Exploring the Detection Process of Gravitational Wave Memory

Interim Report #1

Jillian Doane - Oberlin College

Mentors - Alan Weinstein and Jonah Kanner

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Update For Interim Report #1 - Section 1

Currently I am working on learning how to produce a waveform model in order to calculate the overlap, likelihood, and phase difference. These calculations will help me learn data analysis tools that will also be important when analyzing a gravitational wave signal with memory. While described in more detail in the “Motivation for Memory Detection” (Section 4) , the discovery of new CBC (Compact Binary Coalescence) events makes this a prime time to once again research memory and potentially make progress with the analysis process. The data analysis techniques I am currently working on include plotting data and templates in the time and frequency domain, creating a PSD to view the average noise per each frequency bin, whitening the data to average the noise and reduce the signal at irrelevant frequencies, and creating a matched filter that can help maximize the SNR. So far my approach has been to both use data analysis tutorials to help understand the code, and discussing key concepts with my mentors in order to better understand the signal analysis process. Also working with my peers is a helpful tool as we learn from one another and also get the chance to problem solve before approaching our mentors with questions.

So far I have been working on the objectives listed above while also going through tutorials that involve other important data analysis concepts such as conditioning, band passing, creating ASD plots, etc. I have also been going through some introductory python tutorials in order to better understand the code I am writing and begin to think about how I can write my own code that will be beneficial for my project. I have also started looking into LaTex which will be important for properly formatting my final report.

Thus far I have encountered numerous software challenges that include learning how to reproduce the memory waveform. To resolve this dilemma we have decided to use a new Python package so we will be able to focus on manipulating the waveform. Moving forward I anticipate that there will be further software difficulties that I will have to learn how to de-bug. This project also involves using relatively new techniques such as learning how to stack the SNRs of the signals coherently. This will present a learning curve as the task will not only be very technical, but also relatively undeveloped.

Abstract - Section 2

From studies of general relativity it has already been established that two types of gravitational wave memory exist: linear and nonlinear, also called Christodoulou, memory [27]. These effects are predicted to be detectable in strong-field, highly dynamical regime collisions which until recently have been some of the least explored events in the universe. The linear memory is very small, whereas the nonlinear memory is believed to be large enough to be detected by Advanced LIGO and 3rd generation detectors [11]. The current complications of detecting memory involve the fact that it is extremely small and can easily be overlooked in the gravitational wave signal. Also, memory is not detectable with just one collision; multiple events must be stacked in order to produce a resolution decent enough to detect the memory. The final obstacle is the low-frequency detector noise. While some forms of this noise are manageable, other forms such as quantum and instrument noise, which can look very similar to the memory signal, is difficult to control. As the sensitivity of Advanced LIGO and 3rd generation detectors Previous work has attempted to detect memory with a range of techniques including integrating up the signal to $t = 1/f_{opt}$ where f_{opt} is the optimal frequency of the detector, or using the "effective-one-body" (EOB) approach which will eventually allow one to develop an analytical function for gravitational wave memory. The technique of focus for my project will be stacking multiple events, and we will estimate parameters relevant to this technique in order to infer how well the memory parameter can be measured for future events detected with Advanced LIGO.

Introduction: LIGO, Gravitational Waves, and Memory - Section 3

Gravitational wave detection by LIGO is a developing phenomena that began in 2015 with the detection of a gravitational wave from a binary black hole merger that occured about 1.3

billion years ago [11]. Since this first occurence, there have been 6 other cases of gravitational wave detected by LIGO [14]. With each detection providing new information, there is always a push to analyze the data in hopes of further understanding the source that produced the wave. Along with gaining information concerning some of the universe's most extreme events, the properties of the waves themselves can also be further examined in order to increase our understanding of the effect they have on matter they pass through and our general relativistic predictions.

Motivation for Memory Detection - Section 4

For a binary inspiral there is a non-oscillatory component to the "+" polarization (similar to polarization of light except gravitation wave polarizations are 45 degrees apart) which makes the amplitude of the gravitational waves dampen to a non-zero value [7]. This non-zero amplitude represents the gravitational wave memory, a weak stretching that permanently alters spacetime [4]. There are numerous motivations for memory detection. One reason is that exploring the strong-field, highly-dynamical regime in connection to general relativity is a relatively new and unexplored area of physics. For systems under gravitational influence with unbound components, a linear memory effect is applicable. Linear memory, discovered in the 1970s, arises from near-zero-frequency changes in the time derivatives of the source's multipole moments. Multipole moments are a combination of the mass moment, the extent to which an object resists rotational acceleration about a particular axis, and the mass-current moment which corresponds to the star's spin angular momentum (the star's moment of inertia about its spin axis multiplied by its spin angular frequency Ω) [20, 9, 15]. Linear memory also appears in systems that experience kicks such as a rogue black hole, or systems that eject particles such as neutrinos from supernovae [7]. Non-linear memory grows slowly and is also a non-oscillatory contribution to to the gravitational wave's amplitude. It originates from gravitational waves that are sourced by the previously emitted waves [9]. It is believed that all gravitational waves carry a component of nonlinear memory which means it should be included in LIGO waveform models [25].

Since linear and nonlinear memory depend on the form of Einstein's field equation, a set of 10 general relativity equations that describe gravity as a result of spacetime being curved by mass and energy, it is then possible that different forms of memory could be uncovered if general relativity were to be modified [6]. Studying these two fields side by side is highly intriguing as together they could help us undercover unexplored areas of physics. Since memory is difficult for LIGO to detect, it has been mostly been disregarded by scientists studying gravitational waves. However, the most studied gravitational wave source is currently compact binaries, for example black holes. The memory scales linearily with the black hole's mass, and it is estimated that the memory effect in these cases will have a detectable contribution to the calculated waveform amplitude of the resulting gravitational waves [12]. This memory effect is thought to be so large that the order it enters the waveform is equivalent to the leading-order term, in this case the quadrupole. From this arises the conjecture that the memory effect should not be impossible to detect [7]. Finally, there is little information known about how the memory signal grows and fluctuates throughout the inspiral and ringdown phase, and also post-merger. While there have been multiple studies that have explored memory both with simulations and with real LIGO data, the most recent calculations were published when data from only one LIGO event was released. Now that LIGO has published seven CBC events, there is more data to explore and a greater potential to detect memory. For these reasons one can understand the motivation behind exploring methods to detect memory.

Gravitational Wave Memory Background - Section 5

While there are numerous methods one can utilize to begin to understand gravitational radiation, one helpful analogy is electromagnetic radiation. As electric charges move they create electric and magnetic waves that propagate outward from their source at the speed of light. The waves carry energy and their energy Flux falls off as $1/r^2$ where r is the distance away from the source. They can be detected by the forces they apply to charged particles, or by the amount of energy the source loses from the wave propagation. In a similar fashion, gravitational waves arise when moving masses send out waves that cause the curvature of spacetime to fluctuate. The amplitude of the waves also falls off as 1/r over long distances and they can be detected either by the gravitational strain they apply to groups of massive objects in free fall, or the amount of energy that is lost by the source. While there are strong similarities between gravitational and electromagnetic radiation, the differences become apparent when the strength of the two forces are compared. Due to the weakness of gravity, only very powerful astrophysical interactions are capable of producing gravitational waves that are detectable on earth. Some of these interactions include mergers of neutron stars, black holes, or a combination of both [5]. An additional component that differs between electromagnetism and gravitation is gravitational waves have a large nonlinearity [23].

While the nonlinear component of gravitational radiation is very important, it becomes easier to understand the background physics when only the linear portion is considered at first. In this case, the relevant Einstein field equation reduces to a form similar to that of one of

Maxwell's equations. After taking the time derivatives of the sources multipole moments the resulting equation becomes h(r,t) = $\frac{2G}{c^4 r}\frac{d^2 I_{ij}}{dt^2}(t - r/c)$ [19] where I_{ij} represents the mass quadrupole moment. This is known as the the quadrupole formula of general relativity which is used to calculate energy loss. While this equation is only used in situations without the presence of gravity, it can be altered using a "post-Newtonian approximation" to include gravity's influence. The equation for this quadrupole moment gives rise to an indirect way to detect gravitational waves, that is, by considering a system where motion is measured very accurately. An example of this is a binary neutron star system with one pulsar and one neutron star, where the rate general relativity predicts the system will lose energy can be calculated. This loss of energy will cause the orbit of the neutron star and the pulsar to shrink. The shift in orbital features can be tracked through doppler shift of the arrival time of the pulses. From the formula for energy loss one can then predict what the orbital period of the binary system will be at a particular moment in time. This can be checked with an astronomical measurement of the changing period. These two values were shown to match which means gravitational waves were indirectly detected [9].

In order for one to directly detect gravitational waves, the concept of geodesics must be used. Geodesics refer to the shortest path an object in free fall can follow on a curved surface. The distance between two geodesics changes as a non uniform gravitational field embedded in the curvature of spacetime [10]. If one considers the geodesic deviation equation and focuses on the "weak field slow motion" case, one can integrate twice and obtain the equation $\Delta S^i = \frac{1}{r} S^j P[\Delta Q_{ij}]$ [1] The delta shows that there is a difference between the original and final distance of the geodesic. This is due to gravitational wave memory that has left the wave field at a final non-zero value. The specific component of memory as described above is only the linear aspect, the nonlinear effect makes up the primary portion of memory. Previous work has lead to the conclusion that linear memory arises in situations with unbound components such as ejected neutrinos from a supernova or gamma-ray burst jets, while Christodoulou established that this 'nonlinear' memory is from energy radiated in gravitational radiation [7]. Additionally, while linear memory is a result of fields that do not reach null infinity, nonlinear memory is due to fields that do reach null infinity. Christodoulou then went on to prove that the linear memory effect is relatively small which is why non-linear memory is usually the focus.

Limitations of Memory Detection - Section 6

While we have good reason to believe gravitational memory exists, it is the detection process that has limited us thus far. In understanding why gravitational memory has not yet been directly detected, it is helpful to first examine the details that make finding it difficult.The first reason is due to the extremely small size of the memory effect. The size of the memory is expected to about one-tenth to one-hundredth of that of the gravitational waves. This is why this effect is predicted to mostly be detectable in the most violent collisions in the universe. For example, the first gravitational wave signal, GW150914, caused the LIGO arms to stretch and shrink by about one-five-hundredth of a femtometer. It was predicted this memory effect would only be about one-twentieth of the size of the gravitational waves, which is about one-ten-thousandth of a femtometer [4].

Another way to interpret the difficulty of memory detection is from the presence of low-frequency detector noise. While seismic noise occurs at low frequencies, it is relatively well understod. Quantum noise and instrument noise however are much harder to suppress. Quantum noise arises from the statistical uncertainty of photon arrival time and radiation pressure from the random motion of the mirrors [13]. Instrument noise is a concern because it can overwhelm or mimic the strain pattern that is being looked for. The instrument noise is smallest around a few hundred hertz, but increases sharply at low and high frequencies. At all frequencies there are narrow spikes due to vibrating fibers that suspend the mirrors and test masses in the interferometers [18].

Approach: Potential Methods of Detection - Section 7

Even though there are still numerous challenges to overcome before a clear memory signal is obtained, there are strategies that have allowed the hunt to begin. By analyzing a binary neutron star system one can use the calculated metric perturbation, or disturbance, to find the strain, or the difference in displacement due to the passing of a gravitational wave. The strain is small, on the order of magnitude of 10^-21. Measuring this type of strain means that a length of 1cm is required to be measured to length of 10^-21 cm [3].

However, the memory strain is even smaller than the oscillatory strain. This memory strain monotonically increases with the length of the signal. For signals with relatively short burst lengths, (time shorter than the inverse frequency of the detectors sensitivity), the memory can be approximated by a step function which is proportional to 1/f [16] (Figure 1). To estimate the step function a sine-Gaussian waveform is used.

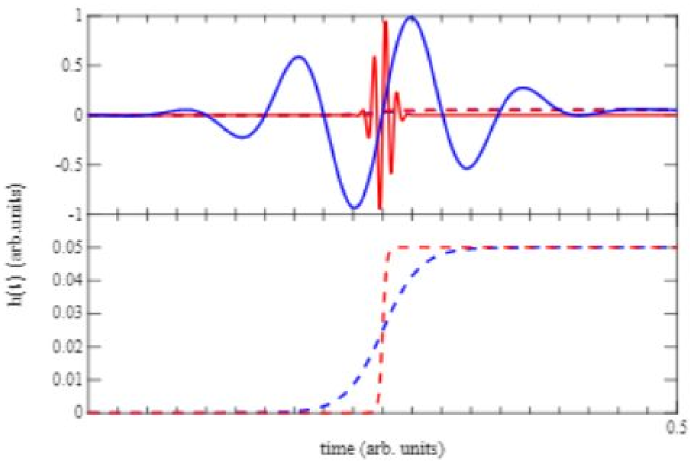

Figure 1 [26]: The top figure shows the sine-Gaussian burst (solid curve) and memory (dashed curve) strains for two bursts with an equal amplitude. The high-frequency burst (red) has a frequency ten times greater than the low-frequency burst (blue). The bottom figure is a close up of the memory time series'. The rise time approaches zero as the frequency increases, and the step function becomes an acceptable approximation of the memory.

From here it has been discovered that the memory from high frequency waves has a low-frequency component whose value is below 1/t where t is the length of the gravitational wave signal. This means that if the source of the burst, also called the parent burst, is of a frequency that is above that of LIGO's observing band, then there may be a detectable orphan memory.

While orphan memory has not been directly detected yet, its theoretical maximum value can still be calculated. The maximum gravitational wave frequency is $f_{max} \sim 1/r_s \sim 1/M_s$ where r_s is the Schwarzschild radius and and M_s is the mass. To find an estimate of the possible maximum memory, we have to assume the mass of the black hole completely radiates away. It follows that the maximum occurs at a frequency equal to $1/E_{gw}$ which leads to a possible maximum memory of $E_{gw}^{1/2}/f^{1/2}d$ [16] which scales in a similar fashion to the oscillatory strain. Also, the maximum memory for edge-on binaries has a scaling relationship proportional to $\sin^2\theta(17 + \cos^2\theta)$ [12] (Figure 2)

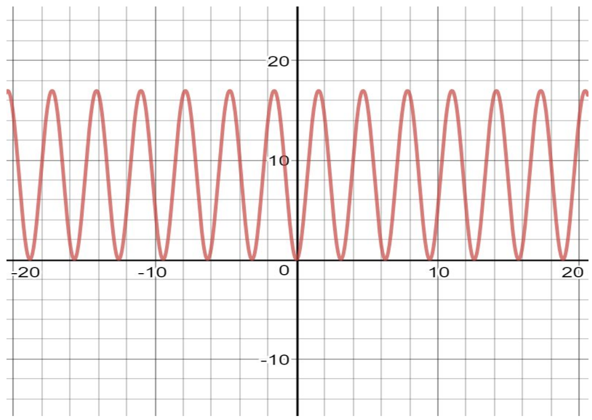

Figure 2: graph of $\sin^2\theta(17 + \cos^2\theta)$

While it is not entirely clear if orphan memory is detectable with Advanced LIGO, there is still potential for its discovery with future detectors.

Another method that could be useful in detecting gravitational memory is to first predict a function that represents it. This model will likely help to check results if gravitational memory is discovered. The first step in finding this representative memory equation is to first use a combination of Bayesian statistics techniques. The next step is to implement the "effective-one-body" (EOB) approach which can model the multipole moments, or the coefficients of a multipole expansion in the inspiral and ringdown phases [7]. A multipole expansion is composed of a group of functions known as spin-weighted spherical harmonics that depend on the angles (usually two) on a curved surface. They are useful for gravitational field study where the fields at distant locations are described in terms of the sources located in a small region of space. The expansion is expressed as a sum of the angles as they approach an increasingly finer degree with the zeroth term named the monopole, the next term the dipole moment (which only varies once around a sphere), and eventually the quadrupole which changes quickly with the angles [17]. Once the angles used for the expansion are combined with a certain distance, then the result is a function that describes three-dimensional space. Next, the minimal-waveform model can be implemented in order to sum the quasi-normal modes. [12]. These quasinormal modes can help us understand the energy loss of dynamical systems such as a black hole mergers. Quasinormal modes of black holes describe the exponential decrease of the asymmetry of the hole in time as it evolves to a spherical shape in the ringdown phase. The result will be an analytical function for gravitational wave memory.

A more general approach to compute Christodoulou memory is to treat the oscillatory component of the wave as a parameter. This means one must look at both the cross and plus polarization of the wave. It can then be integrated over a specific angle and time to calculate the memory. Another benefit of this approach is one can use a series of waveforms from compact binary coalescences instead of a single estimated one. This approach also yields potential to compute high-order memories, or memories produced by memories [22].

A fourth method to detect gravitational wave memory is to integrate along the signal to $t = 1/f_{opt}$ where f_{opt} (optimal frequency) is the frequency at which the detector is the most sensitive to ordinary gravitational wave bursts. If the length of the burst with memory (BMW) is smaller than $1/f_{opt}$, the detector's sensitivity to BMW is practically equivalent to that of bursts without memory that are one cycle long and whose frequency is f_{opt}. A benefit of this method is it has the potential to be implemented despite the type of detector used for the study [2].

A final method of detection is the stacking of events. It is believed that combining information from the mergers could, overtime, boost the detectability of memory enough to obtain a clearer picture. Paul Lasky, an astrophysicist at Monash University in Australia, has predicted that 35 to 90 black hole mergers similar in mass and distance as G150194 may be enough for LIGO to detect memory [4]. Also since LIGO is going through advancements until 2021 which will make it more sensitive, there is potential that fewer mergers than predicted will be needed to detect memory [24]. The most recent attempt of this method was

As a build up to the stacking method mentioned above, I have been working on general data analysis techniques with a normal LIGO waveform in preparation to work with waveforms with memory. My colleague Simona and I first stepped through the process of generating waveforms with merging black holes of both 10 and 10.1 solar masses. To do so we recycled code from the LIGO 2018 open data workshop and adjusted the masses. Then we moved on to calculating the phase difference for the 10 and 10.1 solar mass black hole. This involved creating an array for the cross and plus polarization and making a complex recurring loop. We took advantage of the “angle” function in numpy along with the unwrapping function. Next I went on to calculate the PSD for a set of data which comes in handy when you want to know the average power distribution across each frequency bin.

Currently we are beginning to utilize the matched filtering process to eventually calculate the overlap between one of our templates and the LIGO data. The matched filter is important as it communicates how much data can be found in the template and it is the optimal filter for detecting a stationary Gaussian noise signal. The process of matched filtering includes finding the noise weighted inner product which involves integrating over the two waveforms multiplied, and then dividing by the PSD. From here the overlap is found by normalizing the noise weighted inner product. Matched filtering becomes an important process as we want to maximize our chance of finding the gravitational wave without reducing the SNR.

Objectives For LIGO SURF 2018 Project -Section 8

For my project I plan to first learn background information relevant to gravitational wave memory. This will include learning general signal processing techniques such as how to generate a LIGO waveform in the time domain and the frequency domain using a FFT (Fast Fourier Transform). A FFT is a computer's technique to perform Fourier transform (FT). While a FFT uses discrete values (a list of numbers) a FT will take a data set from the time domain and transform it into the frequency domain using a combination of sines and cosines. Next in the process will likely be creating a power spectral density which is used to calculate the average power in each frequency bin. I will then learn how to clean up the data which includes adjusting the original signal in order to lessen the noise and make the signal from the CBC more noticable. This will include whitening the data which involves using the PSD to divide through the noise in each frequency bin so the average value becomes one. There is also the process of bandpassing the data (filtering out irrelevant frequencies that are not close to that of the signal). The next important step is learning how to model the data using a template. In the normal LIGO data analysis, a template is created to model the predicted data. This template is used in the matched filtering process which involves properly weighting the data and the template and integrating. If part of the data aligns with the template then you will receive a large value when integrated over. In order to numerically characterize how well the template matches the data they calculate the overlap or fitting factor. From here one can calculate the SNR time series which will produce a peak. The higher SNR, the stronger the signal and the easier it is for LIGO to detect. I will be using the processes listed above to first stimulate a gravitational wave event with and without memory. I will calculate the SNR for both in order to show that with either with the right parameters (a close enough distance or large enough mass) or more sensitive equipment, then gravitational wave memory will be possible to detect. Even if the signal can be shown to be detectable, it is still very small potential that it will be observed under the present circumstances. In order to quantify the probability of detection we will calculate liklihood which involves subtracting the template from the data. The result of a very small likelihood will be used to emphasize why memory has not yet been detected. As I progress further with my project the goal is to begin to coherently stack the SNR of the LIGO events in order to allow it to accumulate. This involves learning how to understand the position of the memory in terms of the x-axis and ensuring that all of the SNR are being stacked in the same direction. Memory slowly builds up overtime, which is why eventually the goal is that this process will reveal a component of gravitational wave memory that is deemed statistically significant. Depending on the direction of development for my project I will potentially explore Monte Carlo simulations and Bayesian inference .Th and how to create parameters for memory as part of this model. I will need to study Bayesian analysis in order to establish a procedure that estimates a Bayesian parameter. The stacking procedure will also be used to estimate the accuracy of the memory parameter in order to determine how well it can be measured for future events from Advanced LIGO and 3rd generation detectors. Necessary tools for these procedures include python and previously determined waveform models from the LIGO Algorithm Library [23].

LIGO SURF 2018 Project Schedule - Section 9

The timeline for this project, while subject to change, will first involve me learning the relevant background information as listed above. This involves learning general analysis techniques, reviewing helpful python tactics, and trying to better understand the memory waveform phenomena. This information will be included in my first progress report. I will then focus on learning how to manipulate a waveform with memory and compute a Fourier Transform and its signal to noise ratio (starting with a very near by distance). I will then create a parameter λ which should also peak at the same location as the SNR for the waveform with memory. After this process is complete, I will create an Amplitude Spectral Density (ASD) curve and learn how to change additional parameters to lower the power to create less noise at lower frequencies. This will make lower frequency bands more sensitive which is essential to detecting CBCs. After these first few weeks I will begin to explore the process of stacking the SNRs from the current LIGO events in order to accumulate a higher signal to noise ratio. In the beginning simulated waveforms will be used but as I progress I might begin to handle LIGO data. I will likely perform similar calculations as listed above but might once again focus on the likelihood and show that it is higher with the addition of multiple events opposed to a single event as the previous calculations have used. I will then continue to examine the stacking technique and explore the ways memory manifests itself in the discovered events. The goal will be to create the most foolproof detection approach possible. By looking at the data that has already been collected, I will then examine the parameters for memory we have developed and determine if they are likely to fit future LIGO data. This will allow me to write my second interim report and eventually final report. The last 3 weeks of the program I will begin working on my poster or presentation which will be presented August 24th.

Conclusion - Section 10

There is a continual effort of analyzing gravitational wave signals in hopes of gaining a deeper understanding of their sources. While its effects are not entirely known, memory could serve as a complementary factor that helps us reach this goal. Nonlinear memory continues to be an intriguing research topic due to the interesting information it has revealed thus far, including the way it affects the wave form at a leading order at a level equivalent to that of the quadrupole. This along with other implications are being analyzed so the strong-field, highly dynamical regime can continue to be probed. LIGO in combination with other ground-based gravitational wave interferometers will potentially be able to detect memory after the discovery of dozens of extreme collision events. Additional gravitational wave detectors such as Virgo, KAGRA or LIGO-India will further reduce detection time [12]. There is also great potential that LISA (to be launched in 2030) will be able to greatly reduce the detection time due to its capability to measure much lower frequencies, and also potentially target an area of the data that could point to additional events. Its arms will be much farther apart which is an additional factor that will help with it's time to detect memory. While it has not yet been directly detected, the capability of our technology is one of the many reasons why there is optimism surrounding the potential for gravitational wave memory.

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