SURF Report:
Constructing Echo Waveform from Kerr-like Background

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1 Background

Gravitational wave (GW) echoes, the GW reflected from certain structures outside of horizon, could help us probe Planck scale structure near horizon and therefore is of significant importance in GW physics [1]. Abedi *et al.* claimed to have found evidence of echo signal from LIGO data in 2017[2], although the significance turned out to be low[3]. Still, echo signal is a promising candidate for probing physics beyond General Relativity (GR) and many groups have been trying to search echo signals from LIGO data based on reliable statistical methods [4, 5, 6, 7].

Previous works about GW echoes are mostly for non-rotating Black Holes(BHs) described by Schwarzschild metric. For instance, Mark *et al.* studied echo modes in some Exotic Compact Object(ECO) models by solving Teukolsky equation in Schwarzschild limit with reflecting boundary[8]. Du *et al.* studied energy spectrum of such echoes and showed that they contribute to the stochastic GW background[9]. One notable point is the instability of central ECO under the energy flux of incoming GW[10] and we still don't know if this instability still exists for rotating ECO.

In general situation, astrophysical BHs have both mass and angular momentum and are described by Kerr metric. Estimating spin of BHs from LIGO/VIRGO events can also help understand the formation history of them and their stellar environments[11]. As for echo, some works have been devoted to searching echo signals from spinning ECOs based on phenomenological model[12, 13]. In order to understand spin effects from echo signal, a well-developed theoretical model for generating echo templates is needed. Some works in this direction are [14, 15, 12]. Nakano *et al.* have constructed a model for echoes from spinning ECOs[15], where the asymptotic behavior of solutions to Teukolsky equation is used to analyze reflectivity and echo modes, but the reflecting surface is assumed to be located exactly at the horizon and the incident wave is phenomenological in [15]. We will try to extend [8] to spinning case and develop a more realistic echo model.

2 Objectives

- Provide a more realistic model for echoes from spinning ECOs.
- Generate echo waveform templates and try to incorporate them into current search pipeline.
- Examine stability of spinning ECO against incoming GW.

3 Approach

Teukolsky equation for spin-2 field has been a useful tool for computing GW waveforms, on which our approach is based. We will explain the method to compute Teukolsky-based GW waveform in Sec 3.1. Effective-One-Body(EOB) formalism[16, 17, 18] has been useful tool to study binary inspirals. Wenbiao and Zhoujian have applied EOB formalism and Teukolsky-based waveform to compute GW emission from Extreme-Mass-Ratio Inspiral(EMRI) and Intermediate-Mass-Ratio Inspiral (IMRI) systems[19, 20]. We briefly explain their code in Sec. **??** and in Sec. 3.3 we show how to modify the current method to incorporate echo modes and study the stability. Throughout this article, we will keep the notation consistent with [21].

3.1 Teukolsky Equation

Astrophysical Black Holes with mass M and angular momentum J are described by Kerr metric. Perturbations of spin-0, spin-1 (e.g. electromagnetic fields) and spin-2 (e.g. Gravitational waves) fields in Kerr spacetime are governed by Teukolsky equations[22], which is a set of ordinary differential equations, separated in Boyer-Lindquist coordinates. For GW, the perturbation field ψ_4, decomposed in frequency domain $\psi_4 = \rho^4 \int_{-\infty}^{+\infty} d\omega \sum_{lm} R_{lm\omega}(r) \ {}_{-2}S^{a\omega}_{lm}(\theta) e^{im\phi} e^{-i\omega t}$ with ${}_{-2}S^{a\omega}_{lm}$ being spin-weighted spheroidal harmonics, obeys:

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr}\right) - V(r)R_{lm\omega} = -\mathscr{T}_{lm\omega}(r), \tag{1}$$

where $\mathscr{T}_{lm\omega}(r)$ is the source term, which will be discussed in Sec. 3.4, and the potential is

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{2}$$

where $K = (r^2 + a^2)\omega - ma$, $\lambda = E_{lm} + a^2\omega^2 - 2amw - 2$ and $\Delta = r^2 - 2Mr + a^2$.

First, we consider the homogeneous Teukolsky equation where the source term is zero. We can solve it by analytical expansion, as discussed in [25, 26] and here we don't go into technical details of it. The homogeneous Teukolsky equation allows two independent solutions $R^H_{lm\omega}$, which is purely ingoing at the horizon, and $R^\infty_{lm\omega}$, which is purely outgoing at infinity:

$$\begin{aligned} R^H_{lm\omega} &= B^{hole}_{lm\omega}\Delta^2 e^{-ipr*}, \quad r \to r_+ \\ R^H_{lm\omega} &= B^{out}_{lm\omega} r^3 e^{i\omega r*} + r^{-1} B^{in}_{lm\omega} e^{-i\omega r*}, \quad r \to \infty; \end{aligned} \tag{3}$$

$$\begin{aligned} R^\infty_{lm\omega} &= D^{\rm out}_{lm\omega} e^{ipr*} + \Delta^2 D^{in}_{lm\omega} e^{-ipr*}, \quad r \to r_+ \\ R^\infty_{lm\omega} &= r^3 D^\infty_{lm\omega} e^{-i\omega r*}, \quad r \to \infty, \end{aligned} \tag{4}$$

where $p = \omega - \frac{ma}{2Mr_+}$, $r_+ = M + \sqrt{M^2 - a^2}$ and $r*$ is the tortoise coordinate related to r by $dr*/dr = (r^2 + a^2)/\Delta$

Then, using the homogeneous solutions and proper boundary conditions, we can construct the solution to radial Teukolsky equation with source term. By imposing BH boundary condition, i.e. wave being purely outgoing at infinity and purely ingoing at horizon, the radial function is:

$$R^{BH}_{lm\omega}(r) = \frac{R^\infty_{lm\omega}(r)}{2i\omega B^{in}_{lm\omega} D^\infty_{lm\omega}} \int_{r_+}^{r} dr' \frac{R^H_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} + \frac{R^H_{lm\omega}(r)}{2i\omega B^{in}_{lm\omega} D^\infty_{lm\omega}} \int_{r}^{\infty} dr' \frac{R^\infty_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{5}$$

The asymptotic behavior of this solution near horizon and infinity is:

$$R^{BH}_{lm\omega}(r \to \infty) = Z^{H}_{lm\omega} r^3 e^{i\omega r*}, \tag{6}$$

$$R^{BH}_{lm\omega}(r \to r_+) = Z^{\infty}_{lm\omega} \Delta^2 e^{-ipr*}. \tag{7}$$

By taking the limit at $r \to \infty$ and $r \to r_+$ of the solution (Eq. 5), with the asymptotic behavior of homogeneous solutions (Eq. 3, 4), one can find the amplitudes $Z^{H,\infty}_{lm\omega}$:

$$Z^{H}_{lm\omega} = \frac{1}{2i\omega B^{in}_{lm\omega}} \int_{r_+}^{\infty} dr' \frac{R^{H}_{lm\omega}(r') \mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{8}$$

$$Z^{\infty}_{lm\omega} = \frac{B^{hole}_{lm\omega}}{2i\omega B^{in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r_+}^{\infty} dr' \frac{R^{\infty}_{lm\omega}(r') \mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{9}$$

Finally, from the relation $\psi_4(r \to \infty) \to \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times)$, decompose ψ_4 in frequency domain as we previously introduced and plug in the behavior of radial function at infinity (Eq. 6). The gravitational waveform, observed from distance R, latitude angle Θ and azimuthal angle Φ, is given by:

$$h^{BH}_{+}(R, \Theta, \Phi, t) - ih^{BH}_{\times}(R, \Theta, \Phi, t) = \frac{2}{R} \sum_{lm} \int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} Z^{H}_{lm\omega} \, {}_{-2}S^{a\omega}_{lm}(\Theta) e^{i(m\Phi - \omega[t - r*])}. \tag{10}$$

3.2 Sasaki-Nakamura Equation

The Teukolsky potential is long-ranged, making it hard to numerically extract certain parameters in homogeneous solution, e.g. B^{in}, which is overwhelmed by B^{out} at infinity. Sasaki *et al.* transformed the radial equation so that the potential is short-ranged and the equation become numerical computable[23]. Moreover, asymptotic behavior of solutions to Sasaki-Nakamura equations is purely sinuous, making it easier to generalize to echo construction.

Transformation between Teukolsky function $R_{lm\omega}$ and Sasaki-Nakamura function $X_{lm\omega}$ is given by[21]:

$$R_{lm\omega} = \frac{1}{\eta} \left[\left(\alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta X_{lm\omega}}{\sqrt{r^2 + a^2}} - \frac{\beta}{\Delta} \frac{d}{dr} \frac{\Delta X_{lm\omega}}{\sqrt{r^2 + a^2}} \right]. \tag{11}$$

Taking Eq. (11) into Teukolsky equation, one can find the equation for Sasaki-Nakamura function $X_{lm\omega}$:

$$\frac{d^2 X_{lm\omega}}{dr*^2} - F(r) \frac{dX_{lm\omega}}{dr*} - U(r) X_{lm\omega} = 0. \tag{12}$$

The functions η, α, β and the potentials $F(r), U(r)$ can be found in [21]. The reverse transformation can be found in [24].

The Sasaki-Nakamura equation admits two homogeneous solution having the purely sinuous asymptotic behavior due to the short-rangeness of potential $U(r)$:

$$\begin{aligned} X^{H}_{lm\omega} &= A^{hole}_{lm\omega} e^{-ipr*}, \quad r \to r_+, \\ X^{H}_{lm\omega} &= A^{\text{out}}_{lm\omega} e^{i\omega r*} + A^{\text{in}}_{lm\omega} e^{-i\omega r*}, \quad r \to \infty; \end{aligned} \tag{13}$$

and

$$X^{\infty}_{lm\omega} = C^{\rm out}_{lm\omega} e^{ipr*} + C^{\rm in}_{lm\omega} e^{-ipr*}, \quad r \to r_+,$$
$$X^{\infty}_{lm\omega} = C^{\infty}_{lm\omega} e^{i\omega r*}, \quad r \to \infty, \tag{14}$$

Taking the asymptotic behavior into the transformation (Eq. 11) and comparing with asymptotic behavior of Teukolsky function (Eq. 3,4). One finds the relation between asymptotic amplitudes of Sasaki-Nakamura functions and Teukolsky functions [24]:

$$\begin{aligned} B^{in}_{lm\omega} &= -\frac{1}{4\omega^2} A^{in}_{lm\omega} \\ B^{out}_{lm\omega} &= -\frac{4\omega^2}{c_0} A^{out}_{lm\omega} \\ B^{hole}_{lm\omega} &= -\frac{1}{d_{lm\omega}} A^{hole}_{lm\omega} \end{aligned} \tag{15}$$

$$\begin{aligned} D^{in}_{lm\omega} &= -\frac{1}{d_{lm\omega}} C^{in}_{lm\omega} \\ D^{out}_{lm\omega} &= -\frac{4p\sqrt{2Mr_+}(2Mr_+ p + i\sqrt{M^2 - a^2})}{\eta(r_+)} C^{out}_{lm\omega} \\ D^{\infty}_{lm\omega} &= -\frac{4\omega^2}{c_0} C^{\infty}_{lm\omega} \end{aligned} \tag{16}$$

where the coefficients c_0 and $d_{lm\omega}$ can be found in [21]

The homogeneous solution X^{∞} can be regarded as a incident wave coming out near the horizon with amplitude $C^{out}_{lm\omega}$ scattered off the Sasaki-Nakamura potential. We can thus define the reflection and transmission factors $\mathcal{R}_{BH}$ and $\mathcal{T}_{BH}$:

$$\mathcal{T}_{BH} = \frac{C^{\infty}}{C^{out}}, \mathcal{R}_{BH} = \frac{C^{in}}{C^{out}} \tag{17}$$

3.3 Constructing echo modes

In this section, we construct echo waveforms analytically and the only phenomenological quantity is the reflectivity $\tilde{\mathcal{R}}$ near horizon. First, the method established in [8] will be extended to Teukolsky equations with non-vanishing BH spin. By interpreting a Planck scale potential barrier near horizon to a frequency dependent reflectivity and matching the solution of Teukolsky equation to the reflecting boundary condition, we'll find a procedure to generate echo modes for spinning ECO.

Since the Sasaki-Nakamura equation has short ranged potential and therefore purely sinuous waves near horizon and at infinity, it's easier to impose reflecting boundary for Sasaki-Nakamura equation. On the other hand, we have the method to semi-analytically solve Teukolsky equation and extract waveform from radial Teukolsky function in hand. Therefore, we impose the reflecting boundary condition in Sasaki-Nakamura equation and after getting the relevant coefficient describing additional term induced by reflecting boundary, we transform the Sasaki-Nakamura function into Teukolsky function. Then we can utilize the results of BH waveform based on Teukolsky equation to establish echo waveform.

We consider the combination of X^H and X^{∞}

$$X^{ref}_{lm\omega} = \mathcal{K} X^{\infty}_{lm\omega} + X^H_{lm\omega} \tag{18}$$

which satisfies the reflecting boundary near horizon $r_0 = r_+ + \epsilon$ with reflectivity $\tilde{\mathcal{R}}$

$$X^{ref} \propto e^{-ip(r*-r_0*)} + \tilde{\mathcal{R}} e^{ip(r*-r_0*)} \quad r* \to r_0* \tag{19}$$

Taking the asymptotic behavior of $X^{H,\infty}_{lm\omega}$ (Eq. 13, 14) into the above boundary condition, one can find $\mathcal{K}$:

$$\mathcal{K} = \frac{A^{hole}}{C^\infty} \frac{\tilde{\mathcal{R}} e^{-2ipr_0*} \mathcal{T}_{BH}}{1 - \tilde{\mathcal{R}} e^{-2ipr_0*} \mathcal{R}_{BH}} \quad \text{with} \quad \mathcal{T}_{BH} = \frac{C^\infty}{C^{out}}, \, \mathcal{R}_{BH} = \frac{C^{in}}{C^{out}} \tag{20}$$

The corresponding solution to homogeneous Teukolsky equation by relation (11) is $R^{ref} = \mathcal{K} R^\infty + R^H$. Therefore, with R^∞ satisfying ECO boundary condition at infinity and R^{ref}satisfying ECO boundary at surface, the Green function of Teukolsky with ECO boundary is:

$$g^{ECO}(r, r') = \frac{R^\infty(max\{r, r'\}) R^{ref}(min\{r, r'\})}{2i\omega B^{in} D^\infty} = g^{BH} + \mathcal{K} \frac{R^\infty(r) R^\infty(r')}{2i\omega B^{in} D^\infty} \tag{21}$$

Then the ECO solution will be the BH solution R^{BH} in Eq. 5 with an additional echo contribution:

$$R^{ECO} = R^{BH} + \mathcal{K} \frac{R^\infty(r)}{2i\omega B^{in} D^\infty} \int_{-\infty}^{+\infty} \frac{R^\infty(r') \mathcal{T}(r')}{\Delta(r')} dr' \tag{22}$$

Near ECO surface the solution approaches:

$$R^{ECO} \to \frac{Z^\infty}{B^{hole}} (R^H(r) + \mathcal{K} R^\infty(r)) \quad r \to r_0 \tag{23}$$

The corresponding Sasaki-Nakamura function satisfies the reflecting boundary:

$$X^{ECO} \propto X^H + \mathcal{K} X^\infty \propto e^{-ip(r*-r_0*)} + \tilde{\mathcal{R}} e^{ip(r*-r_0*)} \quad r* \to r_0* \tag{24}$$

At infinity, the solution is purely outgoing:

$$R^{ECO}(r \to \infty) = (Z^H + \mathcal{K} \frac{D^\infty}{B^{hole}} Z^\infty) r^3 e^{i\omega r*} \tag{25}$$

which gives the GW waveform with echoes

$$h_+^{ECO}(R, \Theta, \Phi, t) - i h_\times^{ECO}(R, \Theta, \Phi, t) = \frac{2}{R} \sum_{lm} \int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} (Z^H_{lm\omega} + \mathcal{K}_{lm\omega} \frac{D^\infty_{lm\omega}}{B^{hole}_{lm\omega}} Z^\infty_{lm\omega}) \, {}_{-2}S^{a\omega}_{lm}(\Theta) e^{i(m\Phi - \omega[t-r*])}. \tag{26}$$

Then, we will incorporate this method into our existing code to numerically generate echo waveforms. We expect that the main structure of current code will remain, with some additional subroutines computing the additional term in Eq. 22, the echo contribution to $Z^H_{lm\omega}$ and thus the full waveform seen by a distant observer.

Note that the homogeneous solutions are determined up to two constants, i.e. we can transform $X^H \to PX^H$, $X^\infty \to QX^\infty$ (and consequently $R^H \to PR^H$, $R^\infty \to QR^\infty$) with P, Q being two arbitrary complex number, the final solution with source term (Eq. 5,22) and the waveform (Eq. 10,26) does not change. So we have the freedom to choose $A^{hole} = 1, C^\infty = 1$

3.4 Source Term

Following the formalism of [21], the source term in Teukolsky equation induced by a particle with trajectory $x^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$ is given by [29]:

$$\begin{aligned}\mathscr{T}_{lm\omega}(r') = \int_{-\infty}^{\infty} dt e^{i[\omega t - m\phi(t)]}\Delta(r')^2\{[A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}]\delta(r' - r(t)) \\ \partial_{r'}([A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}]\delta(r' - r(t))) + \partial_{r'}^2[A_{\bar{m}\bar{m}2}\delta(r' - r(t))]\}\end{aligned} \tag{27}$$

where A_{abk} $(a, b = n, \bar{m},\ k = 0, 1, 2)$ are related to $r(t)$ and $\theta(t)$ and can be found in Eq. (4.40) of [21]. Although the original article is considering circular orbits, the derivation about source term (Eq.(4.26) to Eq.(4.41) of [21]) can be easily generalized to generic trajectory by replacing r_0 with $r(t)$. Then the amplitudes $Z_{lm\omega}^{H,\infty}$ are given by:

$$\begin{aligned}Z_{lm\omega}^{H} = \frac{1}{2i\omega B_{lm\omega}^{in}} \int_{-\infty}^{+\infty} dt e^{i[\omega t - m\phi]}\{R_{lm\omega}^{H}(r(t))[A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] \\ -\frac{dR_{lm\omega}^{H}}{dr}|_{r(t)}[A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] + \frac{d^2R_{lm\omega}^{H}}{dr^2}|_{r(t)}A_{\bar{m}\bar{m}2}\}\end{aligned} \tag{28}$$

$$\begin{aligned}Z_{lm\omega}^{\infty} = \frac{B_{lm\omega}^{hole}}{2i\omega D_{lm\omega}^{\infty} B_{lm\omega}^{in}} \int_{-\infty}^{+\infty} dt e^{i[\omega t - m\phi]}\{R_{lm\omega}^{\infty}(r(t))[A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] \\ -\frac{dR_{lm\omega}^{\infty}}{dr}|_{r(t)}[A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] + \frac{d^2R_{lm\omega}^{\infty}}{dr^2}|_{r(t)}A_{\bar{m}\bar{m}2}\}\end{aligned} \tag{29}$$

For ringdown signal, we consider the part of trajectory from particle entering into ISCO at t_{ISCO} to particle plunging into horizon at t_{plunge}. Therefore, the energy-momentum tensor, and thus the integrand of Eq. (28,29), is only non-zero between $t_{\text{ISCO}} < t < t_{\text{plunge}}$.

3.5 Energy Flux

The energy flux at infinity can be derived from the energy carried by the gravitational wave. The energy endowed by "+" and "×" mode of Eq. (26) is:

$$(\frac{dE}{dAdt})|_{R\to\infty} = \frac{1}{16\pi} < (\partial_t h_+)^2 + (\partial_t h_\times)^2 > \tag{30}$$

Take Eq. (26) into it and integrate over a sphere. The energy flux at infinity will be:

$$(\frac{dE}{dt})|_{R\to\infty} = \sum_{l,m}\int_{-\infty}^{+\infty} \frac{|Z_{lm\omega}^{H} + \mathcal{K}_{lm\omega}\frac{D_{lm\omega}^{\infty}}{B_{lm\omega}^{hole}}Z_{lm\omega}^{\infty}|^2}{4\pi\omega^2}d\omega \tag{31}$$

To estimate the energy flux near horizon we first look at the behavior of the radial function $R_{lm\omega}$ near ECO surface and BH horizon. At ECO surface, from Eq. (23):

$$R^{ECO} \to \frac{Z^{\infty}}{B^{hole}}(R^{H}(r) + \mathcal{K}R^{\infty}(r)) = Z^{\infty}(1 + \mathcal{K}\frac{D^{in}}{B^{hole}})\Delta^2 e^{-ipr*} + Z^{\infty}\mathcal{K}\frac{D^{out}}{B^{hole}}e^{ipr*} \quad r \to r_0 \tag{32}$$

Assuming the ECO surface has amplitude transmissivity $\tilde{\mathcal{T}}$, we have at horizon:

$$R^{ECO} \to \tilde{\mathcal{T}}Z^{\infty}(1 + \mathcal{K}\frac{D^{in}}{B^{hole}})\Delta^2 e^{-ipr*} \quad r \to r_+ \tag{33}$$

The energy flux endowed by this ingoing wave is:

$$(\frac{dE}{dt})|_{R\to r_+} = \sum_{l,m}\int_{-\infty}^{+\infty} \alpha_{lm\omega} \frac{|\tilde{\mathcal{T}} Z^{\infty}(1+\mathcal{K}\frac{D^{in}}{B^{hole}})|^2}{4\pi\omega^2} d\omega \tag{34}$$

where $\alpha_{lm\omega}$ can be found in Eq. (4.17) of [21].

4 Stability Problem

4.1 Superradiance

The superradiance for different kinds of perturbation fields around Kerr-like spacetime has been studied in [31, 32, 33, 15]. The manifestation of superradiance includes reflectivity larger than unity for the scattering process [32, 33, 15] and exponentially increasing amplitudes in Quasi-Normal Modes [31], when the frequency of perturbation field drops below a critical value. The superradiance is expected to be quenched when the reflectivity of ECO surface is below unity [31].

We compute $\mathcal{R}_{BH}$ and $\mathcal{T}_{BH}$ by numerical integration of Sasaki-Nakamura equation and Richardson extrapolation to horizon. The reflection and transmission coefficients in frequency domain for different spin are shown in Fig. 1

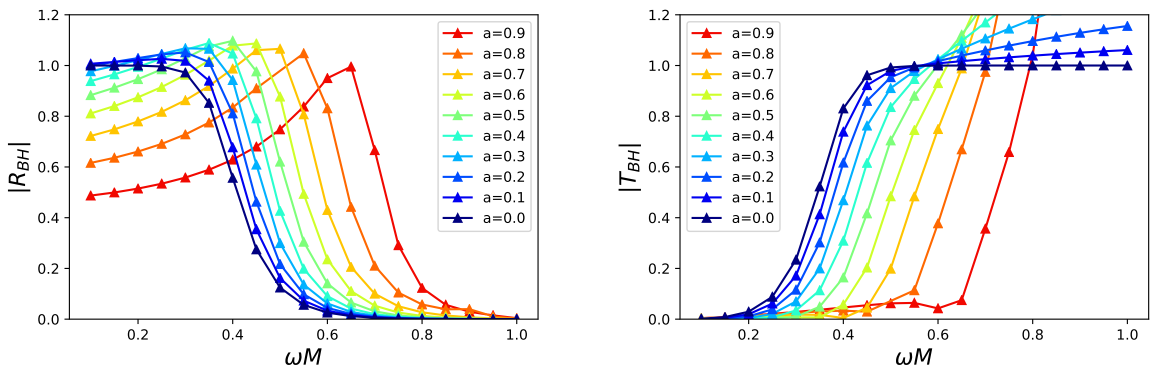

Figure 1: Black Hole reflectivity $\mathcal{R}_{BH}$ and transmissivity $\mathcal{T}_{BH}$ for l=m=2 modes.

Note that $\mathcal{R}_{BH}$ and $\mathcal{T}_{BH}$ do not represent the reflection and transmission of energy. Here, as a sanity check, we consider the energy relation as stated in Nakano *et. al*[15]. In their work, a relation that's quadratic in the asymptotic amplitudes shows:

$$\begin{aligned} |D^{\infty}_{lm\omega}|^2 =& \frac{\omega^3|D^{out}_{lm\omega}|^2}{p(2Mr_+)^3(p^2+4\epsilon_N^2)} - \frac{256\omega^3(2Mr_+)^5 p(p^2+4\epsilon_N^2)(k^2+16\epsilon_N^2)}{|C|^2}|D^{in}_{lm\omega}|^2 \\ |C|^2 =& ((\lambda+2)^2+4ma\omega-4a^2\omega^2)(\lambda^2+36am\omega-36a^2\omega^2) \\ &+48(2\lambda+3)(2a^2\omega^2-am\omega)+144\omega^2(M^2-a^2) \end{aligned} \tag{35}$$

where $\epsilon_N = \frac{\sqrt{M^2-a^2}}{4Mr_+}$. The notation is different from the original paper[15]. To clarify, the symbols in Nakano's paper, $Q, k, Z_{out}, Z_{up}, Z_{down}, \epsilon$, corresponds to our $\lambda+2, p, D^{\infty}_{lm\omega}, D^{in}_{lm\omega}, D^{out}_{lm\omega}, \epsilon_N$, respectively.

Since energy is quadratic in amplitude, Eq.(35) should represent energy balance relation. Therefore, we find energy reflectivity $\mathcal{R}_E$ and transmissivity $\mathcal{T}_E$ as (Eq. (11) of [15]):

$$\mathcal{R}_E^2 = \frac{256p^2(2Mr_+)^8(p^2+4\epsilon_N^2)^2(k^2+16\epsilon_N^2)}{|C|^2}\frac{|D^{in}_{lm\omega}|^2}{|D^{out}_{lm\omega}|^2} \tag{36}$$

$$\mathcal{T}_E^2 = \frac{p(2Mr_+)^3(p^2+4\epsilon_N^2)}{\omega^3}\frac{|D^{\infty}_{lm\omega}|^2}{|D^{out}_{lm\omega}|^2} \tag{37}$$

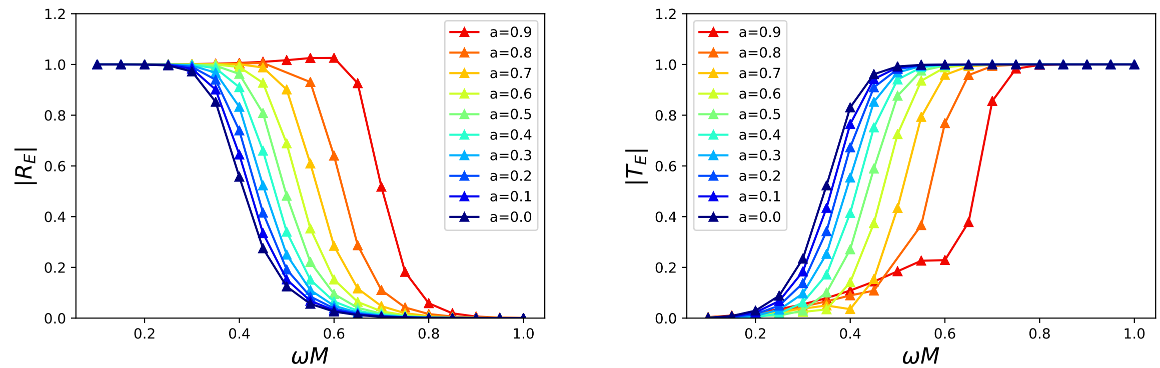

Figure 2: Energy reflectivity $\mathcal{R}_E$ and transmissivity $\mathcal{T}_E$ for l=m=2 modes.

As shown in Fig.2, the energy reflectivity and transmissivity are well-behaved, i.e. transmissivity approaches 1 as frequency increases, reflectivity goes above unity below superradiance frequency and approaches 1 as frequency approaches 0.

4.2 Hoop Conjecture

The incoming energy flux is given by Eq.(**??**).

The spacetime with an expanding horizon is given by Carmeli metric [28].

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5 GW from ECO

5.1 Individual Echoes from Ringdown Signal

The additional reflection contribution $\mathcal{K}$ in Eq.(20) can be understood as a sum of a series of echoes.

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5.2 Interference between Continuous Signal and Subsequent Echoes

For continuous signals like EMRI, the reflected GW will interfere with the main wave. Depending on the phase, the additional amplitude in Eq.(25) can cause enhancement or cancellation of the GW signal in Eq.(26).

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A Derivation of Eq. (15)(16)

The transformation between R functions and X functions is:

$$\begin{aligned} R &= \frac{1}{\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta X}{\sqrt{r^2+a^2}} - \frac{\beta}{\Delta}\frac{d}{dr}\frac{\Delta X}{\sqrt{r^2+a^2}}\right]. \\ &= \frac{1}{\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta X}{\sqrt{r^2+a^2}} - \frac{\beta}{\sqrt{r^2+a^2}}\frac{d}{dr}X - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}X\right] \end{aligned} \tag{38}$$

A.1 relation at infinity

As $r \to \infty$:

$$X^H \to A^{out}e^{i\omega r*} + A^{in}e^{-i\omega r*} \tag{39}$$

$$\frac{dX^H}{dr} = \frac{dr*}{dr}\frac{dX^H}{dr*} \to (\frac{r^2+a^2}{\Delta})(i\omega A^{out}e^{i\omega r*} - i\omega A^{in}e^{-i\omega r*}) \tag{40}$$

Using the relation between R and X:

$$\begin{aligned} R^H \to & \frac{1}{\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} - \frac{\beta\sqrt{r^2+a^2}}{\Delta}i\omega - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]A^{out}e^{i\omega r*} \\ &+ \frac{1}{\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} + \frac{\beta\sqrt{r^2+a^2}}{\Delta}i\omega - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]A^{in}e^{-i\omega r*} \end{aligned} \tag{41}$$

Comparing with equation (3), $R^H \to B^{out}r^3e^{i\omega r*} + r^{-1}B^{in}e^{-i\omega r*}$, we have the relation:

$$B^{out} = \lim_{r\to\infty}\{\frac{1}{r^3\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} - \frac{\beta\sqrt{r^2+a^2}}{\Delta}i\omega - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]\}A^{out} \tag{42}$$

$$B^{in} = \lim_{r\to\infty}\{\frac{r}{\eta}\left[\left(\alpha+\frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} + \frac{\beta\sqrt{r^2+a^2}}{\Delta}i\omega - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]\}A^{in} \tag{43}$$

The first limit is easy to evaluate once we notice that, as $r \to \infty$, $\eta \to c_0 + O(1/r)$, $\Delta \to r^2 + O(r)$, $\beta \to -2i\omega r^4 + O(r^3)$, $\alpha \to -2\omega^2 r^2 + O(r)$, where c_0 can be found in [21]. Replacing them into the first equation, we get $B^{out} = -\frac{4\omega^2}{c_0} A^{out}$.

The second is a little tricky since some nice cancellation should happen so that only $O(1/r)$ term remains in the bracket.

A.2 relation at horizon

As $r \to r_+$:

$$X^\infty \to C^{out} e^{ipr*} + C^{in} e^{-ipr*} \tag{44}$$

$$\frac{dX^H}{dr} = \frac{dr*}{dr}\frac{dX^H}{dr*} \to (\frac{r^2+a^2}{\Delta})(ipC^{out}e^{ipr*} - ipC^{in}e^{-ipr*}) \tag{45}$$

Using the relation between R and X:

$$\begin{aligned} R^\infty \to & \frac{1}{\eta}\left[\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} - \frac{\beta\sqrt{r^2+a^2}}{\Delta}ip - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right] C^{out}e^{ipr*} \\ & + \frac{1}{\eta}\left[\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} + \frac{\beta\sqrt{r^2+a^2}}{\Delta}ip - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right] C^{in}e^{-ipr*} \end{aligned} \tag{46}$$

Comparing with equation (4), $R^\infty \to D^{out}e^{ipr*} + \Delta^2 D^{in} e^{-ipr*}$, we have the relation:

$$D^{out} = \lim_{r\to r_+}\{\frac{1}{\eta}\left[\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} - \frac{\beta\sqrt{r^2+a^2}}{\Delta}ip - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]\}C^{out} \tag{47}$$

$$D^{in} = \lim_{r\to r_+}\{\frac{1}{\Delta^2\eta}\left[\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\frac{\Delta}{\sqrt{r^2+a^2}} + \frac{\beta\sqrt{r^2+a^2}}{\Delta}ip - \frac{\beta}{\Delta}\frac{r^3+a^2r-2Ma^2}{(r^2+a^2)^{3/2}}\right]\}C^{in} \tag{48}$$

When $r \to r_+$, $\Delta \to 0$. Evaluation of the first limit is direct, by noting that $\beta \to 2(-iK + r - M)\Delta + O(\Delta^2)$, $\alpha \to -2iK(-iK + r - M)/\Delta + O(1)$. The result is:

$$D^{out} = -\frac{4p\sqrt{2Mr_+}(2Mr_+p + i\sqrt{M^2-a^2})}{\eta(r_+)}C^{out} \tag{49}$$

Again the second limit needs some nice cancellation so that only $O(\Delta^2)$ terms survives in the bracket.

B Derivation of Eq. (??) (??)