Gravitational Wave Polarizations: a General Relativity Test

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This research proposal aims at assessing with software simulations the possibility of inferring the polarization content of gravitational waves (GWs) within a Bayesian framework. The response of a network of GW interferometric detectors will be studied in order to discriminate between different polarizations. Indeed, it is essential to quantify in advance, through simulations, how GW detector configuration choices affect our ability to measure the GW polarization content, as this measurement can place strong, fundamental constraints on theories of gravity.

I. CONTEXT

Metric theories of gravity alternative to General Relativity (GR) imply the presence of specific GW polarizations other than the tensor ones predicted by GR. A fundamental goal of GW physics is therefore to find viable ways to detect and measure GW polarizations. We begin with a brief summary about detecting and studying GW polarizations with laser interferometeric GW detectors. The notation used is taken from [1].

A. Linearized Metric Theory of Gravity

In the weak-field regime (or in the far-field limit), an appropriate coordinate system can be found in which the full metric $g_{\mu\nu}$ can be expressed as a perturbation $h_{\mu\nu}$ of order ϵ of the Minkowskian metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1}$$

We can introduce a restricted class of coordinate transformations, called *Gauge Transformations*, such that the transformed metric is still separable into a flat *background* and a perturbation. These transformations are induced by a vector field $\zeta^\mu(x^\nu)$, with $\|\zeta^\mu_{,\nu}\| \sim O(\epsilon)$ such that the perturbation in the new coordinates becomes:

$$h'_{\mu\nu} = h_{\mu\nu} - \zeta_{\{\mu,\nu\}} + O(\epsilon^2)\,. \tag{2}$$

In terms of the perturbed metric, the *Riemann Tensor* is written as:

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}(h_{\mu\beta,\nu\alpha} + h_{\nu\alpha,\mu\beta} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu})\,. \tag{3}$$

It is noteworthy to stress that the Riemann Tensor is an invariant under gauge transformations, since it encodes only the information about gravity, without keeping track of the chosen coordinate system.

Fixing a gauge is a way to use the freedom we have in the choice of the vector field ζ^μ to simplify our problem and reduce the ten degrees of freedom of the symmetric rank-2 tensor $h_{\mu\nu}$ down to six. In GR, a common gauge choice in vacuum is the Lorentz gauge (also known as *harmonic gauge*) $\bar{h}^\mu_{\nu,\mu} = 0$, where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$. This gauge is adopted because it simplifies the Einstein tensor. The residual gauge freedom given by field equations (direct consequence of the *Bianchi identity* applied to the Einstein tensor) allows one to further impose $h_{0i} = 0$ and $h = 0$, leaving only two radiative degrees of freedom, the two *tensorial* polarizations. This last choice is often called the *trasverse-traceless gauge*, or TT-gauge. In a non-GR theory, since the field equations are different, we have to take into account all six degrees of freedom.

B. Polarizations and Antenna Patterns

We want to describe the interferometer hit by (plane) GW as a couple of test masses (situated at the end of the two arms of the detector) subject to a metric perturbation: the response of the detector depends on the difference in travel time along the two arms. The behavior of the separation vector ξ^α between the two freely-falling test-particles with 4-velocity u^β is given by the equation of geodesic deviation:

$$\frac{D^2\xi^\mu}{dt^2} = -R^\mu_{\nu\alpha\beta}u^\nu\xi^\alpha u^\beta\,. \tag{4}$$

For slowly moving particles, we can rewrite the previous equation as:

$$\frac{d^2\xi_j}{dt^2} = -c^2R_{0j0k}\xi^k = -\frac{G}{2c^4D}\frac{\partial}{\partial\tau^2}S_{jk}(\tau,\mathbf{N})\xi_k\,. \tag{5}$$

where D is the distance from the source of GW, $\mathbf{N} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is a unit vector which points toward the direction of the source in the sky (direction of propagation of the wave) while the S_{ij} contains the proper time dependent amplitudes of the perturbation. Integrating over time, we obtain the equation of motion for the displacement of the detector arms at first order:

$$\xi^j(t) = \xi^j(0) + \frac{G}{2c^4D}S^{jk}(\tau,\mathbf{N})\xi_k(0)\,. \tag{6}$$

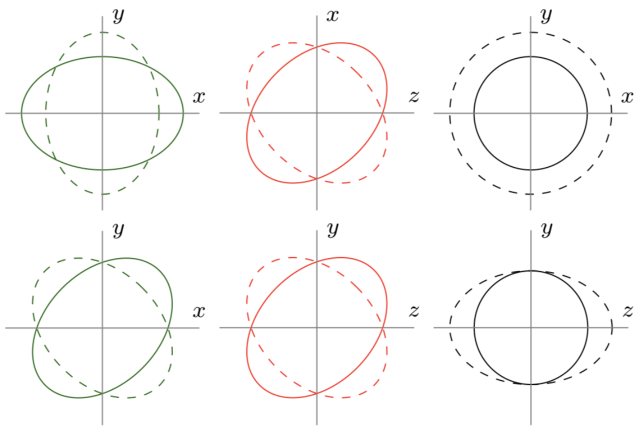


FIG. 1. *Effect of different GW polarizations on a ring of free-falling test particles.* Plus (+) and cross (×) tensor modes (green); vector-x (x) and vector-y (y) modes (red); breathing (b) and longitudinal (l) scalar modes (black). In all of these diagrams the wave propagates in the z direction. This decomposition into polarizations was first proposed for generic metric theories in [2]. (Reproduced from [3].)

Assuming that the wave travels along the z-direction, we have:

$$S_{jk} = \begin{pmatrix} A_b + A_+ & A_\times & A_{Vx} \\ A_\times & A_b - A_+ & A_{Vy} \\ A_{Vx} & A_{Vy} & A_l \end{pmatrix} . \tag{7}$$

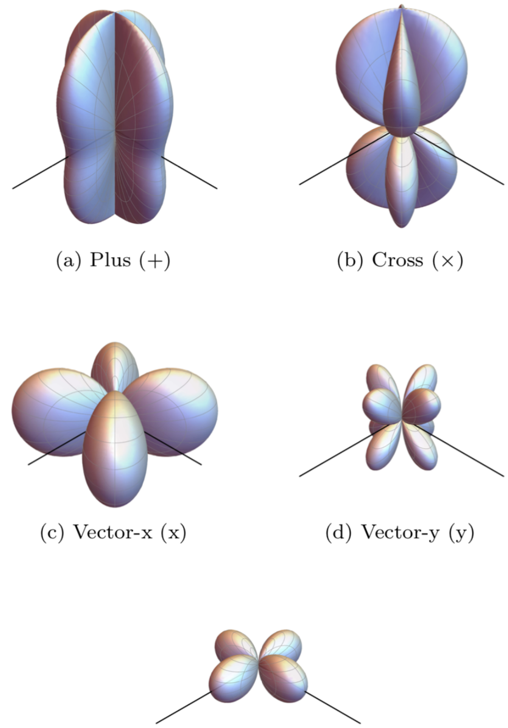
Six degrees of polarization can be identified through their effect on a ring of free-falling test particles (see fig. 1). Two scalar (A_b and A_l), called respectively *breathing* and *longitudinal* modes, one of which (A_b) is transverse with respect to the direction of propagation of the wave. Two vector (A_x and A_y) modes, partly longitudinal and partly transverse. Finally, two tensorial ($A_\times$ and A_+) transverse modes. GR only allows for $A_\times$ and A_+.

From a field-theoretic point of view, polarizations are strictly correlated with the helicity (projection of the spin along the motion) of the *graviton*: a massless graviton has only ±2 helicity, which corresponds to the two tensorial polarizations of GR.

While the amplitudes and the phases of GWs depend crucially on the source dynamic, the response of a quadrupolar antenna to them is determined by the geometry of the system source-detector (up to an overall normalization). Let $\mathbf{e_1}$ and $\mathbf{e_2}$ be the unit vectors aligned with the two arms of the interferometer. The response function is given by:

$$S(t) = \frac{1}{2}(e_1^j e_1^k - e_2^j e_2^k) S_{jk}(\tau, \mathbf{N}) . \tag{8}$$

We can now consider an orthonormal basis $(\mathbf{N}, \mathbf{e_X}, \mathbf{e_Y})$ to fully describe the polarization of the wave, where $(\mathbf{e_X}, \mathbf{e_Y})$ are rotated by an angle ψ (called *angle of polarization*) around $\mathbf{N}$ with respect to the basis $(\mathbf{e'_1}, \mathbf{e'_2})$.



(a) Plus (+) (b) Cross (×) (c) Vector-x (x) (d) Vector-y (y)

FIG. 2. *Angular response of a quadrupolar detector to each GW polarization.* The radial distance represents the response of a single quadrupolar antenna to a unit-amplitude gravitational signal of a tensor (top), vector (middle), or scalar (bottom) polarization, i.e. $|F_P|$ for each polarization P for $\psi = 0$. The polar and azimuthal coordinates correspond to the source location with respect to the detector, which is to be imagined as placed with its vertex at the center of each plot and arms along the x and y-axes. The response is plotted to scale, such that the black lines representing the detector arms have unit length in all plots. The response to breathing and longitudinal modes is identical, so we only display it once and label it scalar. (Reproduced from [4].)

These primed vectors are obtained by two subsequent rotations of $(\mathbf{e_1}, \mathbf{e_2})$ that align the basis of the interferometer along $\mathbf{N}$. We can write the total response as a function of three angles (θ, ϕ, ψ):

$$S(t) = F_P(\theta, \phi, \psi) A_P \tag{9}$$

where $P = \{b, l, x, y, +, \times\}$, the F's are called *antenna pattern functions.* (see fig. 2) and the sum runs over all polarizations P. It can be shown that (apart from a sign) the two scalar polarizations are completely degenerate, therefore they cannot be distinguished and from now on we will consider only a single scalar mode S, with corresponding antenna response F_S.

If we are interested to the sensitivity of a network of N detectors, setting $\psi = 0$ since we are not dealing with any specific source, it is useful to define the effective response vector as:

$$\vec{F}_H(\theta, \phi) := (|F_H^1(\theta, \phi)|, ..., |F_H^N(\theta, \phi)|) \tag{10}$$

where the F_H^i's are the sum in quadrature of the two *antenna patterns* of the i-th detector for each polarization $H = \{s, v, t\}$ (scalar, vector, tensor). We can evaluate the effective sensitivity of the network to non-tensorial polarizations with respect to tensorial ones by computing the *overlap* factor:

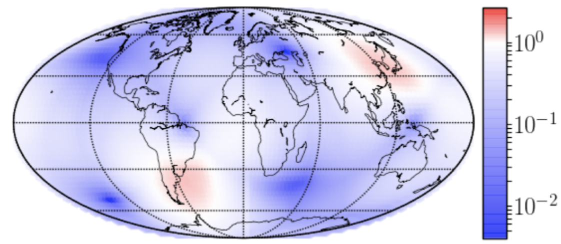
$$\mathcal{F}_{H/t} = \frac{\vec{F}_H(\theta,\phi) \cdot \vec{F}_t(\theta,\phi)}{\vec{F}_t(\theta,\phi) \cdot \vec{F}_t(\theta,\phi)} \,. \tag{11}$$

II. PROPOSAL

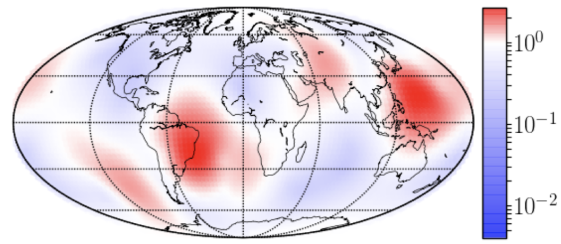
A. Constraints on Detectors Orientation

In the measurement of GW polarizations, eight different unknowns play a role: the six polarization modes and the two angles that identify the position of the source in the sky. However, as mentioned earlier, the response functions of a laser interferometer to the two scalar modes are completely degenerate, and we are therefore left with five independent polarization modes. The number of available quadrupolar antennas is then crucial to measure the polarization content. Futhermore, the orientation of the arms of the instruments plays a fundamental role. If the arms of a pair of interferometers are aligned (as is the case of the twoLIGO detectors), while the sensitivity to gravitational radiation is maximized, there is the downside that the antenna pattern function will be the same, preventing the possibility of distinguishing between different polarizations. Over the next ten years, two new interferometers will be available: KAGRA (in $\sim$ 2020) and LIGO-India. Additionally, the prospect of building a new generation of detectors (so called 3G detectors) is under discussion and investigation (for further references, see [5] and [6]).

It is essential to quantify in advance, through simulations, how GW detector configuration choices affect our ability to measure the GW polarization content, as this measurement can place strong, fundamental constraints on theories of gravity. From the definition of *overlap* given in Eq. (11), we can draw sky-maps of relative sensitivity. This has been done in [3] (see fig. 3) for the three-detector LIGO-Virgo network. We propose to extend the study up to five interferometers, focusing on how the relative effective antenna patterns vary as a function of the orientation of the LIGO-India arms. Averaging over all sky-locations, we can also determine the expected network sensitivity to the different kinds of polarizations. The network will be the least sensitive to scalar modes (since the interferometers are individually less sensitive to these); but the question of how the addition of two more interferometers — one of which (LIGO-India) has not been built yet — will improve our ability to measure the GW polarization content remains unanswered.



(a) Scalar



(b) Vector

FIG. 3. *Overlaps of LIGO-Virgo network effective antenna patterns.* The normalized inner-products of Eq. (27) for the three-instrument network. The top plot compares scalar to tensor $\mathcal{F}_{s/t}$, and the bottom one compares vector to tensor $\mathcal{F}_{v/t}$. Blue (red) marks regions for which the effective non-tensor response is greater (less) than tensor. A map of Earth is overlaid for reference. (Reproduced from [3].)

B. Bayesian Methods

We begin this subsection by summarizing the work carried out in [4]. Given a set of GW data $\mathbf{B}$, we want to test seven possible Bayesian hypotheses regarding its polarization content: it is purely tensor ($\mathcal{H}_t$), purely vector ($\mathcal{H}_v$), purely scalar ($\mathcal{H}_s$), scalar-tensor ($\mathcal{H}_{st}$), vector-tensor ($\mathcal{H}_{vt}$), scalar-vector ($\mathcal{H}_{sv}$), scalar-vector-tensor ($\mathcal{H}_{svt}$). Using Bayes theorem, we can expand the probability $P(\mathcal{H}_S|\mathbf{B})$ that, given the data, a signal hypotheses can be accepted:

$$P(\mathcal{H}_S|\mathbf{B}) = \sum_m P(\mathcal{H}_m) P(\mathbf{B}|\mathcal{H}_S)/P(\mathbf{B}) \,, \tag{12}$$

where $m \in \{t, v, s, st, vt, sv, svt\}$, $P(\mathcal{H}_m)$ is a prior on the model, $P(\mathbf{B}|\mathcal{H}_S)$ is the marginalized likelihood and $P(\mathbf{B})$ is a normalization factor. Therefore, it is crucial to study the priors of our model, i.e., the probability to detect a polarization signal of one kind rather than other.

There are two ways to extract polarization from antenna patterns if the GW signal is transient.

In the first scenario in which the GW signal has an optical counterpart that allows for an accurate determination of the source position [7], it is particularly convenient look for non-GR signal content by constructing *GR null stream(s)*. Indeed, for N detectors, the signal manifold is N-dimensional with N basis vectors, five of which can be chosen along the independent antenna patterns

$F^i_+, F^i_\times, F^i_x, F^i_y, F^i_s$ where the Latin index runs along the N detectors. The remaining $N-5$ vectors will give us null streams, independently from the polarizations of the wave. The j-th detector datastream can be written in tensor notation as:

$$S^j = F^j_P h^P + n^j \,, \tag{13}$$

where n^j is the noise content in the j-th detector. In the case of three detectors, we can define a *GR null stream*, i.e., a stream without tensor modes, in the following way:

$$S_{GR-null} = \frac{e_{ijk} F^j_+ F^k_\times}{|\delta_{ij} F^i_+ F^j_\times|} S^i \,. \tag{14}$$

Depending on the number of interferometers, we can have more than one null stream, and with more than five, one can construct a complete set of null streams that covers all metric theories of gravity. This method is model independent, but it has the disadvantage of requiring an electromagnetic counterpart. It would be interesting to study GR null streams via simulated data with the Bayesian approach suggested in [4].

A second method which doesn't require necessarily an electromagnetic counterpart, using a sine-Gaussian analysis to reconstruct the waveform, one may infer from time delays the source location and then the best fitting combination of antenna patterns for the peak in amplitude. This analysis is independent from the phase evolution and it only needs a well-defined peak. With three interferometers, it is already possible to infer the direction $\mathbf{N}$ of the source in the sky just measuring time delays, which is given by the formula:

$$\delta t_I = \mathbf{N} \cdot \mathbf{x}_I / c \,. \tag{15}$$

where δt_I is the time delay with respect to the geocenter and $\mathbf{x}_I$ joins the geocenter to the detector. With four interferometers, constraints on the propagation velocity of GWs can be placed, providing precious information about the mass of the graviton and, indirectly, on GW polarizations.

In the context of this second method, an interesting study to carry out is to implement a fully Bayesian software analysis like the one suggested in [3] that relies on `LALInference`. In the remainder of this Section, we propose a slight complication of the toy model presented in [3] that can be a useful training in the perspective of having five detectors working simultaneously.

Let's consider a complex GW with all five polarization degrees of freedom. We consider a waveform description that consists of a simple sine-Gaussian, with some characteristic frequency Ω and a damping time τ. The response at each detector I will be:

$$S_I = \mathcal{R}\big[A(F^I_+ + \sum_p \epsilon_p F^I_p)\; e^{i\Omega(t-t_0-\delta t_I)}\big] e^{-(t-t_0-\delta t_I)^2/\tau^2} \,, \tag{16}$$

where we have chosen the plus, tensorial polarization as a reference, F^I_p is the antenna pattern for the $p = \{\times, x, y, s\}$ polarization at the I-th detector, ϵ_p's are complex mixing coefficients to be determined, $A = |A| e^{i\phi_0}$ is a complex amplitude and $\mathcal{R}$ denotes the real part. In a simpler fashion:

$$S_I = \mathcal{A}_I \cos\left[\Omega(t - \Delta t_I) + \Phi_I\right] e^{-(t-\Delta t_I)^2/\tau^2} \tag{17}$$

where we defined the following observables:

$$\mathcal{A}_I = |A| |F^I_+ + \sum_p \epsilon_p F^I_p| \tag{18}$$

$$\Phi_I = \phi_0 + \arctan \frac{\sum_p \mathcal{I}[\epsilon_p] F^I_p}{F^I_+ + \sum_p \mathcal{R}[\epsilon_p] F^I_p} \tag{19}$$

$$\Delta t_I = t_0 + \delta t_I \tag{20}$$

with $\mathcal{I}$ denoting the imaginary part. A simple algebraic argument can be used to show that with five (non-degenerate) detectors, the four complex ϵ_p's can be fully determined. Indeed, with time-delay measurements via Eq. (15), we can obtain the source direction on the sky $\mathbf{N}$. This information allows us to compute the antenna patterns for every detector. From phase differences and ratios of amplitudes, using Eqs. (18) and (19), we obtain eight implicit equations for the four complex ϵ_p's.

So far, we dealt exclusively with transient signals. However, it should be mentioned that if we have a longer lasting signal (possibly detected with 3G detectors or LISA), the motion of the interferometer due to Earth's rotation (or the orbital motion of the detector itself) would allow us to study the evolution in time of antenna patterns, extracting more information from the single detector.

III. CONCLUSION

Since the response of a network of detectors of GWs is strictly correlated to the polarization of the wave through *antenna patterns* of each interferometer, which depend only on the relative geometry between source and detector, an extensive study of the performance of 4 and 5 detector network configurations through simulations (within a Bayesian approach) is essential. We recall that, the detection of GW non-tensor polarizations would be the first, direct evidence of new physics; at the same time, repeated non-detections would allow us to place more and more stringent tests on GR.

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