1} \\
\bar{c}_{44} &= \left\langle c_{44}^{-1} \right\rangle^{-1} \\
\bar{c}_{66} &= \left\langle c_{44} \right\rangle
\end{aligned} \quad (1.2.10)$$

where we have also listed here the shear elements from Eq. (1.2.5) for completeness.

1.3. Stress-free layer in terms of Young's modulus and Poisson ratio

For comparison with more conventional analyses, it is interesting to work through the same calculation in terms of an effective Young's modulus and Poisson ratio. Rather than work out the complete tensor, we focus for simplicity on those elements pertinent to the case of layers with a stress-free normal surface. We also do not compute S_3 since it will not contribute to the energy density when $T_3 = 0$.

Hooks law for in-plane strains of an isotropic medium with $T_4 = T_5 = T_3 = 0$ is

$$\begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} = \frac{1}{Y} \begin{bmatrix} 1 & -\sigma & 0 \\ -\sigma & 1 & 0 \\ 0 & 0 & 2(1+\sigma) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} \quad (1.3.1)$$

Its inverse is

$$\begin{bmatrix} T_1 \\ T_2 \\ T_6 \end{bmatrix} = \frac{Y}{1-\sigma^2} \begin{bmatrix} 1 & \sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & (1-\sigma)/2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_6 \end{bmatrix} \quad (1.3.2)$$

Conveniently, all the fields on the RHS are continuous across interfaces, so that the averaging process is straightforward. We have

$$\begin{bmatrix}\bar{T}_1\\ \bar{T}_2\\ \bar{T}_6\end{bmatrix}=\begin{bmatrix}\left\langle\frac{Y}{1-\sigma^2}\right\rangle & \left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle & 0\\ \left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle & \left\langle\frac{Y}{1-\sigma^2}\right\rangle & 0\\ 0 & 0 & \frac{1}{2}\left\langle\frac{Y}{1+\sigma}\right\rangle\end{bmatrix}\begin{bmatrix}\bar{S}_1\\ \bar{S}_2\\ \bar{S}_6\end{bmatrix} \tag{1.3.3}$$

Inverting this matrix we have

$$\begin{bmatrix}\frac{\left\langle\frac{Y}{1-\sigma^2}\right\rangle}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle^2-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle^2} & \frac{-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle^2-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle^2} & 0\\ * & * & 0\\ 0 & 0 & 2\left\langle\frac{Y}{1+\sigma}\right\rangle^{-1}\end{bmatrix} \tag{1.3.4}$$

where the * represent the components known by symmetry from the others.

Comparing Eq. (1.3.4)with Eq. (1.3.2) we identify the effective medium quantities

$$\begin{aligned}\bar{Y}&=\frac{\left\langle\frac{Y}{1-\sigma^2}\right\rangle^2-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle^2}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle}\\ &=\frac{\left[\left\langle\frac{Y}{1-\sigma^2}\right\rangle-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle\right]\left[\left\langle\frac{Y}{1-\sigma^2}\right\rangle+\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle\right]}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle}=\frac{\left\langle\frac{Y}{1+\sigma}\right\rangle\left\langle\frac{Y}{1-\sigma}\right\rangle}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle}\end{aligned} \tag{1.3.5}$$

and

$$-\frac{\bar{\sigma}}{\bar{Y}}=\frac{-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle^2-\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle^2}\Rightarrow\bar{\sigma}=\frac{\left\langle\frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle\frac{Y}{1-\sigma^2}\right\rangle} \tag{1.3.6}$$

where second form for $\bar{\sigma}$ follows with Eq. (1.3.5) for $\bar{Y}$. If these are consistent, we should have for the shear element

$$\frac{2(1+\bar{\sigma})}{\bar{Y}}=?\;2\left\langle\frac{Y}{1+\sigma}\right\rangle^{-1}\quad\Rightarrow\frac{\bar{Y}}{(1+\bar{\sigma})}=?\;\left\langle\frac{Y}{1+\sigma}\right\rangle \tag{1.3.7}$$

Substituting into Eq. (1.3.7) from Eqs. (1.3.5) and (1.3.6) shows that it in fact is obeyed, so we have for the effective quantities

$$\bar{Y}_1 = \frac{\left\langle \frac{Y}{1+\sigma} \right\rangle \left\langle \frac{Y}{1-\sigma} \right\rangle}{\left\langle \frac{Y}{1-\sigma^2} \right\rangle}, \quad \bar{\sigma}_{12} = \frac{\left\langle \frac{Y\sigma}{1-\sigma^2} \right\rangle}{\left\langle \frac{Y}{1-\sigma^2} \right\rangle} \tag{1.3.8}$$

where we add subscripts as a reminder that these hold for in-plane stresses and strains, while there will be different effective quantities for out-of-plane fields.

2. Thin Effective Medium on a Substrate

One can in a general case include a stiffness tensor in the form of Eq. (1.1.1) with elements calculated from the layers' material properties with Eqs. (1.2.10) into FEM calculations to obtain results for the elastic state of the composite layer. Here we assume that one can adopt the method of GM Harry *et al*, "Thermal noise in interferometric gravitational wave detectors due to dielectric optical coatings," *Class. Quantum Grav.* **19**, 897 (2002), hereafter referred to as "Harry", where the elastic fields in the substrate are calculated independent of the mirror layer, and then the fields in the layer are computed treating those at the surface of the substrate as boundary conditions on the layer.

2.1. Boundary condition treatment of composite layer

Let us assume that the elastic fields in the substrate are known, and in particular that the in-plane strains and the surface-normal stresses in the substrate are S_{1s}, S_{2s}, S_{6s} and T_{3s}, T_{4s}, T_{5s}, respectively. We further assume that the surface of the layer away from the substrate is not subjected to any shear stresses. The continuity of the surface-normal stresses and in-plane strains then allows us to conclude that the surface-normal shears vanish in both the film and the substrate, and therefore that the fields in the film obey

$$\begin{aligned} &T_4 = T_5 = 0,\ T_3 = T_{3s} \\ &S_1 = S_{1s},\ S_2 = S_{2s},\ S_6 = S_{6s} \end{aligned} \tag{2.1.1}$$

We can then use these fields and Eq. (1.1.1) to compute S_3, T_1, T_2, and T_6 in the film. With these fields in hand, the energy in the film can be calculated, either for a thermal noise calculation or to evaluate the energy ratio in a loss measurement.

2.2. Field energy in the layer in terms of boundary fields

With Eqs. (1.1.1) and (2.1.1) we have

$$\begin{aligned} &T_1 = \bar{c}_{11} S_{1s} + \bar{c}_{12} S_{12s} + \bar{c}_{13} S_3 \\ &T_2 = \bar{c}_{12} S_{1s} + \bar{c}_{11} S_{12s} + \bar{c}_{13} S_3 \\ &T_{3s} = \bar{c}_{13} S_{1s} + \bar{c}_{13} S_{2s} + \bar{c}_{33} S_3 \\ &T_6 = \bar{c}_{66} S_{6s} \end{aligned} \tag{2.2.1}$$

The shear stress in the film is obviously trivially obtained, so the work again is in working out the compressional fields. The equation for T_3 yields

$$S_3 = \bar{c}_{33}^{-1}\left[T_{3s} - \bar{c}_{13}(S_{1s} + S_{2s})\right] \tag{2.2.2}$$

With S_3 from Eq. (2.2.2) in Eqs. (2.2.1) for T_1 and T_2 we find

$$\begin{aligned} T_1 &= \left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{1s} + \left(\bar{c}_{12} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{2s} + \frac{\bar{c}_{13}}{\bar{c}_{33}}T_{3s} \\ T_2 &= \left(\bar{c}_{12} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{1s} + \left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{2s} + \frac{\bar{c}_{13}}{\bar{c}_{33}}T_{3s} \end{aligned} \tag{2.2.3}$$

With Eqs. (2.2.2) and (2.2.3), along with Eqs. (2.1.1), we now have all the fields in the layer in terms of those in the substrate, so can calculate the elastic energy density in the film according to

$$\begin{aligned} u &= \frac{1}{2}\sum_{I=1}^{6} T_I S_I \\ &= \frac{1}{2}\left\{\left[\left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{1s} + \left(\bar{c}_{12} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{2s} + \frac{\bar{c}_{13}}{\bar{c}_{33}}T_{3s}\right]S_{1s} + \left[\left(\bar{c}_{12} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{1s} + \left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)S_{2s} + \frac{\bar{c}_{13}}{\bar{c}_{33}}T_{3s}\right]S_{2s}\right. \\ &\qquad \left. + T_{3s}\bar{c}_{33}^{-1}\left[T_{3s} - \bar{c}_{13}(S_{1s} + S_{2s})\right] + \bar{c}_{66}S_{6s}^2\right\} \\ &= \frac{1}{2}\left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)\left(S_{1s}^2 + S_{2s}^2\right) + \left(\bar{c}_{12} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}}\right)\left(S_{1s}S_{2s}\right) + \frac{1}{2\bar{c}_{33}}T_{3s}^2 + \frac{\bar{c}_{66}}{2}S_{6s}^2 \end{aligned} \tag{2.2.4}$$

We are primarily interested in the power dissipated in the film. Here we must be careful and not simply assign complex values to the elastic constants (e.g. the imaginary part of the coefficient $1/\bar{c}_{33}$ will have opposite sign to that of c_{11}, which doesn't make physical sense as both should add to the total loss). To evaluate the dissipated power, we return to the expression for the rate at which work is done on a deformed elastic body (L. D. Landau and E.M. Lifshitz, *Theory of Elasticity*, *Course of Theoretical Physics Vol. 7*, 1975):

$$p = \sigma_{ik}\frac{d\varepsilon_{ik}}{dt} \tag{2.2.5}$$

In Voight notation, and for sinusoidal time dependence exp($i\omega t$), the average dissipated power per unit volume is then

$$p_{diss} = -\frac{\omega}{2}\mathrm{Im}\left[T_I^* S_I\right] \tag{2.2.6}$$

With the fields from Eqs. (2.2.2) and (2.2.3), Eq. (2.2.6) becomes

$$p_{diss} = -\frac{\omega}{2}\mathrm{Im}\Bigg\{\Bigg[\left(\overline{c}_{11} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^* S_{1s}^* + \left(\overline{c}_{12} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^* S_{2s}^* + \left(\frac{\overline{c}_{13}}{\overline{c}_{33}}\right)^* T_{3s}^*\Bigg] S_{1s} + \Bigg[\left(\overline{c}_{12} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^* S_{1s}^* + \left(\overline{c}_{11} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^* S_{2s}^* + \left(\frac{\overline{c}_{13}}{\overline{c}_{33}}\right)^* T_{3s}^*\Bigg] S_{2s}$$
$$+ T_{3s}^* \overline{c}_{33}^{-1}\left[T_{3s} - \overline{c}_{13}(S_{1s} + S_{2s})\right] + \overline{c}_{66}^* S_{6s}^* S_{6s}\Bigg\}$$
$$= -\frac{\omega}{2}\mathrm{Im}\left\{\left(\overline{c}_{11} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^*\left(S_{1s}^2 + S_{2s}^2\right) + 2\left(\overline{c}_{12} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}}\right)^* S_{1s}S_{2s} + c_{33}^{-1}T_{3s}^2 + \left[\left(\frac{\overline{c}_{13}}{\overline{c}_{33}}\right)^* - \left(\frac{\overline{c}_{13}}{\overline{c}_{33}}\right)\right]\left(S_{1s} + S_{2s}\right)T_{3s} + \overline{c}_{66}^* S_{6s}^2\right\} \quad (2.2.7)$$

where the second form follows from taking the negligibly lossy substrate fields as real.

We can evaluate the combinations of effective stiffness elements in Eqs. (2.2.4) and (2.2.7) in terms of the layers' properties according to Eq. (1.2.10). We find

$$\overline{c}_{11} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}} = \left\langle c_{11} - \frac{c_{12}^2}{c_{11}}\right\rangle, \qquad \overline{c}_{12} - \frac{\overline{c}_{13}^2}{\overline{c}_{33}} = \left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle, \qquad \frac{\overline{c}_{13}}{\overline{c}_{33}} = \left\langle \frac{c_{12}}{c_{11}}\right\rangle, \qquad \overline{c}_{33}^{-1} = \left\langle c_{11}^{-1}\right\rangle, \qquad \overline{c}_{66} = \left\langle c_{44}\right\rangle \quad (2.2.8)$$

With Eqs. (2.2.8) for the effective stiffness components in terms of the layer properties, Eq. (2.2.4) for the energy density in the film becomes

$$u = \frac{1}{2}\left\langle c_{11} - \frac{c_{12}^2}{c_{11}}\right\rangle\left(S_{1s}^2 + S_{2s}^2\right) + \left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle\left(S_{1s}S_{2s}\right) + \frac{\left\langle c_{11}^{-1}\right\rangle}{2}T_{3s}^2 + \frac{\left\langle c_{44}\right\rangle}{2}S_{6s}^2$$
$$= \frac{1}{2}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \frac{1}{2}\left\langle c_{11} - c_{12}\right\rangle^* S_{\Delta,s}^2 + \left\langle c_{44}\right\rangle^* S_{6,s}^2 + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2\right\} \quad (2.2.9)$$
$$\xrightarrow{\text{isotropic layers}} \frac{1}{2}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \left\langle c_{44}\right\rangle^*\left(S_{\Delta,s}^2 + S_{6,s}^2\right) + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2\right\}$$

and the dissipated power from Eq. (2.2.7) becomes

$$p_{diss} = -\frac{\omega}{2}\mathrm{Im}\left\{\left\langle c_{11} - \frac{c_{12}^2}{c_{11}}\right\rangle^*\left(S_{1s}^2 + S_{2s}^2\right) + 2\left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle^* S_{1s}S_{2s} + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2 + \left[\left\langle\frac{c_{12}}{c_{11}}\right\rangle^* - \left\langle\frac{c_{12}}{c_{11}}\right\rangle\right]\left(S_{1s} + S_{2s}\right)T_{3s} + \left\langle c_{44}\right\rangle^* S_{6s}^2\right\}$$
$$= -\frac{\omega}{2}\mathrm{Im}\left\{\left(\left\langle c_{11}\right\rangle^* - \left\langle c_{12}\right\rangle^*\right)\left(S_{1s}^2 + S_{2s}^2\right) + \left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle^*\left(S_{1s} + S_{2s}\right)^2 + i\,2\,\mathrm{Im}\left\langle\frac{c_{12}}{c_{11}}\right\rangle^*\left(S_{1s} + S_{2s}\right)T_{3s} + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2 + \left\langle c_{44}\right\rangle^* S_{6s}^2\right\}$$
$$= -\frac{\omega}{2}\mathrm{Im}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \frac{1}{2}\left\langle c_{11} - c_{12}\right\rangle^* S_{\Delta,s}^2 + \left\langle c_{44}\right\rangle^* S_{6,s}^2 + i\,2\,\mathrm{Im}\left\langle\frac{c_{12}}{c_{11}}\right\rangle^* S_{D,s}T_{3s} + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2\right\}$$
$$\xrightarrow{\text{isotropic layers}} -\frac{\omega}{2}\mathrm{Im}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \left\langle c_{44}\right\rangle^*\left(S_{\Delta,s}^2 + S_{6,s}^2\right) + i\,2\,\mathrm{Im}\left\langle\frac{c_{12}}{c_{11}}\right\rangle^* S_{D,s}T_{3s} + \left\langle c_{11}^{-1}\right\rangle T_{3s}^2\right\} \quad (2.2.10)$$

where the third form follows from defining the in-plane dilation as $S_D \equiv S_1 + S_2$ and the anti-symmetric strain as $S_\Delta \equiv S_1 - S_2$, and the last form holds for isotropic layers recalling the isotropy condition for the elastic constants $c_{11} - c_{12} = 2c_{44}$. Noting that the dilation is invariant to coordinate rotations, and that $S_{\Delta,s}^2 + S_{6,s}^2$ is invariant to rotations around z, the final forms of Eqs. (2.2.9) and (2.2.10) can be seen to be explicitly invariant to rotations around the surface normal.

Note that some of the terms in Eq. (2.2.9) involve averages of the stiffness coefficients, while some involve averages of the reciprocal of the coefficients. This is in contradiction to the sometimes-used approach of simply averaging over Young's moduli. For comparison with the literature, recall from Eq. (1.1.1) that $S_6 = 2\varepsilon_{xy} = \partial u_x / \partial y + \partial u_y / \partial x$. The appearance of c_{11}^{-1} rather than its complex conjugate results from its arising from $S_{33} = T_{33}/c_{33}$ and that Eq. (2.2.6) involves conjugating only the stresses; this form correctly results in a positive contribution to the loss from this term.

It is also worth noting that the term proportional to $\mathrm{Im}(c_{12}/c_{11})$ has no corresponding term proportional to $\mathrm{Re}(c_{12}/c_{11})$ in the energy density in Eq. (2.2.9), so it is easier to obtain the correct expression for the losses by using the direct expression Eq. (2.2.10) for the dissipated power rather than simply inserting complex stiffness coefficients into the energy density expression Eq. (2.2.9).

3. Applications of the effective medium approach

3.1. Thermal noise: Isotropic Layers

We can use the result for the elastic energy density in Eq. (2.2.9) to compute the thermal noise in a coated mirror, as was done in Harry. We begin with the substrate fields for a Gaussian pressure field applied to the face of the mirror, with pressure field of the form $(2F/\pi w^2)\exp(-r^2/w^2)$, from his Eqs. (A10):

$$\begin{aligned} S_{rr,s} &= \frac{F}{4\pi(\lambda_s + \mu_s)}\left[\frac{1}{r^2}\left(1 - e^{-2r^2/w^2}\right) - \frac{4}{w^2}e^{-2r^2/w^2}\right] \\ S_{\theta\theta,s} &= -\frac{F}{4\pi(\lambda_s + \mu_s)}\frac{1}{r^2}\left(1 - e^{-2r^2/w^2}\right) \\ T_{zz,s} &= -\frac{F}{2\pi}\left(\frac{4}{w^2}e^{-2r^2/w^2}\right) \end{aligned} \tag{3.1.1}$$

where λ_s and μ_s are the first Lamé constant and the shear modulus of the substrate, respectively.

It is shown in Auld, Appendix 1, that in isotropic media the same form holds for the stiffness tensor for fields given in cylindrical coordinates as for Cartesian coordinates, with the identifications

$$1 \to rr,\;\; 2 \to \theta\theta,\;\; 3 \to zz,\;\; 4 \to \theta z,\;\; 5 \to rz,\;\; 6 \to r\theta \tag{3.1.2}$$

and the same identifications for the stresses. From his arguments, it appears that this connection also holds for cylindrical coordinates in hexagonal (or C_∞) media, though of course not for the tetragonal symmetry media arising from layers of cubic materials.

With Eq. (2.2.10) for the average power dissipated in the film, we have, with the thickness of the coating d,

$$P_{diss} = 2\pi d \int_0^\infty p_{diss}\, r\, dr$$

$$\begin{aligned} P_{diss} = -\frac{\omega}{2} 2\pi d \,\mathrm{Im}\Bigg\{ &\left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right) \int_0^\infty \left(S_{rrs}^2 + S_{\theta\theta s}^2\right) r\, dr + \left\langle c_{12} - \frac{c_{12}^2}{c_{11}} \right\rangle^* \int_0^\infty \left(S_{rrs} + S_{\theta\theta s}\right)^2 r\, dr \\ &+ i\, 2\,\mathrm{Im}\left\langle \frac{c_{12}}{c_{11}} \right\rangle^* \int_0^\infty \left(S_{rrs} + S_{\theta\theta s}\right) T_{zzs}\, r\, dr + \left\langle c_{11}^{-1} \right\rangle \int_0^\infty T_{zzs}^2\, r\, dr \Bigg\} \\ = -\frac{\omega d}{2} \frac{F^2}{\pi w^2} \,\mathrm{Im}\Bigg\{ &\left(\langle c_{11}\rangle^* - \left\langle -\frac{c_{12}^2}{c_{11}} \right\rangle^*\right) \frac{1}{4(\lambda_s + \mu_s)^2} + i\,\mathrm{Im}\left\langle \frac{c_{12}}{c_{11}} \right\rangle^* \frac{1}{(\lambda_s + \mu_s)} + \left\langle c_{11}^{-1} \right\rangle \Bigg\} \end{aligned} \tag{3.1.3}$$

With Eqs. (4.5.4) from the Appendix for the stiffness coefficients in terms of Young's moduli and Poisson ratios, we have from Eq. (3.1.3) the dissipated power in terms of the bulk and shear loss angles

$$\begin{aligned} P_{diss} = \frac{\omega d}{2} \frac{F^2}{\pi w^2} \Bigg\{ &\left\langle Y \left[\frac{(1-2\sigma)}{3(1-\sigma)^2} \phi_K + \frac{2}{3} \frac{1-\sigma+\sigma^2}{(1-\sigma)(1-\sigma^2)} \phi_\mu \right] \right\rangle \frac{1}{4(\lambda_s + \mu_s)^2} + \left\langle \frac{(1-2\sigma)(1+\sigma)}{3(1-\sigma)^2} \left(\phi_K - \phi_\mu\right) \right\rangle \frac{1}{(\lambda_s + \mu_s)} \\ &+ \left\langle \frac{1}{Y} \left[\frac{(1+\sigma)^2 (1-2\sigma)}{3(1-\sigma)^2} \phi_K + \frac{2}{3} \frac{(1+\sigma)(1-2\sigma)^2}{(1-\sigma)^2} \phi_\mu \right] \right\rangle \Bigg\} \end{aligned} \tag{3.1.4}$$

With Eqs. (4.2.1) for λ and μ, Eq. (3.1.4) becomes

$$\begin{aligned} P_{diss} = \frac{\omega d}{2} \frac{F^2}{\pi w^2} \Bigg\{ &\left[\frac{1}{3} \left\langle \frac{(1-2\sigma)}{(1-\sigma)^2} Y \phi_K \right\rangle + \frac{2}{3} \left\langle \frac{1-\sigma+\sigma^2}{(1-\sigma)(1-\sigma^2)} Y \phi_\mu \right\rangle \right] \frac{(1+\sigma_s)^2 (1-2\sigma_s)^2}{Y_s^2} \\ &+ \frac{2}{3} \left\langle \frac{(1-2\sigma)(1+\sigma)}{3(1-\sigma)^2} \left(\phi_K - \phi_\mu\right) \right\rangle \frac{(1+\sigma_s)(1-2\sigma_s)}{Y_s} \\ &+ \left[\frac{1}{3} \left\langle \frac{(1+\sigma)^2 (1-2\sigma)}{(1-\sigma)^2} \frac{1}{Y} \phi_K \right\rangle + \frac{2}{3} \left\langle \frac{(1+\sigma)(1-2\sigma)^2}{(1-\sigma)^2} \frac{1}{Y} \phi_\mu \right\rangle \right] \Bigg\} \end{aligned} \tag{3.1.5}$$

The thermal noise spectral density according to Levin is given by

$$S_x(f) = \frac{4 k_B T}{\pi f} \frac{W_{diss}}{F_0^2} \tag{3.1.6}$$

where $W_{diss} = P_{diss}/(2\pi f)$ is the power dissipated per cycle divided by 2π. Thus, with Eq. (3.1.5) for P_{diss}, we have

$$S_x(f)=\frac{2k_BT}{\pi^2 f}\frac{d}{w^2}\left\{\left[\frac{1}{3}\left\langle\frac{(1-2\sigma)}{(1-\sigma)^2}Y\phi_K\right\rangle+\frac{2}{3}\left\langle\frac{1-\sigma+\sigma^2}{(1-\sigma)(1-\sigma^2)}Y\phi_\mu\right\rangle\right]\frac{(1+\sigma_s)^2(1-2\sigma_s)^2}{Y_s^2}\right.$$
$$+\frac{2}{3}\left\langle\frac{(1-2\sigma)(1+\sigma)}{(1-\sigma)^2}\left(\phi_K-\phi_\mu\right)\right\rangle\frac{(1+\sigma_s)(1-2\sigma_s)}{Y_s} \tag{3.1.7}$$
$$\left.+\left[\frac{1}{3}\left\langle\frac{(1+\sigma)^2(1-2\sigma)}{(1-\sigma)^2}\frac{1}{Y}\phi_K\right\rangle+\frac{2}{3}\left\langle\frac{(1+\sigma)(1-2\sigma)^2}{(1-\sigma)^2}\frac{1}{Y}\phi_\mu\right\rangle\right]\right\}$$

In the limit of equal bulk and shear losses, the noise takes the simpler form

$$S_x(f)=\frac{2k_BT}{\pi^2 f}\frac{d}{w^2}\left\{\left\langle\frac{Y}{(1-\sigma^2)}\phi\right\rangle\frac{(1+\sigma_s)^2(1-2\sigma_s)^2}{Y_s^2}+\left\langle\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}\frac{1}{Y}\phi\right\rangle\right\} \tag{3.1.8}$$

While not required for the thermal noise calculation, we can also evaluate the total energy in the film by integrating Eq. (2.2.9)

$$U=2\pi d\int_0^\infty u(r)\,r\,dr$$
$$=2\pi d\int_0^\infty\left\{\frac{1}{2}\left\langle c_{11}-\frac{c_{12}^2}{c_{11}}\right\rangle\left(S_{rr,s}^2+S_{\theta\theta,s}^2\right)+\left\langle c_{12}-\frac{c_{12}^2}{c_{11}}\right\rangle\left(S_{rr,s}S_{\theta\theta,s}\right)+\frac{\left\langle c_{11}^{-1}\right\rangle}{2}T_{zz,s}^2+\frac{\left\langle c_{44}\right\rangle}{2}S_{6,s}^2\right\}r\,dr \tag{3.1.9}$$

where d is the thickness of the coating.

Noting also that the *xy* (i.e. 6) shear vanishes in the substrate, and, using Eqs. (4.6.1) – (4.6.3) from the Appendix for the integrals in Eq. (3.1.9), we have the energy in the film as

$$U=\frac{F^2 d}{2\pi w^2}\left[\left\langle c_{11}-\frac{c_{12}^2}{c_{11}}\right\rangle\frac{1}{4(\lambda_s+\mu_s)^2}+\left\langle c_{11}^{-1}\right\rangle\right] \tag{3.1.10}$$

In order to compare with Harry, we need to re-express the elastic constants in terms of Young's moduli and Poisson ratio. Using Eqs. (4.1.1) and (4.2.1) to evaluate the coefficients in Eq. (3.1.10), we find

$$U=\frac{F^2 d}{2\pi w^2}\left[\left\langle\frac{Y}{1-\sigma^2}\right\rangle\frac{(1+\sigma_s)^2(1-2\sigma_s)^2}{Y_s^2}+\left\langle\frac{1}{Y}\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)}\right\rangle\right] \tag{3.1.11}$$

where the subscript s denotes a quantity evaluated in the substrate. This result for the elastic energy in the film agrees with that obtained from Harry Eqs. (A17) –(A19), in the limit when the composite layer is truly isotropic, i.e. when the averaging function is not required.

3.1.1. Implications of the film energy expression for thermal noise: reciprocal weighting

By considering the elastic moduli to be complex, various aspects of the aspects of the lossy behavior of the film, and hence the thermal noise, can be explored, e.g. allocating the loss to bulk vs shear contributions, tradeoffs between number of required layers and loss, etc. One quick point worth mentioning here, considering the simple case of a film made up of layers each with isotropic loss, so that the Poisson ratio can be considered real, and neglecting the different weighting of the two terms in Eq. (3.1.11) due to Poisson ratio factors, we see that the averaged loss angle in the first term vs that in the second term of Eq. (3.1.11) would involve

$$\begin{aligned}
&\left\langle \tilde{Y} \right\rangle = \left\langle Y(1+i\phi) \right\rangle = \left\langle Y \right\rangle + i\left\langle Y\phi \right\rangle \;\Rightarrow\; \phi_{avg} = \frac{\left\langle Y\phi \right\rangle}{\left\langle Y \right\rangle} \\
&\left\langle \tilde{Y}^{-1} \right\rangle = \left\langle \left[Y(1+i\phi)\right]^{-1} \right\rangle \approx \left[\left\langle Y^{-1} \right\rangle - i\left\langle \phi / Y \right\rangle\right] = \left\langle Y^{-1} \right\rangle \left[1 - i\frac{\left\langle \phi Y^{-1} \right\rangle}{\left\langle Y^{-1} \right\rangle}\right] \Rightarrow \phi_{avg} = -\frac{\left\langle \phi Y^{-1} \right\rangle}{\left\langle Y^{-1} \right\rangle}
\end{aligned} \tag{3.1.12}$$

We see that even for the simplest case, the importance of the loss in the softer layer will be emphasized in one term in Eq. (3.1.11) for the energy while the loss in the stiffer layer will be emphasized in the other.

3.1.2. Implications of the film energy expression: bulk and shear losses

We see that there are comparable terms involving bulk and shear losses even in the limit of large spot size to film thickness as is assumed here. While a bulk compression would produce only bulk losses, a uniaxial stress as is imposed here will result in both bulk and shear losses. The relative importance of the two depends on the Poisson ratio and ratio of Young's moduli in film and substrate, but in general neither is entirely dominant over the other.

3.2. Thermal noise: Cubic layers

For layers that are made of cubic materials like AlGaAs or AlGaP, there are three independent elastic constants, and we can't make use of the usual description in terms of Young's modulus or Lamé constants of the film. We can, however, use the effective medium result of Eq. (1.2.10) to describe the effective stiffness tensor, and the result in Eq. (2.2.7) for the dissipated power. If we continue to assume that the film is thin enough that Harry's approach of taking the substrate fields to be the same as they would be in the absence of the mirror layers, then the computation remains relatively straightforward.

The first step is expressing in Cartesian coordinates the required strain and stress tensor components given in cylindrical coordinates in Harry. In Harry, we are given $S_{rr,s}$, $S_{\theta\theta,s}$, $S_{r\theta,s} = S_{rz,s} = S_{r\theta,s} = 0$, $T_{zz,s}, T_{rz,s} = T_{r\theta,s} = 0$. The connection between strains in Cartesian and cylindrical coordinates at an angle θ to the *x*-axis is given by

$$\begin{aligned}
\begin{bmatrix} S_{xx} & S_{xy} & S_{xz} \\ S_{yx} & S_{yy} & S_{yz} \\ S_{zx} & S_{zy} & S_{zz} \end{bmatrix} &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{rr} & S_{r\theta} & S_{rz} \\ S_{\theta r} & S_{\theta\theta} & S_{\theta z} \\ S_{zr} & S_{z\theta} & S_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} (c^2 S_{rr} + s^2 S_{\theta\theta}) - 2cs\, S_{r\theta} & cs\,(S_{rr} - S_{\theta\theta}) + (c^2 - s^2)\, S_{r\theta} & c\, S_{rz} - s\, S_{\theta z} \\ cs\,(S_{rr} - S_{\theta\theta}) + (c^2 - s^2)\, S_{r\theta} & (s^2 S_{rr} + c^2 S_{\theta\theta}) + 2cs\, S_{r\theta} & s\, S_{rz} + c\, S_{\theta z} \\ c\, S_{rz} - s\, S_{\theta z} & s\, S_{rz} + c\, S_{\theta z} & S_{zz} \end{bmatrix}
\end{aligned} \tag{3.2.1}$$

where $c \equiv \cos\theta$, $s \equiv \sin\theta$. With Eq. (3.2.1) we then conclude

$$\begin{aligned}
S_1 &= S_{xx} = (c^2 S_{rr} + s^2 S_{\theta\theta}) - 2cs\, S_{r\theta} \\
S_2 &= S_{yy} = s^2 S_{rr} + c^2 S_{\theta\theta} + 2cs\, S_{r\theta} \\
S_3 &= S_{zz} = S_{zz} \\
S_4 &= 2S_{xz} = 2\left(s\, S_{rz} + c\, S_{\theta z}\right) \\
S_5 &= 2S_{yz} = 2\left(c\, S_{rz} - s\, S_{\theta z}\right) \\
S_6 &= 2S_{xy} = 2\left[cs\,(S_{rr} - S_{\theta\theta}) + (c^2 - s^2)\, S_{r\theta}\right]
\end{aligned} \tag{3.2.2}$$

For the given substrate strains, we have

$$\begin{aligned}
S_{1,s} &= c^2 S_{rr,s} + s^2 S_{\theta\theta,s} \\
S_{2,s} &= s^2 S_{rr,s} + c^2 S_{\theta\theta,s} \\
S_{6,s} &= 2cs\,(S_{rr,s} - S_{\theta\theta,s})
\end{aligned} \tag{3.2.3}$$

We only need the stresses,

$$\begin{aligned}
T_{3,s} &= T_{zz,s} \\
T_{4,s} &= T_{yz,s} = 0 \\
T_{5,s} &= T_{xz,s} = 0
\end{aligned} \tag{3.2.4}$$

With Eqs. (3.2.3) and (3.2.4) for the substrate stresses and strains in the expression for the dissipated power density, Eq. (2.2.10), we have

$$\begin{aligned}
p_{diss} = -\frac{\omega}{2}\,\mathrm{Im}\Bigg\{ &\left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)\left(S_{1s}^2 + S_{2s}^2\right) + \left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle^* \left(S_{1s} + S_{2s}\right)^2 + i\,2\,\mathrm{Im}\left\langle \frac{c_{12}}{c_{11}}\right\rangle^* \left(S_{1s} + S_{2s}\right) T_{3s} \\
&+ \left\langle c_{11}^{-1}\right\rangle T_{3s}^2 + \left\langle c_{44}\right\rangle^* S_{6s}^2 \Bigg\}
\end{aligned} \tag{3.2.5}$$

Looking at the combinations of fields individually and expanding them using Eqs. (3.2.3) and (3.2.4) yields:

$$\begin{aligned}
S_{1,s}^2 + S_{2,s}^2 &= \left(c^2 S_{rr,s} + s^2 S_{\theta\theta,s}\right)^2 + \left(s^2 S_{rr,s} + c^2 S_{\theta\theta,s}\right)^2 \\
&= \left(S_{rr,s}^2 + S_{\theta\theta,s}^2\right) - 2c^2 s^2 \left(S_{rr,s} - S_{\theta\theta,s}\right)^2
\end{aligned} \tag{3.2.6}$$

$$\begin{aligned}
\left(S_{1,s} + S_{2,s}\right)^2 &= \left(c^2 S_{rr,s} + s^2 S_{\theta\theta,s} + s^2 S_{rr,s} + c^2 S_{\theta\theta,s}\right)^2 \\
&= \left(S_{rr,s} + S_{\theta\theta,s}\right)^2
\end{aligned} \tag{3.2.7}$$

$$\left(S_{1s} + S_{2s}\right) T_{3s} = \left(S_{rr,s} + S_{\theta\theta,s}\right) T_{zz,s} \tag{3.2.8}$$

$$S_{6,s}^2 = \left[2cs\,(S_{rr,s} - S_{\theta\theta,s})\right]^2$$
$$= 4c^2 s^2 \left(S_{rr,s}^2 - 2S_{rr,s}S_{\theta\theta,s} + S_{\theta\theta,s}^2\right) \tag{3.2.9}$$

With Eqs. (3.2.6) – (3.2.9) in Eq. (3.2.5) we have

$$\begin{aligned} p_{diss} = -\frac{\omega}{2}\mathrm{Im}\Big\{ &\left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)\left[\left(S_{rr,s}^2 + S_{\theta\theta,s}^2\right) - 2c^2 s^2\left(S_{rr,s} - S_{\theta\theta,s}\right)^2\right] \\ &+\left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle^* \left(S_{rr,s} + S_{\theta\theta,s}\right)^2 + i\,2\,\mathrm{Im}\left\langle \frac{c_{12}}{c_{11}}\right\rangle^* \left(S_{rr,s} + S_{\theta\theta,s}\right)T_{zz,s} + \left\langle c_{11}^{-1}\right\rangle T_{zz,s}^2 \\ &+\langle c_{44}\rangle^* 4c^2 s^2 (S_{rr,s} - S_{\theta\theta,s})^2 \Big\} \end{aligned} \tag{3.2.10}$$

Recalling that for isotropic layers, $c_{44} = (c_{11} - c_{12})/2$, we see that as expected p_{diss} becomes independent of θ in that case. We can thus write Eq. (3.2.10) in a more suggestive form, using the trig identity $\sin 2\theta = 2\sin\theta\cos\theta$:

$$\begin{aligned} p_{diss} = -\frac{\omega}{2}\mathrm{Im}\Big\{ &\left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)\left(S_{rr,s}^2 + S_{\theta\theta,s}^2\right) + \sin^2(2\theta)\left[\langle c_{44}\rangle^* - \left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)/2\right]\left(S_{rr,s} - S_{\theta\theta,s}\right)^2 \\ &+\left\langle c_{12} - \frac{c_{12}^2}{c_{11}}\right\rangle^* \left(S_{rr,s} + S_{\theta\theta,s}\right)^2 + i\,2\,\mathrm{Im}\left\langle \frac{c_{12}}{c_{11}}\right\rangle^* \left(S_{rr,s} + S_{\theta\theta,s}\right)T_{zz,s} + \left\langle c_{11}^{-1}\right\rangle T_{zz,s}^2 \Big\} \end{aligned}$$

(3.2.11)

Comparing Eq. (3.2.11) with the first form of Eq. (3.1.3), we see that the result for integrating the dissipated power density over the volume of the films will be the same as for the case of isotropic layers, but with the addition of a term P_{aniso} given by

$$\begin{aligned} P_{aniso} &= -\frac{\omega d}{2}\mathrm{Im}\left[\langle c_{44}\rangle^* - \left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)/2\right]\int_0^{2\pi} d\theta \sin^2(2\theta)\int_0^\infty dr\, r\left(S_{rr,s} - S_{\theta\theta,s}\right)^2 \\ &= -\frac{\omega d}{2}\mathrm{Im}\left[\langle c_{44}\rangle^* - \left(\langle c_{11}\rangle^* - \langle c_{12}\rangle^*\right)/2\right]\pi\left[\frac{F}{4\pi(\lambda_s + \mu_s)}\right]^2 \int_0^\infty dr\, r\left[\frac{2}{r^2}\left(1 - e^{-2r^2/w^2}\right) - \frac{4}{w^2}e^{-2r^2/w^2}\right]^2 \\ &= -\frac{\omega d}{2}\frac{F^2}{\pi w^2}\mathrm{Im}\left[\frac{1}{2}\left(\langle c_{44}\rangle^* - \frac{\langle c_{11}\rangle^* - \langle c_{12}\rangle^*}{2}\right)\right]\frac{1}{4(\lambda_s + \mu_s)^2} \end{aligned}$$

(3.2.12)

where the second form follows from Eqs. (3.1.1) for the substrate fields, and the third form from evaluating the radial integral as $2/w^2$.

Adding Eq. (3.2.12) for the "extra" term in the dissipated power for layers of cubic symmetry to the result in Eq. (3.1.3) for isotropic layers, we find

$$P_{diss,cubic} = -\frac{\omega d}{2}\frac{F^2}{\pi w^2}\operatorname{Im}\left\{\left[\left(\langle c_{11}\rangle^* - \left\langle -\frac{c_{12}^2}{c_{11}}\right\rangle^*\right) + \frac{1}{2}\left(\langle c_{44}\rangle^* - \frac{\langle c_{11}\rangle^* - \langle c_{12}\rangle^*}{2}\right)\right]\frac{1}{4(\lambda_s+\mu_s)^2} + i\operatorname{Im}\left\langle\frac{c_{12}}{c_{11}}\right\rangle^*\frac{1}{(\lambda_s+\mu_s)} + \left\langle c_{11}^{-1}\right\rangle\right\} \tag{3.2.13}$$

With Eq. (3.2.13) for the dissipated power in Eq. (3.1.6) for the thermal noise spectral density, we find

$$S_x(f) = -\frac{2k_BTd}{\pi^2 f w^2}\operatorname{Im}\left\{\left[\left(\langle c_{11}\rangle^* - \left\langle -\frac{c_{12}^2}{c_{11}}\right\rangle^*\right) + \frac{1}{2}\left(\langle c_{44}\rangle^* - \frac{\langle c_{11}\rangle^* - \langle c_{12}\rangle^*}{2}\right)\right]\frac{1}{4(\lambda_s+\mu_s)^2} + i\operatorname{Im}\left\langle\frac{c_{12}}{c_{11}}\right\rangle^*\frac{1}{(\lambda_s+\mu_s)} + \left\langle c_{11}^{-1}\right\rangle\right\} \tag{3.2.14}$$

where again the correction term for cubic vs isotropic layers explicitly vanishes when the shear stiffness obeys the symmetry condition required for isotropy.

3.3. Loss measurements with resonator Q's

3.3.1. Analysis with effective stiffness approach

The same general considerations as were considered in computing the energy in the film for the thermal noise problem are applicable to compute the connection between modal Q's and the losses in the multilayer. The difference between the film energy in this case and that in section 3.1, are that the surface of the resonator is in general stress free, so that $T_{3,s} = T_3 = 0$, but the in-plane shears will not necessarily vanish, i.e. $S_{6,s} = S_6 \neq 0$. In this case the energy density in the film from Eq. (2.2.9) takes the form

$$\begin{aligned} u &= \frac{1}{2}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle S_{D,s}^2 + \frac{1}{2}\langle c_{11} - c_{12}\rangle S_{\Delta,s}^2 + \langle c_{44}\rangle S_{6,s}^2\right\} \\ &\xrightarrow[\text{isotropic layers}]{} \frac{1}{2}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle S_{D,s}^2 + \langle c_{44}\rangle\left(S_{\Delta,s}^2 + S_{6,s}^2\right)\right\} \end{aligned} \tag{3.3.1}$$

The dissipated power according to the third form of Eq. (2.2.10) with $T_{3,s}$ =0, is

$$\begin{aligned} p_{diss} &= -\frac{\omega}{2}\operatorname{Im}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \frac{1}{2}\langle c_{11} - c_{12}\rangle^* S_{\Delta,s}^2 + \langle c_{44}\rangle^* S_{6,s}^2\right\} \\ &\xrightarrow[\text{isotropic layers}]{} -\frac{\omega}{2}\operatorname{Im}\left\{\frac{1}{2}\left\langle c_{11} + c_{12} - 2\frac{c_{12}^2}{c_{11}}\right\rangle^* S_{D,s}^2 + \langle c_{44}\rangle^*\left(S_{\Delta,s}^2 + S_{6,s}^2\right)\right\} \end{aligned} \tag{3.3.2}$$

The in-plane dilation and the asymmetric strain were defined after Eq. (2.2.10) as $S_{D,s} \equiv S_{1,s} + S_{2,s}$ and $S_{\Delta,} \equiv S_{1,s} - S_{2,s}$, respectively. This energy density and dissipated power of course cannot be integrated over the strain fields until the fields of the mode in the substrate are known, presumably from FEA.

For isotropic layers the second form of Eqs. (3.3.1) and (3.3.2) can be expressed in terms of Young's modulus and Poisson ratio. Eq. (3.3.1) for the energy density in the film becomes with Eqs. (4.1.1) for the elastic constants.

$$\begin{aligned} u &= \frac{1}{4}\left[\left\langle \frac{Y}{1-\sigma}\right\rangle S_{D,s}^2 + \left\langle \frac{Y}{1+\sigma}\right\rangle \left(S_{\Delta,s}^2 + S_{6,s}^2\right)\right] \\ &= \frac{1}{2}\left[\left\langle \frac{Y}{1-\sigma^2}\right\rangle \left(S_{1,s}^2 + S_{2,s}^2\right) + 2\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle S_{1,s}S_{2,s} + \frac{1}{2}\left\langle \frac{Y}{1+\sigma}\right\rangle S_{6,s}^2\right] \end{aligned} \tag{3.3.3}$$

where the second form follows by expanding S_D and S_Δ as described after Eq. (3.3.2). Eq. (3.3.2) for the dissipated power becomes with Eqs. (4.5.4) for the imaginary parts of the elastic constants in terms of the bulk and shear losses

$$p_{diss} = \frac{\omega}{2}\left\{\left\langle \frac{Y}{6(1-\sigma)^2}\left[2(1-2\sigma)\phi_K + (1+\sigma)\phi_\mu\right]\right\rangle S_{D,s}^2 + \left\langle \frac{Y}{2(1+\sigma)}\phi_\mu\right\rangle \left(S_{\Delta,s}^2 + S_{6,s}^2\right)\right\} \tag{3.3.4}$$

In this case, there are no coefficients involving averaging of inverse stiffness coefficients, unlike Eqs. (3.1.10) or (3.1.11), which can be traced back to the absence of any out-of-plane stresses in this case. We see by comparison with Eq. (3.1.9) that the combination of elastic losses obtained with *Q*-measurements is not the same as that which enters into the calculation of thermal noise.

3.3.2. Analysis for isotropic layers via effective Young's modulus and Poisson ratio

We can also obtain the energy density via the effective Young's modulus and Poisson ratio approach. We of course must ultimately get the same result as in Eq. (3.3.3) if the methods are both carried out correctly. The energy density is given by

$$u = \frac{1}{2}\left[\overline{T}_1 S_1 + \overline{T}_2 S_2 + \overline{T}_6 S_6\right] \tag{3.3.5}$$

With the stresses in terms of the strains via the constitutive relation in Eq. (1.3.2) we have

$$\begin{aligned} u &= \frac{1}{2}\frac{\overline{Y}}{1-\overline{\sigma}^2}\left[(S_1 + \overline{\sigma}S_2)S_1 + (S_2 + \overline{\sigma}S_1)S_2 + \frac{1-\overline{\sigma}}{2}S_6^2\right] \\ &= \frac{1}{2}\frac{\overline{Y}}{1-\overline{\sigma}^2}\left[S_1^2 + S_2^2 + 2\overline{\sigma}S_1S_2 + \frac{1-\overline{\sigma}}{2}S_6^2\right] \end{aligned} \tag{3.3.6}$$

Consider the coefficients individually. With the effective $\overline{Y}, \overline{\sigma}$ from Eqs. (1.3.8), we have

$$1-\bar{\sigma}^2 = 1-\left(\frac{\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right)^2$$

$$= \left[1-\frac{\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right]\left[1+\frac{\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right]$$

$$= \left[\frac{\left\langle \frac{Y}{1-\sigma^2}\right\rangle - \left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right]\left[\frac{\left\langle \frac{Y}{1-\sigma^2}\right\rangle + \left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right] \tag{3.3.7}$$

$$= \left[\frac{\left\langle \frac{Y}{1+\sigma}\right\rangle \left\langle \frac{Y}{1-\sigma}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle^2}\right]$$

and hence with $\bar{Y}$ from Eq. (1.3.8) in Eq. (3.3.7)

$$\frac{\bar{Y}}{1-\bar{\sigma}^2} = \left\langle \frac{Y}{1-\sigma^2}\right\rangle \tag{3.3.8}$$

We also have with Eqs. (1.3.8)

$$\frac{1}{2}\frac{\bar{Y}}{1-\bar{\sigma}^2}(1-\bar{\sigma}) = \frac{1}{2}\left\langle \frac{Y}{1-\sigma^2}\right\rangle\left[1-\frac{\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle}{\left\langle \frac{Y}{1-\sigma^2}\right\rangle}\right]$$

$$= \frac{1}{2}\left[\left\langle \frac{Y}{1-\sigma^2}\right\rangle - \left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle\right] \tag{3.3.9}$$

$$= \frac{1}{2}\left\langle \frac{Y}{1+\sigma}\right\rangle$$

and

$$2\bar{\sigma}\frac{\bar{Y}}{1-\sigma^2} = 2\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle \tag{3.3.10}$$

With Eqs. (3.3.8), (3.3.9), and (3.3.10) in Eq. (3.3.6) we have the energy density as

$$u = \frac{1}{2}\left[\left\langle \frac{Y}{1-\sigma^2}\right\rangle (S_1^2+S_2^2) + 2\left\langle \frac{Y\sigma}{1-\sigma^2}\right\rangle S_1 S_2 + \frac{1}{2}\left\langle \frac{Y}{1+\sigma}\right\rangle S_6^2\right] \tag{3.3.11}$$

Comparing Eqs. (3.3.11) and (3.3.3), we see that the same result is obtained with the effective Young's modulus and Poisson ratio as with the stiffness method.

4. Appendix

A variety of forms are used for the elastic constants of isotropic media. We collect some useful results here.

4.1. Stiffness in terms of Young's modulus and Poisson ratio

The components of the stiffness tensor in terms of Young's modulus and Poisson ratio are

$$c_{11} = Y\frac{1-\sigma}{(1+\sigma)(1-2\sigma)},\ c_{12} = Y\frac{\sigma}{(1+\sigma)(1-2\sigma)},\ c_{44} = Y\frac{1}{2(1+\sigma)} \tag{4.1.1}$$

4.2. First Lamé constant and shear modulus in terms of Young's modulus and Poisson ratio

The first Lamé constant and shear modulus in terms of Young's modulus and Poisson ratio are

$$\lambda = Y\frac{\sigma}{(1+\sigma)(1-2\sigma)},\ \mu = Y\frac{1}{2(1+\sigma)} \tag{4.2.1}$$

4.3. Stiffness in terms of bulk and shear moduli

The components of the stiffness tensor in terms of the bulk K and shear μ moduli are

$$c_{11} = \frac{3K+4\mu}{3},\ c_{12} = \frac{3K-2\mu}{3},\ c_{44} = \mu \tag{4.3.1}$$

4.4. Young's modulus and Poisson ratio in terms of bulk and shear moduli

The Young's modulus and Poisson ratio in terms of the bulk and shear moduli are

$$Y = \frac{9K\mu}{3K+\mu},\ \sigma = \frac{3K-2\mu}{6K+2\mu} \tag{4.4.1}$$

The inverse is

$$K = \frac{Y}{3(1-2\sigma)},\quad \mu = Y\frac{1}{2(1+\sigma)} \tag{4.4.2}$$

These can be expanded into their real and imaginary parts to find the loss angle for the Young's modulus and Poisson ratio in terms of those for the bulk ϕ_K and shear moduli ϕ_μ (which are perhaps the most physically meaningful loss quantities). It is easiest to obtain the result for ϕ_σ by taking the ratio of the two Eqs. (4.4.2) and then expanding. ϕ_Y is then obtained straightforwardly. We find

$$\phi_Y = \frac{1}{3}\left[(1-2\sigma)\phi_K + 2(1+\sigma)\phi_\mu\right],\ \phi_\sigma = \frac{(1-2\sigma)(1+\sigma)}{3\sigma}(\phi_K - \phi_\mu) \tag{4.4.3}$$

Note that the loss angle for the Poisson ratio vanishes only if the bulk and shear losses are equal. These results are consistent with Hong *et al*, *Phys. Rev. D*, **87**, 082001 (2013), Eqs. (53) and (54).

4.5. Coefficients in mirror energy equation, Eq. (3.1.11)

Two combinations of stiffness coefficients appear in the energy of the mirror layer in the thermal noise calculation in Eq. (3.1.3). These can be computed in terms of bulk and shear moduli using the relations in Eqs. (4.3.1)

$$c_{11} - \frac{c_{12}^2}{c_{11}} = 4\mu\frac{3K+\mu}{3K+4\mu},\ c_{11}^{-1} = \frac{3}{3K+4\mu} \tag{4.5.1}$$

Or, with Eqs. (4.1.1), in terms of the Young's modulus and Poisson ratio:

$$c_{11} - \frac{c_{12}^2}{c_{11}} = \frac{Y}{1-\sigma^2},\quad c_{11}^{-1} = \frac{1}{Y}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma},\quad \frac{c_{12}}{c_{11}} = \frac{\sigma}{1-\sigma} \tag{4.5.2}$$

Using the imaginary part of Young's modulus and Poisson ratio in terms of bulk and shear loss angles from Eqs. (4.4.3), we have

$$\begin{aligned}
&\tilde{c}_{11} - \frac{\tilde{c}_{12}^2}{\tilde{c}_{11}} = \frac{Y}{1-\sigma^2}(1+i\phi_D), && \phi_D \equiv \frac{(1-2\sigma)(1+\sigma)}{3(1-\sigma)}\phi_K + \frac{2}{3}\frac{1-\sigma+\sigma^2}{1-\sigma}\phi_\mu \\
&\tilde{c}_{11}^{-1} = \frac{1}{Y}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma}(1-i\phi_E), && \phi_E \equiv \frac{1+\sigma}{3(1-\sigma)}\phi_K + \frac{2}{3}\frac{1-2\sigma}{1-\sigma}\phi_\mu \\
&\frac{\tilde{c}_{12}}{\tilde{c}_{11}} = \frac{\sigma}{1-\sigma}\left(1+i\,\phi_F\right) && \phi_F \equiv \frac{(1-2\sigma)(1+\sigma)}{3\sigma(1-\sigma)}\left(\phi_K - \phi_\mu\right) \\
&\tilde{c}_{44} = \frac{Y}{2(1+\sigma)}(1+i\phi_\mu) && \\
&\tilde{c}_{11} + \tilde{c}_{12} - 2\frac{\tilde{c}_{12}^2}{\tilde{c}_{11}} = \frac{Y}{1-\sigma}(1+i\phi_G) && \phi_G \equiv \frac{2}{3}\frac{(1-2\sigma)}{(1-\sigma)}\phi_K + \frac{1}{3}\frac{(1+\sigma)}{(1-\sigma)}\phi_\mu
\end{aligned} \tag{4.5.3}$$

In some cases it will be more convenient to have the total imaginary part of the coefficients, which we obtain from Eqs. (4.5.2) and (4.5.3) as

$$\mathrm{Im}\left[c_{11}-\frac{c_{12}^2}{c_{11}}\right]=\frac{Y}{1-\sigma^2}\phi_D=Y\left[\frac{(1-2\sigma)}{3(1-\sigma)^2}\phi_K+\frac{2}{3}\frac{1-\sigma+\sigma^2}{(1-\sigma)^2(1+\sigma)}\phi_\mu\right]$$
$$\mathrm{Im}\left[c_{11}^{-1}\right]=-\frac{1}{Y}\frac{(1+\sigma)(1-2\sigma)}{1-\sigma}\phi_E=-\frac{1}{Y}\left[\frac{(1+\sigma)^2(1-2\sigma)}{3(1-\sigma)^2}\phi_K+\frac{2}{3}\frac{(1+\sigma)(1-2\sigma)^2}{(1-\sigma)^2}\phi_\mu\right]$$
$$\mathrm{Im}\left[\frac{\tilde{c}_{12}}{\tilde{c}_{11}}\right]=\frac{\sigma}{1-\sigma}\phi_F=\frac{(1-2\sigma)(1+\sigma)}{3(1-\sigma)^2}\left(\phi_K-\phi_\mu\right)$$
$$\mathrm{Im}[\tilde{c}_{44}]=\frac{Y}{2(1+\sigma)}\phi_\mu$$
$$\mathrm{Im}\left[\tilde{c}_{11}+\tilde{c}_{12}-2\frac{\tilde{c}_{12}^2}{\tilde{c}_{11}}\right]=\frac{Y}{3(1-\sigma)^2}\left[2(1-2\sigma)\phi_K+(1+\sigma)\phi_\mu\right]$$

(4.5.4)

4.6. Evaluating the integrals in the thermal noise calculation

Let us consider the integrals in Eq. (3.1.9) separately:

$$2\pi\int_0^\infty S_{rr,s}^2 r\,dr=\frac{F^2}{8\pi(\lambda_s+\mu_s)^2}\left(\frac{2-\ln 4}{w^2}\right),\qquad 2\pi\int_0^\infty S_{\theta\theta,s}^2 r\,dr=\frac{F^2}{8\pi(\lambda_s+\mu_s)^2}\frac{\ln 4}{w^2}$$
$$\Rightarrow 2\pi\int_0^\infty\left(S_{rr,s}^2+S_{\theta\theta,s}^2\right)r\,dr=\frac{F^2}{4\pi w^2(\lambda_s+\mu_s)^2} \tag{4.6.1}$$

The integral of the product of $S_{rr,s}S_{\theta\theta,s}$ is tricky due to the logarithmic convergence. It is easier to evaluate as $2S_{rr,s}S_{\theta\theta,s}=(S_{rr,s}+S_{\theta\theta,s})^2-(S_{rr,s}^2+S_{\theta\theta,s}^2)$.

$$2\pi\int_0^\infty\left(S_{rr,s}+S_{\theta\theta,s}\right)^2 r\,dr=\frac{F^2}{4\pi(\lambda_s+\mu_s)^2 w^2}$$
$$\Rightarrow 2\pi\int_0^\infty\left(S_{rr,s}S_{\theta\theta,s}\right)r\,dr=0 \tag{4.6.2}$$

where the second result in Eq. (4.6.2) follows from Eq. (4.6.1).

The integral of the stress is

$$2\pi\int_0^\infty T_{zz,s}^2 r\,dr=\frac{F^2}{\pi w^2} \tag{4.6.3}$$

Finally, the integral of the combined stress and strain fields is

$$2\pi\int_0^\infty T_{zz,s}\left(S_{1s}+S_{2s}\right)r\,dr=2\pi\frac{2F^2}{\pi^2(\lambda_s+\mu_s)w^4}\int_0^\infty e^{-4r^2/w^2}r\,dr$$
$$=\frac{F^2}{2\pi(\lambda_s+\mu_s)w^2} \tag{4.6.4}$$