

# **SURF Progress Report #1: Constructing Echo Waveform from Kerr-like Background**

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## **Summary of Progresses:**

- Reflectivity and Transmissivity of Teukolsky Potential are scrutinized. Energy conservation is satisfied. We see superradiance in the spectrum of reflectivity. The behavior that reflectivity goes to unity at low frequency and transmissivity goes to unity at high frequency is recovered. See Sec.III A.
- We derived the formulation for energy flux carried by Gravitational Wave(GW) emission with echoes.
- We derived the formulation for GW spectrum with echoes.
- Numerical calculation based on our formulation is on the way.  $|h| \propto \omega^{-7/6}$  trend in the inspiral part has been recovered.

## **Plan for future work:**

- Finish the code. Check the result of Schwarzschild case.
- Study Echo spectrum of ringdown signal in various cases
- Consider energy flux enhancement by interference of reflected signal in EMRI system, which will be important in future LISA analysis.
- Study the stability of Kerr-like ECO based on hoop conjecture and energy flux from ringdown signal

## I. Introduction

Gravitational wave (GW) echoes, the GW reflected from certain structures outside of horizon, could help us probe Planck scale structure near horizon and therefore is of significant importance in GW physics [1]. Abedi *et al.* claimed to have found evidence of echo signal from LIGO data in 2017[2], although the significance turned out to be low[3]. Still, echo signal is a promising candidate for probing physics beyond General Relativity (GR) and many groups have been trying to search echo signals from LIGO data based on reliable statistical methods [4–7].

Previous works about GW echoes are mostly for non-rotating Black Holes(BHs) described by Schwarzschild metric. For instance, Mark *et al.* studied echo modes in some Exotic Compact Object(ECO) models by solving Teukolsky equation in Schwarzschild limit with reflecting boundary[8]. Du *et al.* studied energy spectrum of such echoes and showed that they contribute to the stochastic GW background[9]. One notable point is the instability of central ECO under the energy flux of incoming GW[10] and we still don't know if this instability still exists for rotating ECO.

In general situation, astrophysical BHs have both mass and angular momentum and are described by Kerr metric. Estimating spin of BHs from LIGO/VIRGO events can also help understand the formation history of them and their stellar environments[11]. As for echo, some works have been devoted to searching echo signals from spinning ECOs based on phenomenological model[12, 13]. In order to understand spin effects from echo signal, a well-developed theoretical model for generating echo templates is needed. Some works in this direction are [12, 14, 15]. Nakano *et al.* have constructed a model for echoes from spinning ECOs[15], where the asymptotic behavior of solutions to Teukolsky equation is used to analyze reflectivity and echo modes, but the reflecting surface is assumed to be located exactly at the horizon and the incident wave is phenomenological in [15]. We will try to extend [8] to spinning case and develop a more realistic echo model.

## II. Formulation for Echo Construction

Teukolsky equation for spin-2 field has been a useful tool for computing GW waveforms, on which our approach is based. We will explain the method to compute Teukolsky-based GW waveform in Sec 3.1. Effective-One-Body(EOB) formalism[16–18] has been useful tool to study binary inspirals. Wenbiao and Zhoujian have applied EOB formalism and Teukolsky-based waveform to compute GW emission from Extreme-Mass-Ratio Inspiral(EMRI) and Intermediate-Mass-Ratio Inspiral (IMRI) systems[19, 20]. We briefly explain their code in Sec. ?? and in Sec. 3.3 we show how to modify the current method to incorporate echo modes and study the stability. Throughout this article, we will keep the notation consistent with [21].

### A. Teukolsky Equation

Astrophysical Black Holes with mass  $M$  and angular momentum  $J$  are described by Kerr metric. Perturbations of spin-0, spin-1 (e.g. electromagnetic fields) and spin-2 (e.g. Gravitational waves) fields in Kerr spacetime are governed by Teukolsky equations[22], which is a set of ordinary differential equations, separated in Boyer-Lindquist coordinates. For GW, the perturbation field  $\psi_4$ , decomposed in frequency domain  $\psi_4 = \rho^4 \int_{-\infty}^{+\infty} d\omega \sum_{lm} R_{lm\omega}(r) {}_{-2}S_{lm}^{a\omega}(\theta) e^{im\phi} e^{-i\omega t}$  with  ${}_{-2}S_{lm}^{a\omega}$  being spin-weighted spheroidal harmonics, obeys:

$$\Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR_{lm\omega}}{dr} \right) - V(r)R_{lm\omega} = -\mathcal{T}_{lm\omega}(r), \quad (1)$$

where  $\mathcal{T}_{lm\omega}(r)$  is the source term, which will be discussed in Sec. **II D**, and the potential is

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \quad (2)$$

where  $K = (r^2 + a^2)\omega - ma$ ,  $\lambda = E_{lm} + a^2\omega^2 - 2am\omega - 2$  and  $\Delta = r^2 - 2Mr + a^2$ .

First, we consider the homogeneous Teukolsky equation where the source term is zero. We can solve it by analytical expansion, as discussed in [23, 24] and here we don't go into technical details of it. The homogeneous Teukolsky equation allows two independent solutions  $R_{lm\omega}^H$ , which is purely ingoing at the horizon, and  $R_{lm\omega}^\infty$ , which is purely outgoing at infinity:

$$\begin{aligned} R_{lm\omega}^H &= B_{lm\omega}^{hole} \Delta^2 e^{-ipr^*}, \quad r \rightarrow r_+ \\ R_{lm\omega}^H &= B_{lm\omega}^{out} r^3 e^{i\omega r^*} + r^{-1} B_{lm\omega}^{in} e^{-i\omega r^*}, \quad r \rightarrow \infty; \end{aligned} \quad (3)$$

$$\begin{aligned} R_{lm\omega}^\infty &= D_{lm\omega}^{out} e^{ipr^*} + \Delta^2 D_{lm\omega}^{in} e^{-ipr^*}, \quad r \rightarrow r_+ \\ R_{lm\omega}^\infty &= r^3 D_{lm\omega}^\infty e^{i\omega r^*}, \quad r \rightarrow \infty, \end{aligned} \quad (4)$$

where  $p = \omega - \frac{ma}{2Mr_+}$ ,  $r_+ = M + \sqrt{M^2 - a^2}$  and  $r^*$  is the tortoise coordinate related to  $r$  by  $dr^*/dr = (r^2 + a^2)/\Delta$

Then, using the homogeneous solutions and proper boundary conditions, we can construct the solution to radial Teukolsky equation with source term. By imposing BH boundary condition, i.e. wave being purely outgoing at infinity and purely ingoing at horizon, the radial function is:

$$R_{lm\omega}^{BH}(r) = \frac{R_{lm\omega}^\infty(r)}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_{r_+}^r dr' \frac{R_{lm\omega}^H(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2} + \frac{R_{lm\omega}^H(r)}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_r^\infty dr' \frac{R_{lm\omega}^\infty(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2} \quad (5)$$

The asymptotic behavior of this solution near horizon and infinity is:

$$R_{lm\omega}^{BH}(r \rightarrow \infty) = Z_{lm\omega}^H r^3 e^{i\omega r^*}, \quad (6)$$

$$R_{lm\omega}^{BH}(r \rightarrow r_+) = Z_{lm\omega}^\infty \Delta^2 e^{-ipr^*}. \quad (7)$$

By taking the limit at  $r \rightarrow \infty$  and  $r \rightarrow r_+$  of the solution (Eq. 5), with the asymptotic behavior of homogeneous solutions (Eq. 3, 4), one can find the amplitudes  $Z_{lm\omega}^{H,\infty}$ :

$$Z_{lm\omega}^H = \frac{1}{2i\omega B_{lm\omega}^{in}} \int_{r_+}^\infty dr' \frac{R_{lm\omega}^H(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2} \quad (8)$$

$$Z_{lm\omega}^\infty = \frac{B_{lm\omega}^{hole}}{2i\omega B_{lm\omega}^{in} D_{lm\omega}^\infty} \int_{r_+}^\infty dr' \frac{R_{lm\omega}^\infty(r') \mathcal{T}_{lm\omega}(r')}{\Delta(r')^2} \quad (9)$$

Finally, from the relation  $\psi_4(r \rightarrow \infty) \rightarrow \frac{1}{2}(\ddot{h}_+ - i\dot{h}_\times)$ , decompose  $\psi_4$  in frequency domain as we previously introduced and plug in the behavior of radial function at infinity (Eq. 6). The gravitational waveform, observed from distance  $R$ , latitude angle  $\Theta$  and azimuthal angle  $\Phi$ , is given by:

$$h_+^{BH}(R, \Theta, \Phi, t) - ih_\times^{BH}(R, \Theta, \Phi, t) = \frac{2}{R} \sum_{lm} \int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} Z_{lm\omega}^H {}_{-2}S_{lm}^{\omega}(\Theta) e^{i(m\Phi - \omega[t - r^*])}. \quad (10)$$

## B. Sasaki-Nakamura Equation

The Teukolsky potential is long-ranged, making it hard to numerically extract certain parameters in homogeneous solution, e.g.  $B^{in}$ , which is overwhelmed by  $B^{out}$  at infinity. Sasaki *et al.* transformed the radial equation so that the potential is short-ranged and the equation become numerical computable[25]. Moreover, asymptotic behavior of solutions to Sasaki-Nakamura equations is purely sinuous, making it easier to generalize to echo construction.

Transformation between Teukolsky function  $R_{lm\omega}$  and Sasaki-Nakamura function  $X_{lm\omega}$  is given by[21]:

$$R_{lm\omega} = \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta, r}{\Delta} \right) \frac{\Delta X_{lm\omega}}{\sqrt{r^2 + a^2}} - \frac{\beta}{\Delta} \frac{d}{dr} \frac{\Delta X_{lm\omega}}{\sqrt{r^2 + a^2}} \right]. \quad (11)$$

Taking Eq. (11) into Teukolsky equation, one can find the equation for Sasaki-Nakamura function  $X_{lm\omega}$ :

$$\frac{d^2 X_{lm\omega}}{dr^{*2}} - F(r) \frac{dX_{lm\omega}}{dr^*} - U(r) X_{lm\omega} = 0. \quad (12)$$

The functions  $\alpha, \beta, \eta$  and the potentials  $F(r), U(r)$  can be found in [21]. The reverse transformation can be found in [26].

The Sasaki-Nakamura equation admits two homogeneous solution having the purely sinuous asymptotic behavior due to the short-rangeness of potential  $U(r)$ :

$$\begin{aligned} X_{lm\omega}^H &= A_{lm\omega}^{hole} e^{-ipr^*}, \quad r \rightarrow r_+, \\ X_{lm\omega}^H &= A_{lm\omega}^{out} e^{i\omega r^*} + A_{lm\omega}^{in} e^{-i\omega r^*}, \quad r \rightarrow \infty; \end{aligned} \quad (13)$$

and

$$\begin{aligned} X_{lm\omega}^\infty &= C_{lm\omega}^{out} e^{ipr^*} + C_{lm\omega}^{in} e^{-ipr^*}, \quad r \rightarrow r_+, \\ X_{lm\omega}^\infty &= C_{lm\omega}^\infty e^{i\omega r^*}, \quad r \rightarrow \infty, \end{aligned} \quad (14)$$

Taking the asymptotic behavior into the transformation (Eq. 11) and comparing with asymptotic behavior of Teukolsky function (Eq. 3,4). One finds the relation between asymptotic amplitudes of Sasaki-Nakamura functions and Teukolsky functions [26]:

$$\begin{aligned} B_{lm\omega}^{in} &= -\frac{1}{4\omega^2} A_{lm\omega}^{in} \\ B_{lm\omega}^{out} &= -\frac{4\omega^2}{c_0} A_{lm\omega}^{out} \\ B_{lm\omega}^{hole} &= -\frac{1}{d_{lm\omega}} A_{lm\omega}^{hole} \end{aligned} \quad (15)$$

$$\begin{aligned} D_{lm\omega}^{in} &= -\frac{1}{d_{lm\omega}} C_{lm\omega}^{in} \\ D_{lm\omega}^{out} &= -\frac{4p\sqrt{2Mr_+}(2Mr_+p + i\sqrt{M^2 - a^2})}{\eta(r_+)} C_{lm\omega}^{out} \\ D_{lm\omega}^\infty &= -\frac{4\omega^2}{c_0} C_{lm\omega}^\infty \end{aligned} \quad (16)$$

where the coefficients  $c_0$  and  $d_{lm\omega}$  can be found in [21]

The homogeneous solution  $X^\infty$  can be regarded as a incident wave coming out near the horizon with amplitude  $C_{lm\omega}^{out}$  scattered off the Sasaki-Nakamura potential. We can thus define the reflection and transmission factors  $\mathcal{R}_{BH}$  and  $\mathcal{T}_{BH}$ :

$$\mathcal{T}_{BH} = \frac{C^\infty}{C^{out}}, \mathcal{R}_{BH} = \frac{C^{in}}{C^{out}} \quad (17)$$

### C. Constructing echo modes

In this section, we construct echo waveforms analytically and the only phenomenological quantity is the reflectivity  $\tilde{\mathcal{R}}$  near horizon. First, the method established in [8] will be extended to Teukolsky equations with non-vanishing BH spin. By interpreting a Planck scale potential barrier near horizon to a frequency dependent reflectivity and matching the solution of Teukolsky equation to the reflecting boundary condition, we'll find a procedure to generate echo modes for spinning ECO.

Since the Sasaki-Nakamura equation has short ranged potential and therefore purely sinuous waves near horizon and at infinity, it's easier to impose reflecting boundary for Sasaki-Nakamura equation. On the other hand, we have the method to semi-analytically solve Teukolsky equation and extract waveform from radial Teukolsky function in hand. Therefore, we impose the reflecting boundary condition in Sasaki-Nakamura equation and after getting the relevant coefficient describing additional term induced by reflecting boundary, we transform the Sasaki-Nakamura function into Teukolsky function. Then we can utilize the results of BH waveform based on Teukolsky equation to establish echo waveform.

We consider the combination of  $X^H$  and  $X^\infty$

$$X_{lm\omega}^{ref} = \mathcal{K}X_{lm\omega}^\infty + X_{lm\omega}^H \quad (18)$$

which satisfies the reflecting boundary near horizon  $r_0 = r_+ + \epsilon$  with reflectivity  $\tilde{\mathcal{R}}$

$$X^{ref} \propto e^{-ip(r^*-r_0^*)} + \tilde{\mathcal{R}}e^{ip(r^*-r_0^*)} \quad r^* \rightarrow r_0^* \quad (19)$$

Taking the asymptotic behavior of  $X_{lm\omega}^{H,\infty}$  (Eq. 13, 14) into the above boundary condition, one can find  $\mathcal{K}$ :

$$\mathcal{K} = \frac{A^{hole}}{C^\infty} \frac{\tilde{\mathcal{R}}e^{-2ipr_0^*}\mathcal{T}_{BH}}{1 - \tilde{\mathcal{R}}e^{-2ipr_0^*}\mathcal{R}_{BH}} \quad \text{with} \quad \mathcal{T}_{BH} = \frac{C^\infty}{C^{out}}, \mathcal{R}_{BH} = \frac{C^{in}}{C^{out}} \quad (20)$$

The corresponding solution to homogeneous Teukolsky equation by relation (11) is  $R^{ref} = \mathcal{K}R^\infty + R^H$ . Therefore, with  $R^\infty$  satisfying ECO boundary condition at infinity and  $R^{ref}$  satisfying ECO boundary at surface, the Green function of Teukolsky with ECO boundary is:

$$g^{ECO}(r, r') = \frac{R^\infty(\max\{r, r'\})R^{ref}(\min\{r, r'\})}{2i\omega B^{in} D^\infty} = g^{BH} + \mathcal{K} \frac{R^\infty(r)R^\infty(r')}{2i\omega B^{in} D^\infty} \quad (21)$$

Then the ECO solution will be the BH solution  $R^{BH}$  in Eq. 5 with an additional echo contribution:

$$R^{ECO} = R^{BH} + \mathcal{K} \frac{R^\infty(r)}{2i\omega B^{in} D^\infty} \int_{-\infty}^{+\infty} \frac{R^\infty(r')\mathcal{T}(r')}{\Delta(r')} dr' \quad (22)$$

Near ECO surface the solution approaches:

$$R^{ECO} \rightarrow \frac{Z^\infty}{B^{hole\epsilon}} (R^H(r) + \mathcal{K}R^\infty(r)) \quad r \rightarrow r_0 \quad (23)$$

The corresponding Sasaki-Nakamura function satisfies the reflecting boundary:

$$X^{ECO} \propto X^H + \mathcal{K}X^\infty \propto e^{-ip(r^*-r_0^*)} + \tilde{\mathcal{R}}e^{ip(r^*-r_0^*)} \quad r^* \rightarrow r_0^* \quad (24)$$

At infinity, the solution is purely outgoing:

$$R^{ECO}(r \rightarrow \infty) = (Z^H + \mathcal{K} \frac{D^\infty}{B^{hole}} Z^\infty) r^3 e^{i\omega r^*} \quad (25)$$

which gives the GW waveform with echoes

$$h_+^{ECO}(R, \Theta, \Phi, t) - ih_\times^{ECO}(R, \Theta, \Phi, t) = \frac{2}{R} \sum_{lm} \int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} (Z_{lm\omega}^H + \mathcal{K}_{lm\omega} \frac{D_{lm\omega}^\infty}{B_{lm\omega}^{hole}} Z_{lm\omega}^\infty) {}_{-2}S_{lm}^{a\omega}(\Theta) e^{i(m\Phi - \omega[t - r^*])}. \quad (26)$$

Then, we will incorporate this method into our existing code to numerically generate echo waveforms. We expect that the main structure of current code will remain, with some additional subroutines computing the additional term in Eq. 22, the echo contribution to  $Z_{lm\omega}^H$  and thus the full waveform seen by a distant observer.

Note that the homogeneous solutions are determined up to two constants, i.e. we can transform  $X^H \rightarrow PX^H$ ,  $X^\infty \rightarrow QX^\infty$  (and consequently  $R^H \rightarrow PR^H$ ,  $R^\infty \rightarrow QR^\infty$ ) with  $P, Q$  being two arbitrary complex number, the final solution with source term (Eq. 5,22) and the waveform (Eq. 10,26) does not change. So we have the freedom to choose  $A^{hole} = 1, C^\infty = 1$

#### D. Source Term

Following the formalism of [21], the source term in Teukolsky equation induced by a particle with trajectory  $x^\mu(\tau) = (t(\tau), r(\tau), \theta(\tau), \phi(\tau))$  is given by [27]:

$$\begin{aligned} \mathcal{T}_{lm\omega}(r') = & \int_{-\infty}^{\infty} dt e^{i[\omega t - m\phi(t)]} \Delta(r')^2 \{ [A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] \delta(r' - r(t)) \\ & \partial_{r'} ([A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] \delta(r' - r(t))) + \partial_{r'}^2 [A_{\bar{m}\bar{m}2} \delta(r' - r(t))] \} \end{aligned} \quad (27)$$

where  $A_{abk}$  ( $a, b = n, \bar{m}, k = 0, 1, 2$ ) are related to  $r(t)$  and  $\theta(t)$  and can be found in Eq. (4.40) of [21]. Although the original article is considering circular orbits, the derivation about source term (Eq.(4.26) to Eq.(4.41) of [21]) can be easily generalized to generic trajectory by replacing  $r_0$  with  $r(t)$ . Then the amplitudes  $Z_{lm\omega}^{H,\infty}$  are given by:

$$\begin{aligned} Z_{lm\omega}^H = & \frac{1}{2i\omega B_{lm\omega}^{in}} \int_{-\infty}^{+\infty} dt e^{i[\omega t - m\phi]} \{ R_{lm\omega}^H(r(t)) [A_{nn0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] \\ & - \frac{dR_{lm\omega}^H}{dr} \Big|_{r(t)} [A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] + \frac{d^2 R_{lm\omega}^H}{dr^2} \Big|_{r(t)} A_{\bar{m}\bar{m}2} \} \end{aligned} \quad (28)$$

$$Z_{lm\omega}^\infty = \frac{B_{lm\omega}^{hole}}{2i\omega D_{lm\omega}^\infty B_{lm\omega}^{in}} \int_{-\infty}^{+\infty} dt e^{i[\omega t - m\phi]} \{ R_{lm\omega}^\infty(r(t)) [A_{nm0} + A_{n\bar{m}0} + A_{\bar{m}\bar{m}0}] - \frac{dR_{lm\omega}^\infty}{dr} \Big|_{r(t)} [A_{n\bar{m}1} + A_{\bar{m}\bar{m}1}] + \frac{d^2 R_{lm\omega}^\infty}{dr^2} \Big|_{r(t)} A_{\bar{m}\bar{m}2} \} \quad (29)$$

For ringdown signal, we consider the part of trajectory from particle entering into ISCO at  $t_{ISCO}$  to particle plunging into horizon at  $t_{plunge}$ . Therefore, the energy-momentum tensor, and thus the integrand of Eq. (28,29), is only non-zero between  $t_{ISCO} < t < t_{plunge}$ .

### E. Energy Flux

The energy flux at infinity can be derived from the energy carried by the gravitational wave. The energy endowed by “+” and “ $\times$ ” mode of Eq. (26) is:

$$\left( \frac{dE}{dAdt} \right) \Big|_{R \rightarrow \infty} = \frac{1}{16\pi} \langle (\partial_t h_+)^2 + (\partial_t h_\times)^2 \rangle \quad (30)$$

Take Eq. (26) into it and integrate over a sphere. The energy flux at infinity will be:

$$\left( \frac{dE}{dt} \right) \Big|_{R \rightarrow \infty} = \sum_{l,m} \int_{-\infty}^{+\infty} \frac{|Z_{lm\omega}^H + \mathcal{K}_{lm\omega} \frac{D_{lm\omega}^\infty}{B_{lm\omega}^{hole}} Z_{lm\omega}^\infty|^2}{4\pi\omega^2} d\omega \quad (31)$$

To estimate the energy flux near horizon we first look at the behavior of the radial function  $R_{lm\omega}$  near ECO surface and BH horizon. At ECO surface, from Eq. (23):

$$R^{ECO} \rightarrow \frac{Z^\infty}{B^{hole}} (R^H(r) + \mathcal{K} R^\infty(r)) = Z^\infty (1 + \mathcal{K} \frac{D^{in}}{B^{hole}}) \Delta^2 e^{-ipr^*} + Z^\infty \mathcal{K} \frac{D^{out}}{B^{hole}} e^{ipr^*} \quad r \rightarrow r_0 \quad (32)$$

Assuming the ECO surface has amplitude transmissivity  $\tilde{\mathcal{T}}$ , we have at horizon:

$$R^{ECO} \rightarrow \tilde{\mathcal{T}} Z^\infty (1 + \mathcal{K} \frac{D^{in}}{B^{hole}}) \Delta^2 e^{-ipr^*} \quad r \rightarrow r_+ \quad (33)$$

The energy flux endowed by this ingoing wave is:

$$\left( \frac{dE}{dt} \right) \Big|_{R \rightarrow r_+} = \sum_{l,m} \int_{-\infty}^{+\infty} \alpha_{lm\omega} \frac{|\tilde{\mathcal{T}} Z^\infty (1 + \mathcal{K} \frac{D^{in}}{B^{hole}})|^2}{4\pi\omega^2} d\omega \quad (34)$$

where  $\alpha_{lm\omega}$  can be found in Eq. (4.17) of [21].

## III. Stability Problem

### A. Superradiance

The superradiance for different kinds of perturbation fields around Kerr-like spacetime has been studied in [15, 28–30]. The manifestation of superradiance includes reflectivity larger than unity for the scattering process [15, 29, 30] and exponentially increasing amplitudes in Quasi-Normal Modes [28], when the frequency

of perturbation field drops below a critical value. The superradiance is expected to be quenched when the reflectivity of ECO surface is below unity [28].

We compute  $\mathcal{R}_{BH}$  and  $\mathcal{T}_{BH}$  by numerical integration of Sasaki-Nakamura equation and Richardson extrapolation to horizon. The reflection and transmission coefficients in frequency domain for different spin are shown in Fig. 1

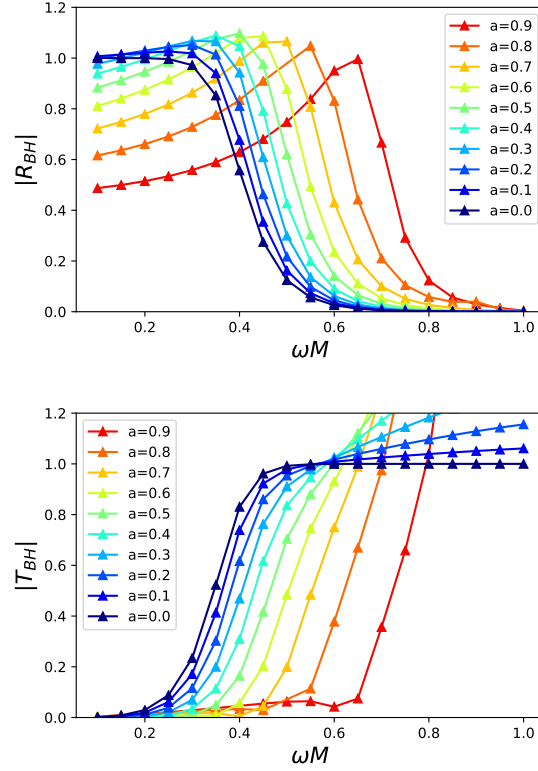


FIG. 1. Black Hole reflectivity  $\mathcal{R}_{BH}$  and transmissivity  $\mathcal{T}_{BH}$  for  $l=m=2$  modes.

Note that  $\mathcal{R}_{BH}$  and  $\mathcal{T}_{BH}$  do not represent the reflection and transmission of energy. Here, as a sanity check, we consider the energy relation as stated in Nakano *et. al*[15]. In their work, a relation that's quadratic in the asymptotic amplitudes shows:

$$|D_{lm\omega}^\infty|^2 = \frac{\omega^3 |D_{lm\omega}^{out}|^2}{p(2Mr_+)^3 (p^2 + 4\epsilon_N^2)} - \frac{256\omega^3 (2Mr_+)^5 p(p^2 + 4\epsilon_N^2)(k^2 + 16\epsilon_N^2)}{|C|^2} |D_{lm\omega}^{in}|^2 \quad (35)$$

$$|C|^2 = ((\lambda + 2)^2 + 4ma\omega - 4a^2\omega^2)(\lambda^2 + 36am\omega - 36a^2\omega^2) + 48(2\lambda + 3)(2a^2\omega^2 - am\omega) + 144\omega^2(M^2 - a^2)$$

where  $\epsilon_N = \frac{\sqrt{M^2 - a^2}}{4Mr_+}$ . The notation is different from the original paper[15]. To clarify, the symbols in Nakano's paper,  $Q$ ,  $k$ ,  $Z_{out}$ ,  $Z_{up}$ ,  $Z_{down}$ ,  $\epsilon$ , corresponds to our  $\lambda+2$ ,  $p$ ,  $D_{lm\omega}^\infty$ ,  $D_{lm\omega}^{in}$ ,  $D_{lm\omega}^{out}$ ,  $\epsilon_N$ , respectively.



Since energy is quadratic in amplitude, Eq.(35) should represent energy balance relation. Therefore, we find energy reflectivity  $\mathcal{R}_E$  and transmissivity  $\mathcal{T}_E$  as (Eq. (11) of [15]):

$$\mathcal{R}_E^2 = \frac{256p^2(2Mr_+)^8(p^2 + 4\epsilon_N^2)^2(k^2 + 16\epsilon_N^2)}{|C|^2} \frac{|D_{lm\omega}^{in}|^2}{|D_{lm\omega}^{out}|^2} \quad (36)$$

$$\mathcal{T}_E^2 = \frac{p(2Mr_+)^3(p^2 + 4\epsilon_N^2)}{\omega^3} \frac{|D_{lm\omega}^\infty|^2}{|D_{lm\omega}^{out}|^2} \quad (37)$$

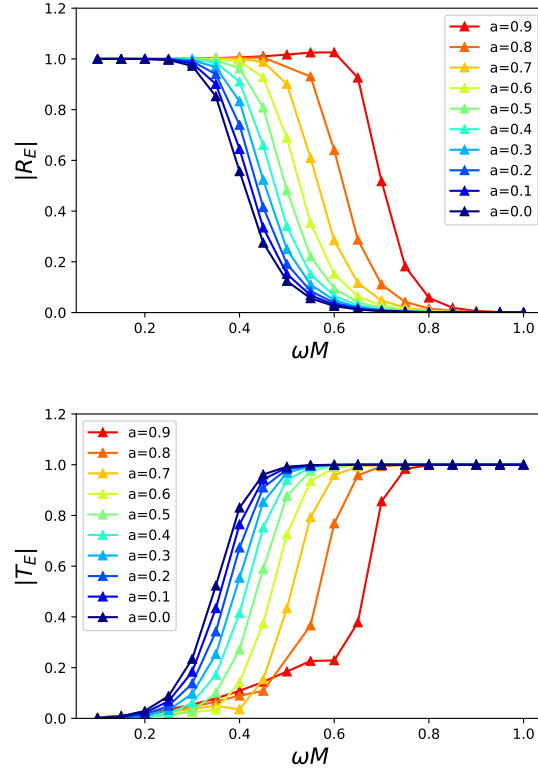


FIG. 2. Energy reflectivity  $\mathcal{R}_E$  and transmissivity  $\mathcal{T}_E$  for  $l=m=2$  modes.

As shown in Fig.2, the energy reflectivity and transmissivity are well-behaved, i.e. transmissivity approaches 1 as frequency increases, reflectivity goes above unity below superradiance frequency and approaches 1 as frequency approaches 0.

## B. Hoop Conjecture

The incoming energy flux is given by Eq.(??).

The spacetime with an expanding horizon is given by Carmeli metric [31].

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## IV. GW from ECO

### A. Individual Echoes from Ringdown Signal

The additional reflection contribution  $\mathcal{K}$  in Eq.(20) can be understood as a sum of a series of echoes.

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### B. Interference between Continuous Signal and Subsequent Echoes

For continuous signals like EMRI, the reflected GW will interfere with the main wave. Depending on the phase, the additional amplitude in Eq.(25) can cause enhancement or cancellation of the GW signal in Eq.(26).

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- [1] Vitor Cardoso and Paolo Pani. Tests for the existence of black holes through gravitational wave echoes. *Nature Astronomy*, 1(9):586, 2017.
  - [2] Jahed Abedi, Hannah Dykaar, and Niayesh Afshordi. Echoes from the abyss: Tentative evidence for planck-scale structure at black hole horizons. *Phys. Rev. D*, 96:082004, Oct 2017.
  - [3] Julian Westerweck, Alex B. Nielsen, Ofek Fischer-Birnholtz, Miriam Cabero, Collin Capano, Thomas Dent, Badri Krishnan, Grant Meadors, and Alexander H. Nitz. Low significance of evidence for black hole echoes in gravitational wave data. *Phys. Rev. D*, 97:124037, Jun 2018.
  - [4] Andrea Maselli, Sebastian H. Völkel, and Kostas D. Kokkotas. Parameter estimation of gravitational wave echoes from exotic compact objects. *Phys. Rev. D*, 96:064045, Sep 2017.
  - [5] Ka Wa Tsang, Michiel Rollier, Archisman Ghosh, Anuradha Samajdar, Michalis Agathos, Katerina Chatziioannou, Vitor Cardoso, Gaurav Khanna, and Chris Van Den Broeck. A morphology-independent data analysis method for detecting and characterizing gravitational wave echoes. *Phys. Rev. D*, 98:024023, Jul 2018.
  - [6] Alex B Nielsen, Collin D Capano, and Julian Westerweck. Parameter estimation for black hole echo signals and their statistical significance. *arXiv preprint arXiv:1811.04904*, 2018.
  - [7] R Ka Lok Lo, Tjonnie GF Li, and Alan J Weinstein. Template-based gravitational-wave echoes search using bayesian model selection. *arXiv preprint arXiv:1811.07431*, 2018.
  - [8] Zachary Mark, Aaron Zimmerman, Song Ming Du, and Yanbei Chen. A recipe for echoes from exotic compact objects. *Phys. Rev. D*, 96:084002, Oct 2017.
  - [9] Song Ming Du and Yanbei Chen. Searching for near-horizon quantum structures in the binary black-hole stochastic gravitational-wave background. *Phys. Rev. Lett.*, 121:051105, Aug 2018.
  - [10] Baoyi Chen, Yanbei Chen, Yiqiu Ma, Ka-Lok R Lo, and Ling Sun. Instability of exotic compact objects and its implications for gravitational-wave echoes. *arXiv preprint arXiv:1902.08180*, 2019.
  - [11] Ben Farr, Daniel E. Holz, and Will M. Farr. Using spin to understand the formation of LIGO and virgo’s black holes. *The Astrophysical Journal*, 854(1):L9, feb 2018.
  - [12] Randy S. Conklin, Bob Holdom, and Jing Ren. Gravitational wave echoes through new windows. *Phys. Rev. D*, 98:044021, Aug 2018.
  - [13] Qingwen Wang and Niayesh Afshordi. Black hole echology: The observer’s manual. *Phys. Rev. D*, 97:124044, Jun 2018.

- [14] C. P. Burgess, Ryan Plestid, and Markus Rummel. Effective field theory of black hole echoes. *Journal of High Energy Physics*, 2018(9):113, Sep 2018.
- [15] Hiroyuki Nakano, Takahiro Tanaka, Norichika Sago, and Hideyuki Tagoshi. Black hole ringdown echoes and howls. *Progress of Theoretical and Experimental Physics*, 2017(7), 07 2017.
- [16] A. Buonanno and T. Damour. Effective one-body approach to general relativistic two-body dynamics. *Phys. Rev. D*, 59:084006, Mar 1999.
- [17] Yi Pan, Alessandra Buonanno, Ryuichi Fujita, Etienne Racine, and Hideyuki Tagoshi. Post-newtonian factorized multipolar waveforms for spinning, nonprecessing black-hole binaries. *Phys. Rev. D*, 83:064003, Mar 2011.
- [18] Enrico Barausse and Alessandra Buonanno. Improved effective-one-body hamiltonian for spinning black-hole binaries. *Phys. Rev. D*, 81:084024, Apr 2010.
- [19] Wen-Biao Han and Zhoujian Cao. Constructing effective one-body dynamics with numerical energy flux for intermediate-mass-ratio inspirals. *Phys. Rev. D*, 84:044014, Aug 2011.
- [20] Wen-Biao Han. Gravitational waves from extreme-mass-ratio inspirals in equatorially eccentric orbits. *International Journal of Modern Physics D*, 23(07):1450064, 2014.
- [21] Scott A. Hughes. Evolution of circular, nonequatorial orbits of kerr black holes due to gravitational-wave emission. *Phys. Rev. D*, 61:084004, Mar 2000.
- [22] Saul A Teukolsky. Perturbations of a rotating black hole. 1. fundamental equations for gravitational electromagnetic and neutrino field perturbations. *Astrophys. J.*, 185:635–647, 1973.
- [23] Hideyuki Tagoshi, Shuhei Mano, and Eiichi Takasugi. Post-Newtonian Expansion of Gravitational Waves from a Particle in Circular Orbits around a Rotating Black Hole: Effects of Black Hole Absorption. *Progress of Theoretical Physics*, 98(4):829–850, 10 1997.
- [24] Ryuichi Fujita and Hideyuki Tagoshi. New Numerical Methods to Evaluate Homogeneous Solutions of the Teukolsky Equation. *Progress of Theoretical Physics*, 112(3):415–450, 09 2004.
- [25] Misao Sasaki and Takashi Nakamura. A class of new perturbation equations for the kerr geometry. *Physics Letters A*, 89(2):68–70, 1982.
- [26] Yasushi Mino, Misao Sasaki, Masaru Shibata, Hideyuki Tagoshi, and Takahiro Tanaka. Chapter 1. Black Hole Perturbation. *Progress of Theoretical Physics Supplement*, 128:1–121, 03 1997.
- [27] Scott A. Hughes. Erratum: Evolution of circular, nonequatorial orbits of kerr black holes due to gravitational-wave emission [phys. rev. d 61, 084004 (2000)]. *Phys. Rev. D*, 65:069902, Mar 2002.
- [28] Elisa Maggio, Vitor Cardoso, Sam R. Dolan, and Paolo Pani. Ergoregion instability of exotic compact objects: Electromagnetic and gravitational perturbations and the role of absorption. *Phys. Rev. D*, 99:064007, Mar 2019.
- [29] Rodrigo Vicente, Vitor Cardoso, and Jorge C. Lopes. Penrose process, superradiance, and ergoregion instabilities. *Phys. Rev. D*, 97:084032, Apr 2018.
- [30] Mauricio Richartz, Silke Weinfurtner, A. J. Penner, and W. G. Unruh. Generalized superradiant scattering. *Phys. Rev. D*, 80:124016, Dec 2009.
- [31] M Carmeli and M Kaye. Gravitational field of a radiating rotating body. *Annals of Physics*, 103(1):97 – 120, 1977.

### A. Derivation of Eq. (15)(16)

The transformation between  $R$  functions and  $X$  functions is:

$$\begin{aligned}
 R &= \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta X}{\sqrt{r^2 + a^2}} - \frac{\beta}{\Delta} \frac{d}{dr} \frac{\Delta X}{\sqrt{r^2 + a^2}} \right] \\
 &= \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta X}{\sqrt{r^2 + a^2}} - \frac{\beta}{\sqrt{r^2 + a^2}} \frac{d}{dr} X - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} X \right]
 \end{aligned} \tag{A1}$$

#### 1. relation at infinity

As  $r \rightarrow \infty$ :

$$X^H \rightarrow A^{out} e^{i\omega r^*} + A^{in} e^{-i\omega r^*} \quad (\text{A2})$$

$$\frac{dX^H}{dr} = \frac{dr^*}{dr} \frac{dX^H}{dr^*} \rightarrow \left(\frac{r^2 + a^2}{\Delta}\right) (i\omega A^{out} e^{i\omega r^*} - i\omega A^{in} e^{-i\omega r^*}) \quad (\text{A3})$$

Using the relation between  $R$  and  $X$ :

$$\begin{aligned} R^H \rightarrow & \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} - \frac{\beta \sqrt{r^2 + a^2}}{\Delta} i\omega - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] A^{out} e^{i\omega r^*} \\ & + \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} + \frac{\beta \sqrt{r^2 + a^2}}{\Delta} i\omega - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] A^{in} e^{-i\omega r^*} \end{aligned} \quad (\text{A4})$$

Comparing with equation (3),  $R^H \rightarrow B^{out} r^3 e^{i\omega r^*} + r^{-1} B^{in} e^{-i\omega r^*}$ , we have the relation:

$$B^{out} = \lim_{r \rightarrow \infty} \left\{ \frac{1}{r^3 \eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} - \frac{\beta \sqrt{r^2 + a^2}}{\Delta} i\omega - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] \right\} A^{out} \quad (\text{A5})$$

$$B^{in} = \lim_{r \rightarrow \infty} \left\{ \frac{r}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} + \frac{\beta \sqrt{r^2 + a^2}}{\Delta} i\omega - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] \right\} A^{in} \quad (\text{A6})$$

The first limit is easy to evaluate once we notice that, as  $r \rightarrow \infty$ ,  $\eta \rightarrow c_0 + O(1/r)$ ,  $\Delta \rightarrow r^2 + O(r)$ ,  $\beta \rightarrow -2i\omega r^4 + O(r^3)$ ,  $\alpha \rightarrow -2\omega^2 r^2 + O(r)$ , where  $c_0$  can be found in [21]. Replacing them into the first equation, we get  $B^{out} = -\frac{4\omega^2}{c_0} A^{out}$ .

The second is a little tricky since some nice cancellation should happen so that only  $O(1/r)$  term remains in the bracket.

## 2. relation at horizon

As  $r \rightarrow r_+$ :

$$X^\infty \rightarrow C^{out} e^{ipr^*} + C^{in} e^{-ipr^*} \quad (\text{A7})$$

$$\frac{dX^H}{dr} = \frac{dr^*}{dr} \frac{dX^H}{dr^*} \rightarrow \left(\frac{r^2 + a^2}{\Delta}\right) (ipC^{out} e^{ipr^*} - ipC^{in} e^{-ipr^*}) \quad (\text{A8})$$

Using the relation between  $R$  and  $X$ :

$$\begin{aligned} R^\infty \rightarrow & \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} - \frac{\beta \sqrt{r^2 + a^2}}{\Delta} ip - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] C^{out} e^{ipr^*} \\ & + \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} + \frac{\beta \sqrt{r^2 + a^2}}{\Delta} ip - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] C^{in} e^{-ipr^*} \end{aligned} \quad (\text{A9})$$

Comparing with equation (4),  $R^\infty \rightarrow D^{out} e^{ipr^*} + \Delta^2 D^{in} e^{-ipr^*}$ , we have the relation:

$$D^{out} = \lim_{r \rightarrow r_+} \left\{ \frac{1}{\eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} - \frac{\beta \sqrt{r^2 + a^2}}{\Delta} ip - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] \right\} C^{out} \quad (\text{A10})$$

$$D^{in} = \lim_{r \rightarrow r_+} \left\{ \frac{1}{\Delta^2 \eta} \left[ \left( \alpha + \frac{\beta_{,r}}{\Delta} \right) \frac{\Delta}{\sqrt{r^2 + a^2}} + \frac{\beta \sqrt{r^2 + a^2}}{\Delta} ip - \frac{\beta}{\Delta} \frac{r^3 + a^2 r - 2Ma^2}{(r^2 + a^2)^{3/2}} \right] \right\} C^{in} \quad (\text{A11})$$

When  $r \rightarrow r_+$ ,  $\Delta \rightarrow 0$ . Evaluation of the first limit is direct, by noting that  $\beta \rightarrow 2(-iK + r - M)\Delta + O(\Delta^2)$ ,  $\alpha \rightarrow -2iK(-iK + r - M)/\Delta + O(1)$ . The result is:

$$D^{out} = -\frac{4p\sqrt{2Mr_+}(2Mr_+p + i\sqrt{M^2 - a^2})}{\eta(r_+)} C^{out} \quad (\text{A12})$$

Again the second limit needs some nice cancellation so that only  $O(\Delta^2)$  terms survives in the bracket.