SURF Final Report:
Constructing Echo Waveforms from Spinning Exotic Compact Objects

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Abstract: Quantum structures resulted from exotic matters near where the horizon would have been or modified gravity may set a reflecting boundary for gravitational waves (GWs) and result in echoes. The GW echoes, if exist, would allow testing theories of gravity in the near-horizon region. We construct echo waveforms based on Teukolsky-Sasaki-Nakamura formalism for spinning exotic compact objects (ECOs), by simulating ringdown signals with plunging particles and Gaussian sources. The delay time between individual echoes is calculated, taking into consideration the additional shifts due to the frequency dependent reflectivity from the Teukolsky potential. The impact of superradiance and the stability of spinning ECOs are discussed. The resulting echo waveforms can be used in the searches for echo signals from detected compact binary coalescences and other GW sources, including extreme-mass-ratio inspirals, accessible to future detectors.

I. INTRODUCTION

LIGO's detection of binary black hole(BH) merger event GW150914 [1] and binary neutron star merger GW170817 [2] has opened up the era of gravitational wave (GW) astronomy. Together with Virgo[3], the LIGO-Virgo Collaboration has confirmed over ten gravitational wave events[4]. With the new messenger to observe signals from strong gravity region, we are able to study theory of gravity in a way that was never accessible before. In the third observation run starting this year, the upgraded detectors will see further in space and probably observe more abundant details in the signal.

Black holes have been the paradigm for compact objects, but the spacetime near horizon could have a different structure if the compact object is formed by exotic matter or the spacetime is governed by a modified theory of gravity. The exotic matter may modify the metric near horizon, leading to a potential barrier which reflects GWs [5]. When the spacetime is described by wormholes, the GW can be reflected between Teukolsky potentials in the two worlds connected by the wormhole [6]. As results of quantum theories of gravity, quantum structures near horizon may also reflect GWs [7, 8].

Gravitational wave echoes, the GW reflected from certain boundaries outside of horizon, could help us probe Planck scale structure near horizon and therefore is of significant importance in GW physics [9].Although there's still no significant evidence for echoes in current data[10, 11], echo signal is a promising candidate for probing physics beyond General Relativity (GR) and many groups have been trying to search echo signals from LIGO data based on reliable statistical methods [12–15].

Previous works about GW echoes are mostly from non-rotating spacetime background described by Schwarzschild metric. For instance, Mark $et\,al.$ studied echo modes of scalar waves in some exotic compact pbject(ECO) models by solving scalar perturbation equations with reflecting boundary[16]. Du $et\,al.$ solved GW echo modes based on Sasaki-Nakamura formalism and studied its contribution to stochastic background[17]. Huang $et\,al.$ developed Fredholm method and a diagrammatic representation of echo solution for general wave equations[18].

In general situation, astrophysical objects have both mass and angular momentum and the spacetime around them is described by Kerr metric. Estimating spin of BHs from LIGO/VIRGO events can also help understand the formation history of them and their stellar environments[19]. As for echo, some works have been devoted to searching echo signals from spinning ECOs based on phenomenological model[5, 20]. In order to understand spin effects from echo signal, a well-developed theoretical model for generating echo templates is needed. Some works in this direction are [5, 21, 22]. Nakano $et\,al.$ have constructed a model for echoes from spinning ECOs[22], where the asymptotic behavior of solutions to Teukolsky equation is used to analyze reflectivity and echo modes, but the reflecting surface is assumed to be located exactly at the horizon and the incident wave is phenomenological in [22]. We will try to extend [16] to GW perturbation and Kerr cases based on Teukolsky formalism and develop a more realistic echo model.

Whether ECO can exist stably and generate echoes is still an open problem. Different instabilities could potentially kill echo signals or make the generation of echo unphysical. Energy flux by incoming GW could make the ECO collapse [23]. Spinning ECO without a horizon may also be killed by superradiance. The superradiance for different kinds of perturbation fields around Kerr-like spacetime has been studied in [22, 24–26]. The manifestation of superradiance is reflectivity larger than unity for the scattering process [22, 25, 26]. The superradiance is expected to be quenched when the reflectivity of ECO surface is small [24].

In this article, we present the methods for constructing echo waveforms from spinning ECOs based on Teukolsky equations. In Sec. II, we formulate the problem, start from BH background and derive the waveform for ECO solution by solving Teukolsky equations. Sec. III briefly introduces the results from this study. The full description of the echo waveforms from different sources, and the discussions about superradiance and the energy carried by echoes are given in Appendices B-C, which partly depends on some ongoing work and will be completed in our future paper. At last, we summarize and conclude in Sec. IV.

II. FORMULATION FOR ECHO CONSTRUCTION

We develop our formalism for gravitational wave echoes based on black hole perturbation theory. We briefly review how GW waveform is solved based on BH perturbation theory in Sec. II A. Sasaki-Nakamura method is used for solving Teukolsky equation, as described in Appendix A. The code implementation is already developed by Han and Cao [27][28], which will be incorporated in our study. We construct echo waveforms for spinning ECOs assuming that a reflecting boundary is located at r_0^* with reflectivity $\tilde{\mathcal{R}}(\omega)$ as shown in Sec. II B. Throughout this article, we will keep the notation consistent with [29].

A. GW from BH background

We compute the GW waveform based on BH perturbation theory. Consider the projection of Weyl tensor ψ_4, which has the asymptotic behavior at infinity $\psi_4(r\to\infty)\to\frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times)$. After the decomposition $\psi_4=(\frac{1}{r-ia\cos\theta})^4\int_{-\infty}^{+\infty}d\omega\sum_{lm}R_{lm\omega}(r)\ {}_{-2}S_{lm}^{a\omega}(\theta)e^{im\phi}e^{-i\omega t}$ where ${}_{-2}S_{lm}^{a\omega}$ is spin-weighted spheroidal harmonic, the radial function $R_{lm\omega}$ satisfies Teukolsky master equation, which is an ordinary differential equation.

When the central object is a BH, we impose the boundary conditions so that the wave is only ingoing at horizon and only out going at infinity. The solution has the following asymptotic behavior:

$$R_{lm\omega}^{\rm BH}(r\to\infty)=Z_{lm\omega}^{\rm H}r^3e^{i\omega r*}, \tag{1}$$

$$R_{lm\omega}^{\rm BH}(r\to r_+)=Z_{lm\omega}^{\infty}\Delta^2e^{-ipr*}. \tag{2}$$

where $Z_{lm\omega}^{\rm H,\infty}$ are amplitudes and their detailed expression can be found in Appendix. At infinity, ψ_4 is related to familiar polarizations of GW by $\psi_4(r\to\infty)\to\frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times)$. Therefore, the gravitational waveform, observed from distance R, latitude angle Θ and azimuthal angle Φ, is given by:

$$h_+^{\rm BH}(R,\Theta,\Phi,t)-ih_\times^{\rm BH}(R,\Theta,\Phi,t)=\frac{2}{R}\sum_{lm}\int_{-\infty}^{+\infty}d\omega\frac{1}{\omega^2}Z_{lm\omega}^{\rm H}\ {}_{-2}S_{lm}^{a\omega}(\Theta)e^{i(m\Phi-\omega[t-r*])}. \tag{3}$$

B. Constructing Echo Modes

The ECO surface, or the "quantum structure" near horizon is regarded as a reflecting boundary located at $r^*=r_0^*$ (or, equivalently, at $r=r_0=r_+(1+\epsilon)$ where ϵ is "compactness") with reflectivity $\tilde{\mathcal{R}}(\omega)$. Similar to $\mathcal{R}_{\rm BH},\mathcal{T}_{\rm BH}$ in Eq. (A17), we define the reflectivity of ECO boundary in terms of Sasaki-Nakamura function. Namely, we consider one solution of homogeneous Sasaki-Nakamura equation $X^{\rm ref}$ satisfying the reflecting boundary condition:

$$X^{\rm ref}\propto e^{-ip(r*-r_0*)}+\tilde{\mathcal{R}}e^{ip(r*-r_0*)}\quad r*\to r_0* \tag{4}$$

The corresponding Teukolsky function is $R^{\rm ref}$, defined by the transformation law (Eq. A11). Then we impose that the ECO solution to Teukolsky equation $R^{\rm ECO}$ be proportional to $R^{\rm ref}$ near r_0:

$$R^{\rm ECO}\propto R^{\rm ref}\quad r\to r_0 \tag{5}$$

Similar to the BH solution, given the boundary condition at horizon given above and the only outgoing condition at infinity, ECO solution can be obtained using Green function method. The counterpart of Eq. (A3) for $R^{\rm ECO}$ is:

$$R^{\rm ECO}_{lm\omega}(r) = \frac{R^{\infty}_{lm\omega}(r)}{2i\omega B^{\rm in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r_+}^{r} dr' \frac{R^{\rm ref}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} + \frac{R^{\rm ref}_{lm\omega}(r)}{2i\omega B^{\rm in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r}^{\infty} dr' \frac{R^{\infty}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{6}$$

To get the gravitational waveforms, we see the solution at infinity:

$$R^{\rm ECO}_{lm\omega}(r\to\infty) = Z^{\rm ref}_{lm\omega} r^3 e^{i\omega r*} \tag{7}$$

where the amplitude is given by:

$$Z^{\rm ref}_{lm\omega} = \frac{1}{2i\omega B^{\rm in}_{lm\omega}} \int_{r_+}^{\infty} dr' \frac{R^{\rm ref}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{8}$$

This gives the GW waveform from ECO:

$$h^{\rm ECO}_{+}(R,\Theta,\Phi,t) - ih^{\rm ECO}_{\times}(R,\Theta,\Phi,t) = \frac{2}{R}\sum_{lm}\int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} Z^{\rm ref}_{lm\omega}\, {}_{-2}S^{a\omega}_{lm}(\Theta) e^{i(m\Phi - \omega[t-r*])}. \tag{9}$$

Then we will express $Z^{\rm ref}$ by $Z^{H,\infty}$ of BH solution with some transfer function. First, we see $X^{\rm ref}_{lm\omega}$ is related to $X^{H,\infty}_{lm\omega}$ by:

$$X^{\rm ref}_{lm\omega} = \mathcal{K} X^{\infty}_{lm\omega} + X^{\rm H}_{lm\omega}, \tag{10}$$

$$\mathcal{K} = \frac{A^{\rm hole}}{C^{\infty}} \frac{\tilde{\mathcal{R}} e^{-2ipr_0*}\mathcal{T}_{\rm BH}}{1 - \tilde{\mathcal{R}} e^{-2ipr_0*}\mathcal{R}_{\rm BH}}, \tag{11}$$

which can be directly verified by taking Eq. (A13,A14) into equations above.

Due to linearity of Sasaki-Nakamura transformation, we also have $R^{\rm ref}_{lm\omega} = \mathcal{K} R^{\infty}_{lm\omega} + R^{\rm H}_{lm\omega}$. Taking into Eq. (8), we get the relation between $Z^{\rm ref}$ and $Z^{H,\infty}$:

$$Z^{\rm ref}_{lm\omega} = \frac{D^{\infty}_{lm\omega}}{B^{\rm hole}_{lm\omega}} \mathcal{K} Z^{\infty}_{lm\omega} + Z^{\rm H}_{lm\omega} \tag{12}$$

Note that the homogeneous solutions are determined up to two constants, i.e. we can transform $X^{\rm H} \to PX^{\rm H}$, $X^{\infty} \to QX^{\infty}$ (and consequently $R^{\rm H} \to PR^{\rm H}$, $R^{\infty} \to QR^{\infty}$) with P, Q being two arbitrary complex number, the final solution with source term (Eq. A3,6) and the waveform (Eq. 3,9) does not change. So we have the freedom to choose $A^{\rm hole} = 1, C^{\infty} = 1$

III. MAJOR RESULTS

We derive echo waveforms from plunging particles, Gaussian sources, and ingoing and outgoing wave packets. The impacts of superradiance and energy flux are discussed. Further studies are being carried out to understand the ingoing energy spectra near the horizon from multiple formalisms. The methods of integrating the resulting waveforms into echo searches are being investigated. Hence the full results and discussions are temporarily given in Appendices B and C, which will be added into the major sections in this paper when the ongoing work is completed.

IV. CONCLUSION

In this article we present construction of echo waveforms from spinning ECOs based on Teukolsky equation and Sasaki-Nakamura formalism. ECOs are regarded as reflecting boundaries near horizon and the sources of GW are plunging particle or Gaussian distributions. We show some examples of echo signals from different sources. The shapes of each echoes are compared and the frequency of late-time echo is lower, which is consistent with the findings in [30]. Also we point out that there are shifts of delay time between each echoes due to the frequency dependence of $\mathcal{R}_{\rm BH}, \mathcal{T}_{\rm BH}$.

Stability of ECO is an important problem. Apart from the superradiance discussed in our work, there are other kinds of instabilities. The energy of spinning can be dissipated due to Penrose process in ergo-region. Also recently it's pointed out that highly compact ECO may collapse due to the incoming energy of GW[23]. We will also study further on the stability of ECO in the future.

Although no significant evidence for echo is found in data yet, lots of data analysis work is still ongoing to find possible sources of echoes. With more data and higher accuracy in the third observation run of advanced LIGO, we should be able to stronger constraint on ECO assumption for each event or possibly find echoes. Since BH merger is major source of event for LIGO and therefore the final product of merger is usually spinning, waveforms for spinning ECO are needed to perform a comprehensive search. This work will help echo searches and we'll perform data analysis based on spinning ECO model soon.

With the planned launch of Laser Interferometer Space Antenna in 2030s, we will be able to probe GW from Extreme-Mass-Ratio Inspirals (EMRIs). If the central object of the EMRI system is ECO instead of BH, we would expect echoes of continuous signal. The interference between direct signal and subsequent echoes will result in enhancement or reduction in energy flux dissipated by the test particle. This will change the evolution path of the system or contribute a phase shift, which are sensitive parameters in data analysis. Preliminary analysis in this direction is also in order.

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A. Teukolsky-Sasaki-Nakamura Formalism

Astrophysical black holes with mass M and angular momentum J are described by Kerr metric. Perturbations of spin-0, spin-1 (e.g. electromagnetic fields) and spin-2 (e.g. Gravitational waves) fields in Kerr spacetime are governed by Teukolsky equations[31], which is a set of ordinary differential equations, separated in Boyer-Lindquist coordinates. For GW, the perturbation field ψ_4, decomposed in frequency domain $\psi_4 = \rho^4 \int_{-\infty}^{+\infty} d\omega \sum_{lm} R_{lm\omega}(r)\ {}_{-2}S^{a\omega}_{lm}(\theta) e^{im\phi} e^{-i\omega t}$ where ${}_{-2}S^{a\omega}_{lm}$ is spin-weighted spheroidal harmonic with eigenvalue E_{lm}, obeys:

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{lm\omega}}{dr}\right) - V(r) R_{lm\omega} = -\mathscr{T}_{lm\omega}(r), \tag{A1}$$

where $\mathscr{T}_{lm\omega}(r)$ is the source term and the potential is

$$V(r) = -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda, \tag{A2}$$

where $K = (r^2+a^2)\omega - ma,\ \lambda = E_{lm} + a^2\omega^2 - 2amw - 2$ and $\Delta = r^2 - 2Mr + a^2$.

For black holes, we impose the boundary conditions that $R^{\rm BH}_{lm\omega}$ is only ingoing at horizon and only out going at infinity. Using method of Green function, the solution of the radial Teukolsky equation is given by:

$$R^{\rm BH}_{lm\omega}(r) = \frac{R^{\infty}_{lm\omega}(r)}{2i\omega B^{\rm in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r_+}^{r} dr' \frac{R^{\rm H}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} + \frac{R^{\rm H}_{lm\omega}(r)}{2i\omega B^{\rm in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r}^{\infty} dr' \frac{R^{\infty}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{A3}$$

where $R^{\rm H}, R^{\infty}$ are solutions to homogeneous equation. $R^{\rm H}$ satisfies the boundary condition at horizon and R^{∞} satisfies the boundary condition at infinity, with following asymptotic amplitudes:

$$\begin{aligned} R^{\rm H}_{lm\omega} &= B^{\rm hole}_{lm\omega}\Delta^2 e^{-ipr*}, \quad r \to r_+ \\ R^{\rm H}_{lm\omega} &= B^{\rm out}_{lm\omega} r^3 e^{i\omega r*} + r^{-1} B^{\rm in}_{lm\omega} e^{-i\omega r*}, \quad r \to \infty; \end{aligned} \tag{A4}$$

$$\begin{aligned} R^{\infty}_{lm\omega} &= D^{\rm out}_{lm\omega} e^{ipr*} + \Delta^2 D^{\rm in}_{lm\omega} e^{-ipr*}, \quad r \to r_+ \\ R^{\infty}_{lm\omega} &= r^3 D^{\infty}_{lm\omega} e^{i\omega r*}, \quad r \to \infty, \end{aligned} \tag{A5}$$

The solution $R^{\rm BH}_{lm\omega}$ is only ingoing at horizon and only outgoing at infinity:

$$R^{\rm BH}_{lm\omega}(r \to \infty) = Z^{\rm H}_{lm\omega} r^3 e^{i\omega r*}, \tag{A6}$$

$$R^{\rm BH}_{lm\omega}(r \to r_+) = Z^{\infty}_{lm\omega} \Delta^2 e^{-ipr*}. \tag{A7}$$

By taking the limit at $r \to \infty$ and $r \to r_+$ of the solution (Eq. A3), with the asymptotic behavior of homogeneous solutions (Eq. A4, A5), one can find the amplitudes $Z^{H,\infty}_{lm\omega}$:

$$Z^{\rm H}_{lm\omega} = \frac{1}{2i\omega B^{\rm in}_{lm\omega}} \int_{r_+}^{\infty} dr' \frac{R^{\rm H}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{A8}$$

$$Z^{\infty}_{lm\omega} = \frac{B^{\rm hole}_{lm\omega}}{2i\omega B^{\rm in}_{lm\omega} D^{\infty}_{lm\omega}} \int_{r_+}^{\infty} dr' \frac{R^{\infty}_{lm\omega}(r')\mathscr{T}_{lm\omega}(r')}{\Delta(r')^2} \tag{A9}$$

At infinity, ψ_4 is related to familiar polarizations of GW by $\psi_4(r \to \infty) \to \frac{1}{2}(\ddot{h}_+ - i\ddot{h}_\times)$. Therefore, the gravitational waveform, observed from distance R, latitude angle Θ and azimuthal angle Φ, is given by:

$$h^{\rm BH}_+(R,\Theta,\Phi,t) - ih^{\rm BH}_\times(R,\Theta,\Phi,t) = \frac{2}{R}\sum_{lm}\int_{-\infty}^{+\infty} d\omega \frac{1}{\omega^2} Z^{\rm H}_{lm\omega}\, {}_{-2}S^{a\omega}_{lm}(\Theta) e^{i(m\Phi - \omega[t-r*])}. \tag{A10}$$

Computing GW waveform thus boils down to solving homogeneous Teukolsky equation and determining the asymptotic amplitudes $B^{\rm hole}, B^{\rm in}, B^{\rm out}, D^{\rm in}, D^{\rm out}, D^{\infty}$. However, Teukolsky potential is long-ranged, making it hard to numerically extract certain parameters in homogeneous solution, e.g. $B^{\rm in}$, which is overwhelmed by $B^{\rm out}$ at infinity. Sasaki $et\,al.$ transformed the radial equation so that the potential is short-ranged and the equation become numerical computable[32]. Moreover, asymptotic behavior of solutions to Sasaki-Nakamura equation are purely sinuous, making it easier to generalize to echo construction.

Transformation between Teukolsky function $R_{lm\omega}$ and Sasaki-Nakamura function $X_{lm\omega}$ is given by[29]:

$$R_{lm\omega} = \frac{1}{\eta}\left[\left(\alpha + \frac{\beta_{,r}}{\Delta}\right)\frac{\Delta X_{lm\omega}}{\sqrt{r^2+a^2}} - \frac{\beta}{\Delta}\frac{d}{dr}\frac{\Delta X_{lm\omega}}{\sqrt{r^2+a^2}}\right]. \tag{A11}$$

Taking Eq. (A11) into Teukolsky equation, one can find the equation for Sasaki-Nakamura function $X_{lm\omega}$:

$$\frac{d^2 X_{lm\omega}}{dr*^2} - F(r)\frac{dX_{lm\omega}}{dr*} - U(r)X_{lm\omega} = 0. \tag{A12}$$

The detailed expressions of the functions α, β, η and the potentials $F(r), U(r)$ can be found in [29].

The Sasaki-Nakamura equation admits two homogeneous solution having the purely sinuous asymptotic behavior due to the short-rangeness of potential $U(r)$:

$$\begin{aligned} X^{\rm H}_{lm\omega} &= A^{\rm hole}_{lm\omega} e^{-ipr*}, \quad r \to r_+, \\ X^{\rm H}_{lm\omega} &= A^{\rm out}_{lm\omega} e^{i\omega r*} + A^{\rm in}_{lm\omega} e^{-i\omega r*}, \quad r \to \infty; \end{aligned} \tag{A13}$$

and

$$\begin{aligned} X^{\infty}_{lm\omega} &= C^{\rm out}_{lm\omega} e^{ipr*} + C^{\rm in}_{lm\omega} e^{-ipr*}, \quad r \to r_+, \\ X^{\infty}_{lm\omega} &= C^{\infty}_{lm\omega} e^{i\omega r*}, \quad r \to \infty, \end{aligned} \tag{A14}$$

$X^{H,\infty}$ are related to $R^{H,\infty}$ by Eq. (A11) and thus the asymptotic amplitudes $A^{\rm hole}, A^{\rm in}, A^{\rm out}, C^{\rm in}, C^{\rm out}, C^{\infty}$

and $B^{\rm hole}, B^{\rm in}, B^{\rm out}, D^{\rm in}, D^{\rm out}, D^{\infty}$ have following relations[33]:

$$\begin{aligned}
B^{\rm in}_{lm\omega} =& -\frac{1}{4\omega^2}A^{\rm in}_{lm\omega} \\
B^{\rm out}_{lm\omega} =& -\frac{4\omega^2}{-12i\omega M+\lambda(\lambda+2)-12a\omega(a\omega-m)}A^{\rm out}_{lm\omega} \\
B^{\rm hole}_{lm\omega} =& \{\sqrt{2Mr_+}[(8-24iM\omega-16M^2\omega^2)r_+^2+(12iam-16M+16amM\omega+ \\
& 24iM^2\omega)r_+-4a^2m^2-12iamM+8M^2]\}^{-1}A^{\rm hole}_{lm\omega}
\end{aligned} \tag{A15}$$

$$\begin{aligned}
D^{\rm in}_{lm\omega} =& \{\sqrt{2Mr_+}[(8-24iM\omega-16M^2\omega^2)r_+^2+(12iam-16M+16amM\omega+ \\
& 24iM^2\omega)r_+-4a^2m^2-12iamM+8M^2]\}^{-1}C^{\rm in}_{lm\omega} \\
D^{\rm out}_{lm\omega} =& -\frac{4p\sqrt{2Mr_+}(2Mr_+p+i\sqrt{M^2-a^2})}{\eta(r_+)}C^{\rm out}_{lm\omega} \\
D^{\infty}_{lm\omega} =& -\frac{4\omega^2}{-12i\omega M+\lambda(\lambda+2)-12a\omega(a\omega-m)}C^{\infty}_{lm\omega}
\end{aligned} \tag{A16}$$

The homogeneous solution X^{∞} can be regarded as a incident wave coming out near the horizon with amplitude $C^{\rm out}_{lm\omega}$ scattered off the Sasaki-Nakamura potential. We can thus define the reflection and transmission factors $\mathcal{R}_{\rm BH}$ and $\mathcal{T}_{\rm BH}$, which reduces to energy reflectivity and transmissivity in Schwarzschild case:

$$\mathcal{T}_{\rm BH}=\frac{C^{\infty}}{C^{\rm out}},\, \mathcal{R}_{\rm BH}=\frac{C^{\rm in}}{C^{\rm out}} \tag{A17}$$

To summarize, we numerically integrate Sasaki-Nakamura equation (A12) to get the solutions $X^{H,\infty}$ and asymptotic amplitudes. Then by the transformation law (Eq. A11,A15,A16), we have the solutions to homogeneous Teukolsky equation $R^{H,\infty}$ and asymptotic amplitudes. Integrating $R^{H,\infty}$ with source term, we get $Z^{H,\infty}$ from Eq. (A8,A9). Finally, for black holes, the gravitational waveform observed at infinity is given by Eq. (3).