Controls Design — 2025/07/17

- Start with a Time Series → DFT :

cross spectral density
$$C^{(1)}_{xy}(f_k) = \frac{2}{f_s \sum_n^{N-1} \omega^2(n)} \left(\sum_n^{N-1} w(n)\, x(n)\, e^{-i 2\pi f_k t_n} \right) \left(\sum_n^{N-1} w(n)\, y(n)\, e^{-i 2\pi f_k t_n} \right)^*$$

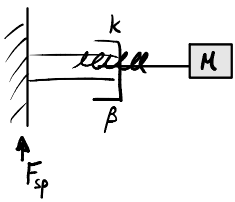
power spectral density
$$P^{(1)}_{xx}(f_k) = C^{(1)}_{xy}(f_k), \quad y \to x, \quad A^{(1)}_{xx}(f_k) = \sqrt{P^{(1)}_{xx}(f_k)}$$

- $Y(f_k) = H(f_k)\, X(f_k)$:

Transfer Functions
$$H^{(1)}_m(f_k) = C^{(1)}_{xy}(f_k) \,/\, P^{(1)}_{xx}(f_k) \quad \text{(x-susceptible)}$$
$$H^{(2)}_m(f_k) = P^{(1)}_{yy}(f_k) \,/\, C^{(1)}_{yx}(f_k) \quad \text{(y-susceptible)}$$

- $$\gamma^2 \equiv \text{coherence} = \frac{H^{(1)}_m(f_k)}{H^{(2)}_m(f_k)}$$

Pendulum Review



$$F_{sp} = +M\ddot{x}_p + \beta \dot{x}_p + k x_p$$
$$x_p = A e^{+i\omega t}$$
$$\dot{x}_p = i\omega x_p$$
$$\ddot{x}_p = -\omega^2 x_p$$

$$F_{sp}(\omega) = m x_p \left(-\omega^2 + i\omega \beta/m + k/m \right)$$

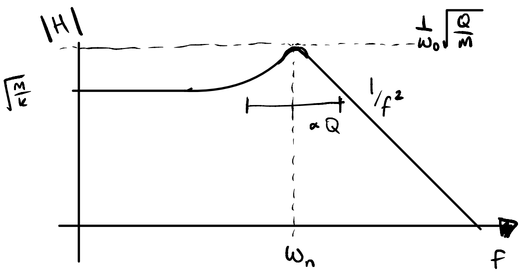
$$\omega_0^2 = \frac{k}{m}, \quad \frac{\beta}{m} = \omega_0/Q \;\longrightarrow\; Q = \frac{m\omega_0}{\beta} = \frac{1}{2\xi}$$

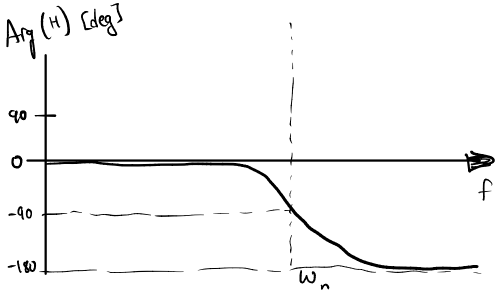
Q: quality factor; β: damping coeff; ξ: damping ratio

$$|H| = \sqrt{H\, H^*}$$
$$\mathrm{Arg}(H) = \tan^{-1}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)} \right)$$

$$\boxed{\frac{X_p}{F_{sp}}(\omega) = H(\omega) = \frac{1}{m} \frac{1}{(\omega_0^2 - \omega^2) + i\omega\omega_0/Q}}$$

$$\omega_n = \omega_0 \sqrt{1 - \xi^2}$$





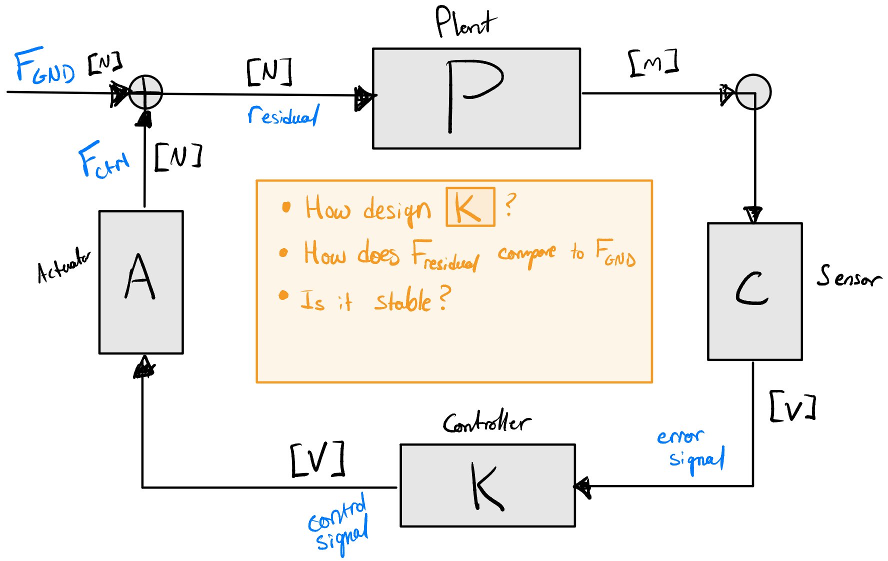
Wrapping a loop around the system

"Controls is about compromise"

"Sensors (+ readouts) have noise and sometimes have frequency response"

"Actuators (+ drivers) have noise and sometimes have a frequency response + can saturate"

"Creating a sensor/actuator array = $$$"



$$F_{res} = F_{GND} + F_{ctrl}$$

$\hookrightarrow F_{ctrl} = V_{ctrl} A$

$\hookrightarrow V_{ctrl} = V_e K$

$\hookrightarrow V_e = X_e C$

$\hookrightarrow X_e = F_{res} P$

$$F_{res} = F_{GND} + F_{res}(PCKA)$$

$$\boxed{F_{res} = \frac{1}{1-G} F_{GND}, \quad G = PCKA \equiv \text{Open Loop Gain}, \quad \frac{1}{1-G} \equiv \text{Loop Suppression}}$$

Stability

$$F_{res} = \frac{1}{1-G} F_{GND}$$

Note: $G \in \mathbb{C}$

When $|G| = 1$, $\angle G$ CANNOT be 0! unstable

$$\frac{X_{err}}{F_{GND}} = \frac{P}{1-G} \equiv \text{Loop Suppressed Plant}$$

$$\frac{F_{ctrl}}{F_{GND}} = \frac{G}{1-G} \equiv \text{Closed Loop Gain}$$

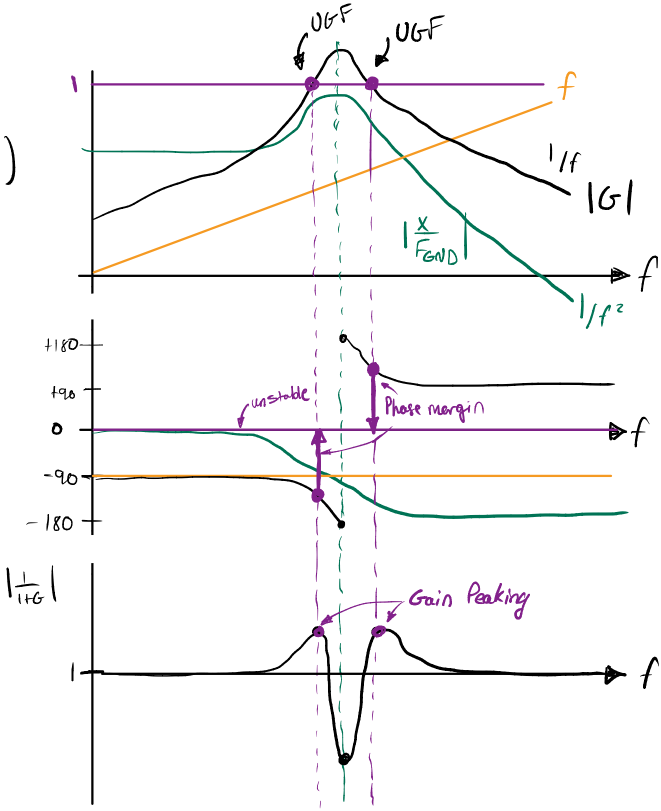
Pendulum Scenario

Goal: $\frac{1}{1+G} = 0.1$ (factor of 10)

Assume $G = P \cdot K$

$K = -i\omega$

UGF = unity gain freq.



Lower UGF = "rising into $|G| = 1$"
Upper UGF = "falling into $|G| = 1$"