Detecting gravitational waves isn't as easy as it sounds

an introduction to the detectors

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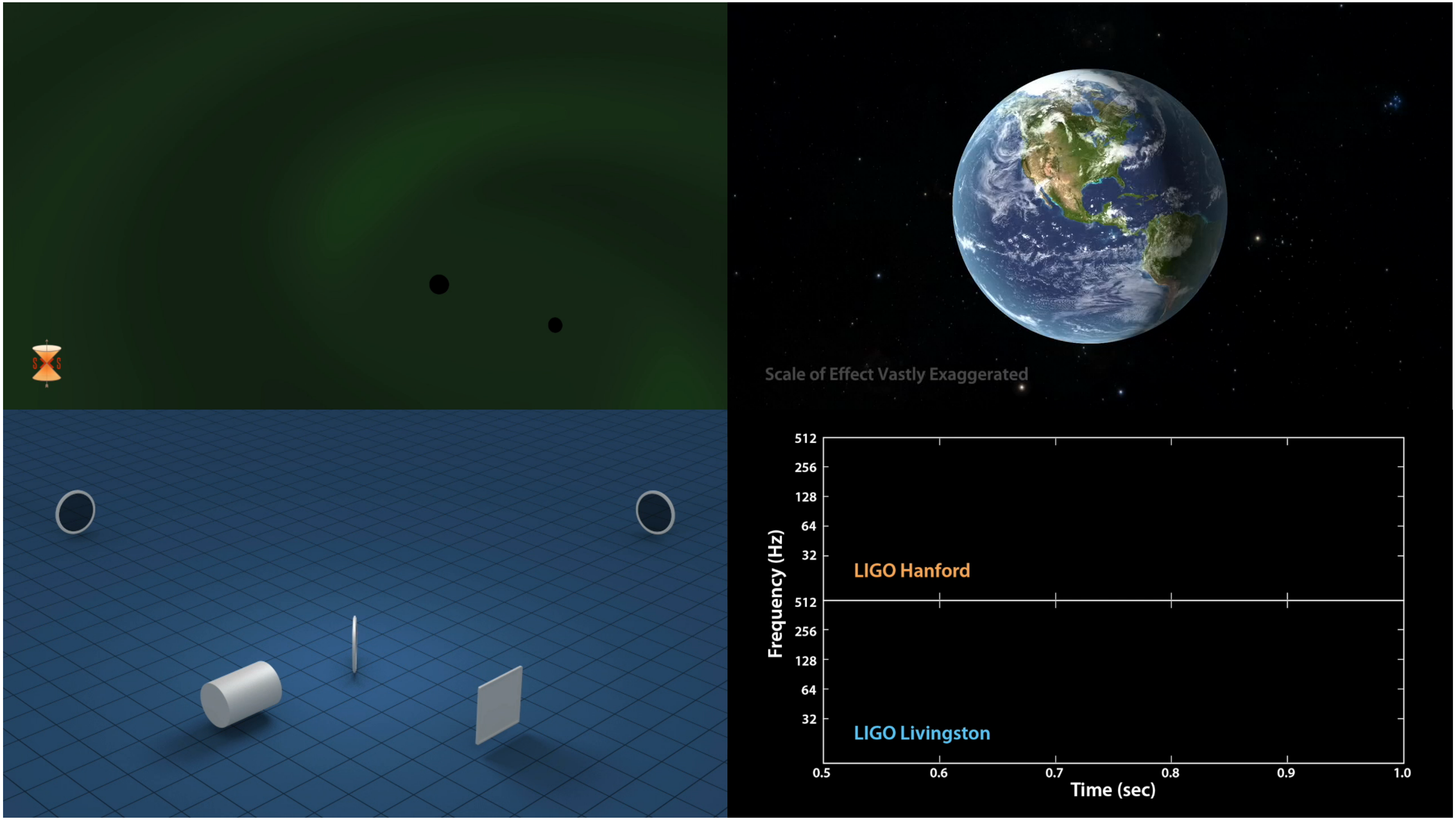
SURF lectures 2017

Let's look behind the scene



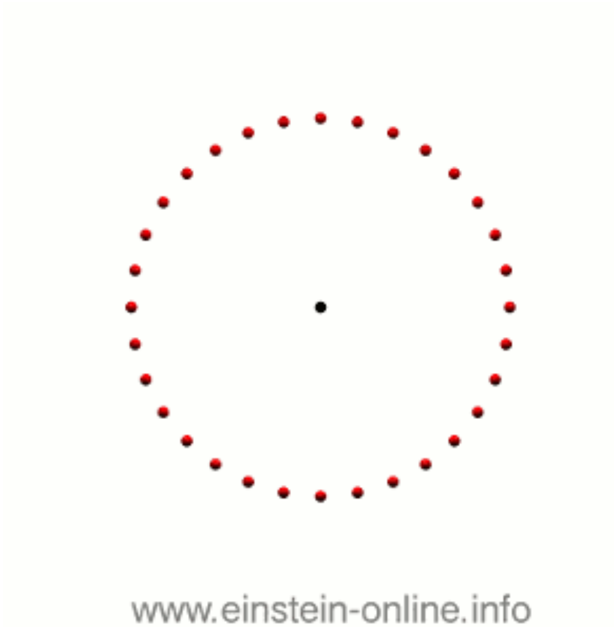
The interferometer as a transducer

- Gravitational waves create a differential change in the distance between free falling masses

$$\delta L = \pm h\, L$$

- Interferometers are good **transducers** to convert differential displacements into optical signals

$$\delta\phi = G\, \delta L$$

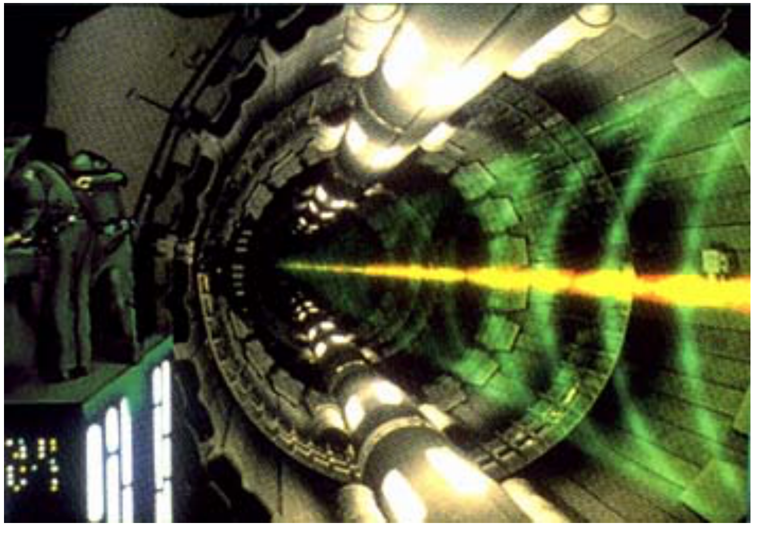


- G is the **optical gain** of the instrument
- For the moment let's neglect how to measure optical phases

Laser beams as plane waves

- The laser field is described by the Maxwell equations
- It is polarized, we can use a scalar wave description in terms of amplitude and complex phase (the physical field is the real part)

$$\Psi = Ae^{i\phi}$$



- The propagation in vacuum is described in terms of plane waves

$$\begin{aligned} \Psi(t,z) &= \Psi_0 e^{i\omega t - ikz} \\ \omega &= \frac{2\pi c}{\lambda} \\ k &= \frac{2\pi}{\lambda} = \frac{\omega}{c} \end{aligned}$$

$\lambda = 1064$ nm
$\omega = 2\pi \times 3\times10^{14}$ rad/s

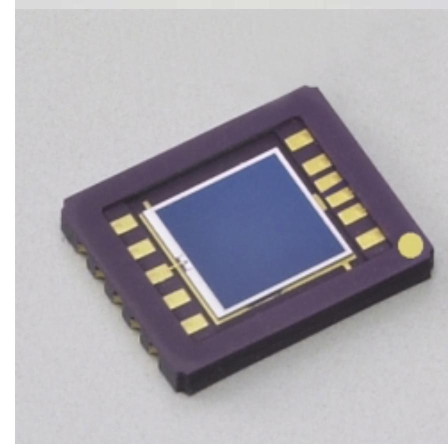
- In the real worlds there is also diffraction…

Detectors

- Photodiodes can only read the power in the laser beam:

$$P(t,z) = |\Psi(t,z)|^2$$

- We use units such that the squared field gives the power
- Photodiode can not read the absolute field phase, but only the amplitude
- We can't directly use a photodiode to extract the phase induced by a GW, without using clever tricks (more later on)



Mirrors

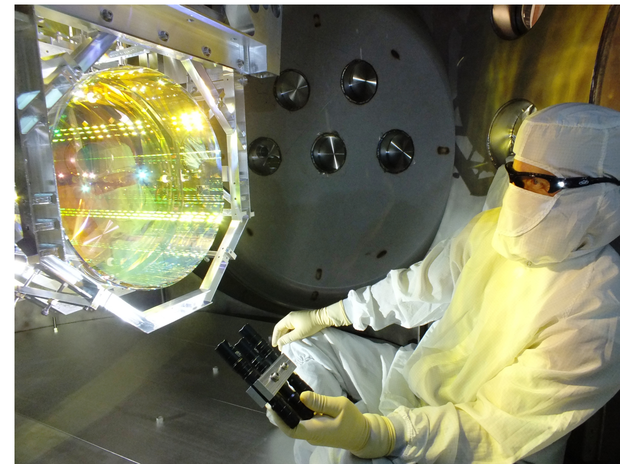
- Interferometers use semi-transparent mirrors
- Reflectivity and trasmissivity are clearly linked by energy conservation

$$\begin{aligned} |\Psi_r| &= r|\Psi_i| \\ |\Psi_t| &= t|\Psi_i| \end{aligned} \qquad r^2 + t^2 = 1 - L$$

- There is still some freedom in the choice of reflected and transmitted field phases. Let's choose:

$$\boxed{\begin{aligned} \Psi_r &= ir\Psi_i \\ \Psi_t &= t\Psi_i \end{aligned}}$$

- There is a phase jump in the reflected field and nothing in the transmitted one



Signal sidebands

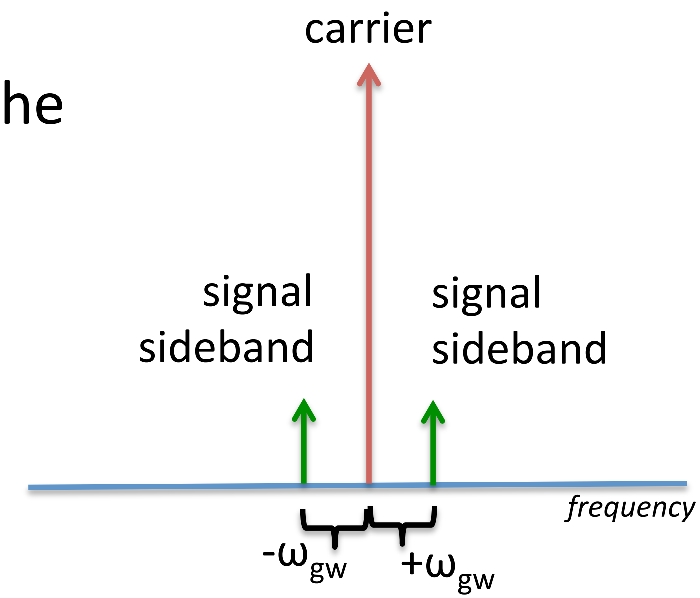
- When a GW travels perpendicularly to the laser propagation axis:

$$\Psi_L = \Psi_0 \cdot e^{-ik(L+h(t)L)} = \Psi_L^{(0)} e^{-ikL\,h(t)}$$

- If the GW is monochromatic

$$h(t) = h_0 \cos(\omega_{gw} t)$$

- The effect is at first order the creation of signal sidebands:

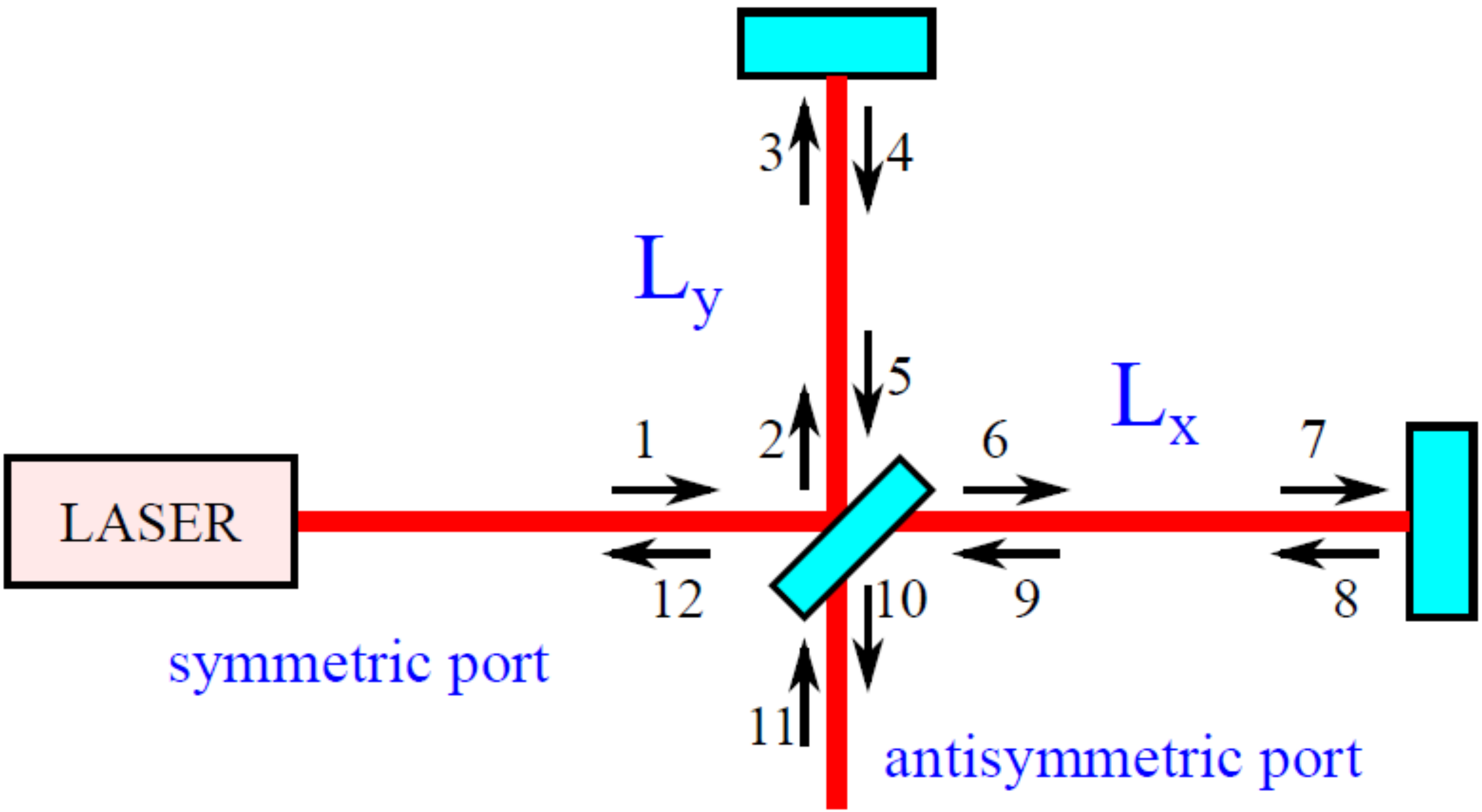


Rotating frame: all field phases are referred to the fast rotating carrier phase.

$$\Psi_L = \Psi_0 e^{-ikL}\left(1 + \frac{ikLh_0}{2} e^{i\omega_{gw} t} + \frac{ikLh_0}{2} e^{-i\omega_{gw} t}\right)$$

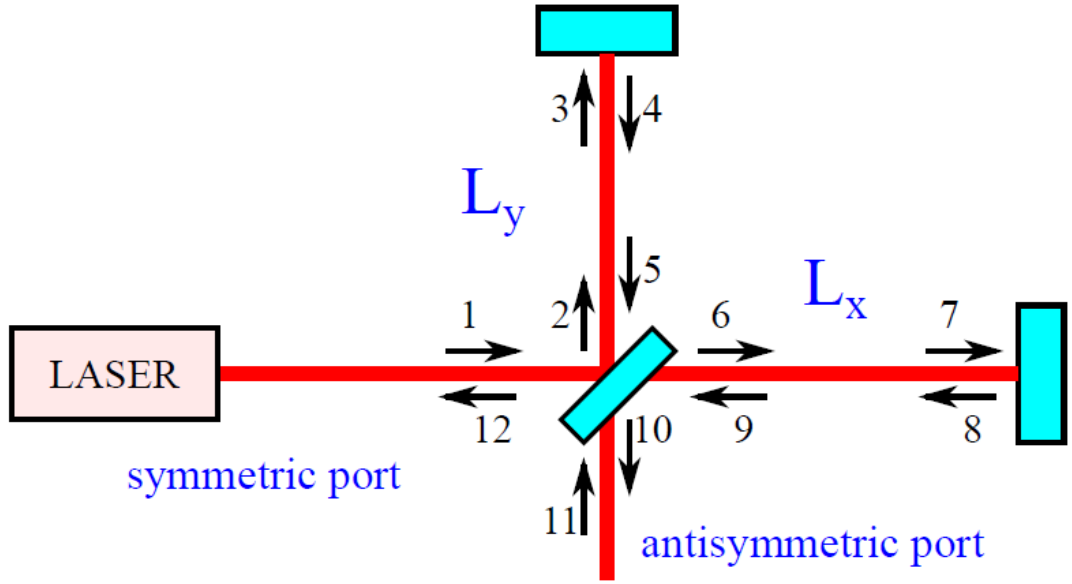
The same happens for any variation of the length (for example a moving mirror)

The Michelson interferometer



Field equations

$$\Psi_2 = \frac{i}{\sqrt{2}}\Psi_1 + \frac{1}{\sqrt{2}}\Psi_{11}$$
$$\Psi_5 = e^{-2ikL_Y} i r_Y \Psi_2$$
$$\Psi_6 = \frac{1}{\sqrt{2}}\Psi_1 + \frac{i}{\sqrt{2}}\Psi_{11}$$
$$\Psi_9 = e^{-2ikL_X} i r_X \Psi_6$$
$$\Psi_{10} = \frac{i}{\sqrt{2}}\Psi_9 + \frac{1}{\sqrt{2}}\Psi_5$$
$$\Psi_{12} = \frac{1}{\sqrt{2}}\Psi_9 + \frac{i}{\sqrt{2}}\Psi_5$$

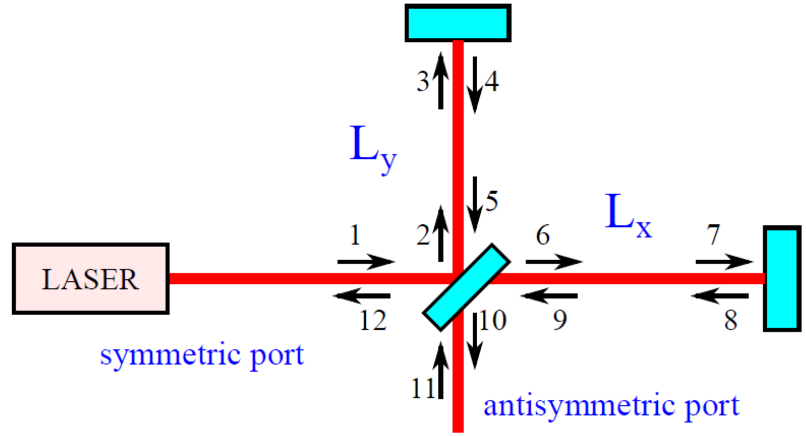


- We are interested in the symmetric (REFL) and anti-symmetric (AS) port fields:

$$\Psi_{ASY} = -\frac{1}{2}\left(r_X e^{-2ikL_X} + r_Y e^{-2ikL_Y}\right)\Psi_1$$
$$\Psi_{SYM} = \frac{i}{2}\left(r_X e^{-2ikL_X} - r_Y e^{-2ikL_Y}\right)\Psi_1$$

- Mirror and length asymmetries are possible

$$r_X = r + \frac{\delta r}{2} \qquad L_X = L + \frac{\delta L}{2}$$
$$r_Y = r - \frac{\delta r}{2} \qquad L_Y = L - \frac{\delta L}{2}$$



$$\Psi_{ASY} = -re^{-2ikL}\left(\cos k\delta L - \frac{i}{2}\frac{\delta r}{r}\sin k\delta L\right)\Psi_1$$

- The minimum power at AS (dark fringe condition) is obtained if

$$\cos k\delta L = 0 \longrightarrow \delta L = (2n+1)\frac{\lambda}{2}$$

- We distinguish between **common** and **differential** motions (microscopic variation around the working point)

$$\delta L = \delta L_0 + d \qquad L = L_0 + c$$

- The response can be computed deriving the ASY field:

$$\frac{d\Psi_{ASY}}{dL} = 2ikr\left[\cos k\delta L_0 - \frac{i}{2}\frac{\delta r}{r}\sin k\delta L_0\right]\Psi_1$$

$$\frac{d\Psi_{ASY}}{d\delta L} = kre^{-2ikL_0}\left[\sin k\delta L_0 + \frac{i}{2}\frac{\delta r}{r}\cos k\delta L_0\right]\Psi_1$$

- If the dark fringe condition is enforced, the ASY field has the maximum sensitivity to differential signals and the minimum to common ones

$$\Psi_{ASY} = re^{-2ikL_0} \frac{2\pi}{\lambda} \left(d - \frac{\delta r}{r} c \right) \Psi_1$$

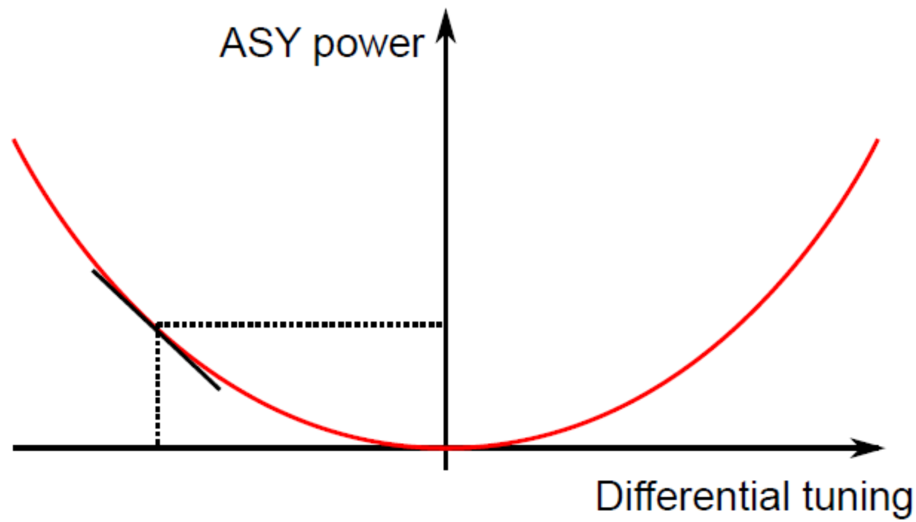
- Asymmetries make the common signal leak into ASY (*common mode rejection ratio*)

$$P_{ASY} = \left[\left(r^2 - \frac{\delta r^2}{4r^2} \right) \cos^2 k\delta L + \frac{\delta r^2}{4r^2} \right] P_{input}$$

- And we have a *contrast defect*

DC readout

- Photodiodes can't detect phase, but GW signal is encoded in the field phase…
- If the Michelson is tuned exactly at dark fringe, then the power at the ASY port is quadratic in the signal: not good!
- Work at a slight detuning: gain a linear dependency of power on the differential motion of the IFO



$$\Psi_{ASY} = \Psi_{DC} + x_0 \left[G(\omega) e^{i\omega t} + G(-\omega) e^{-i\omega t} \right]$$

$$P(t) = \cdots + \left[\Psi_{DC}^* G(\omega) + \Psi_{DC} G^*(-\omega) \right] x_0 e^{i\omega t} + \text{ c.c.} + \ldots$$

Shot noise

- The optical gain is not all the story
- We must compare the signal due to GW to the fundamental limits in the measurement of power
- Shot noise is due to the quantum fluctuation of the laser light (counting statistics of photons)

$$n_S = \sqrt{2 h_P \nu P}$$

- Depends on the power reaching our photodiode
- One might say that at dark fringe the power is zero
- However we can't measure the signal in this case!
- Need to compute the signal to noise ratio

- Power signal due to motion x(t) of one of the mirrors (or GW)

$$P_{AS} = P_0 r^2 \sin^2 k(\delta L + x) + \Delta P$$

$$\frac{dP_{AS}}{dx} = kP_0 r^2 \sin 2k\delta L$$

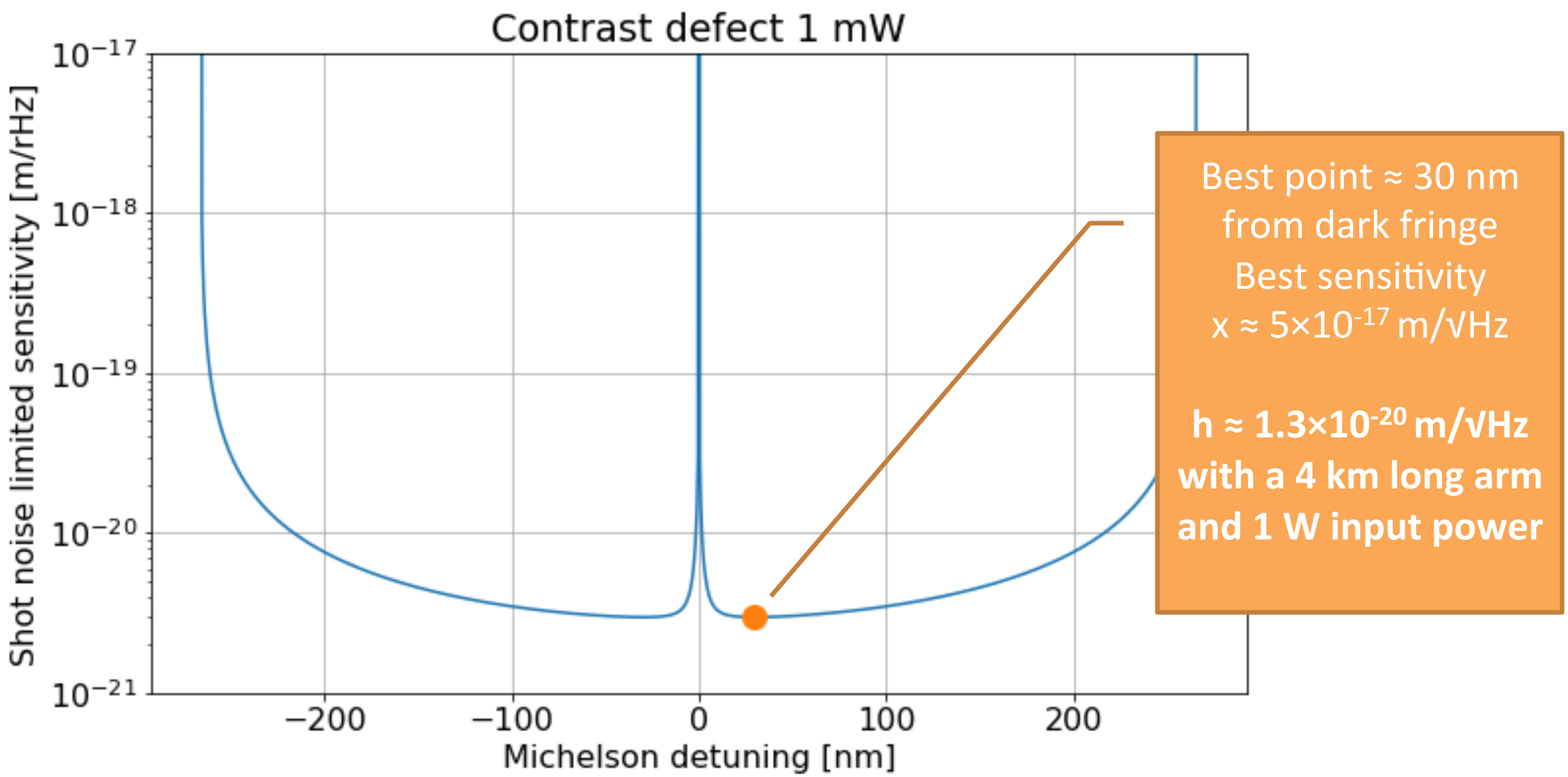
$$\delta P_x = kP_0 r^2 \sin 2k\delta L \cdot x$$

- Power signal due to shot noise

$$\delta P_{sn} = \sqrt{2h\nu \left[P_0 r^2 \sin^2 k\delta L + \Delta P\right]}$$

- Signal to noise ratio

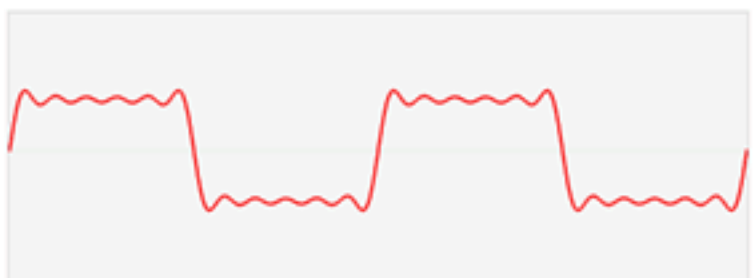
$$SNR = \frac{P_x}{P_{sn}} = \frac{2\pi}{c}\sqrt{\frac{\nu}{2h}} r^2 \sqrt{P_0} \frac{\sin 2k\delta L \; x}{r^2 \sin^2 k\delta L + \Delta P / P_0}$$



$$SNR = \frac{P_x}{P_{sn}} = \frac{2\pi}{c}\sqrt{\frac{\nu}{2h}} r^2 \sqrt{P_0} \frac{\sin 2k\delta L \; x}{r^2 \sin^2 k\delta L + \Delta P / P_0}$$

What is that m/√Hz

- Signals have non trivial distribution of power over various frequencies
- We use a Fourier transform to estimate the amount of power at each frequency
- Easy for signals made of a finite number of sinusoids:



https://www.quora.com/Whats-the-use-of-Fast-Fourier-Transform

What is that m/√Hz

- A bit more complicated for signals that do not have well defined periodicity

$f(x)$

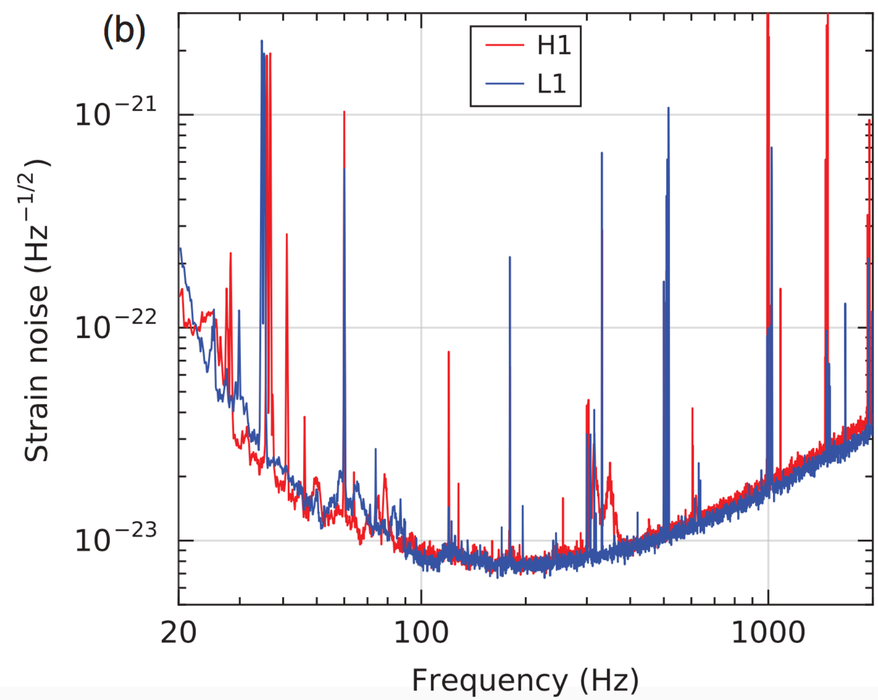
https://www.quora.com/Whats-the-use-of-Fast-Fourier-Transform

What is that m/√Hz

- We start from a signal (in meters), and use the Fourier transform to compute the power distribution with respect to frequency (in meters²/Hz)
  - It's a distribution usually called Power Spectra Density PSD(f)
  - Meaning that the total power in a range $f_1 < f < f_2$ is

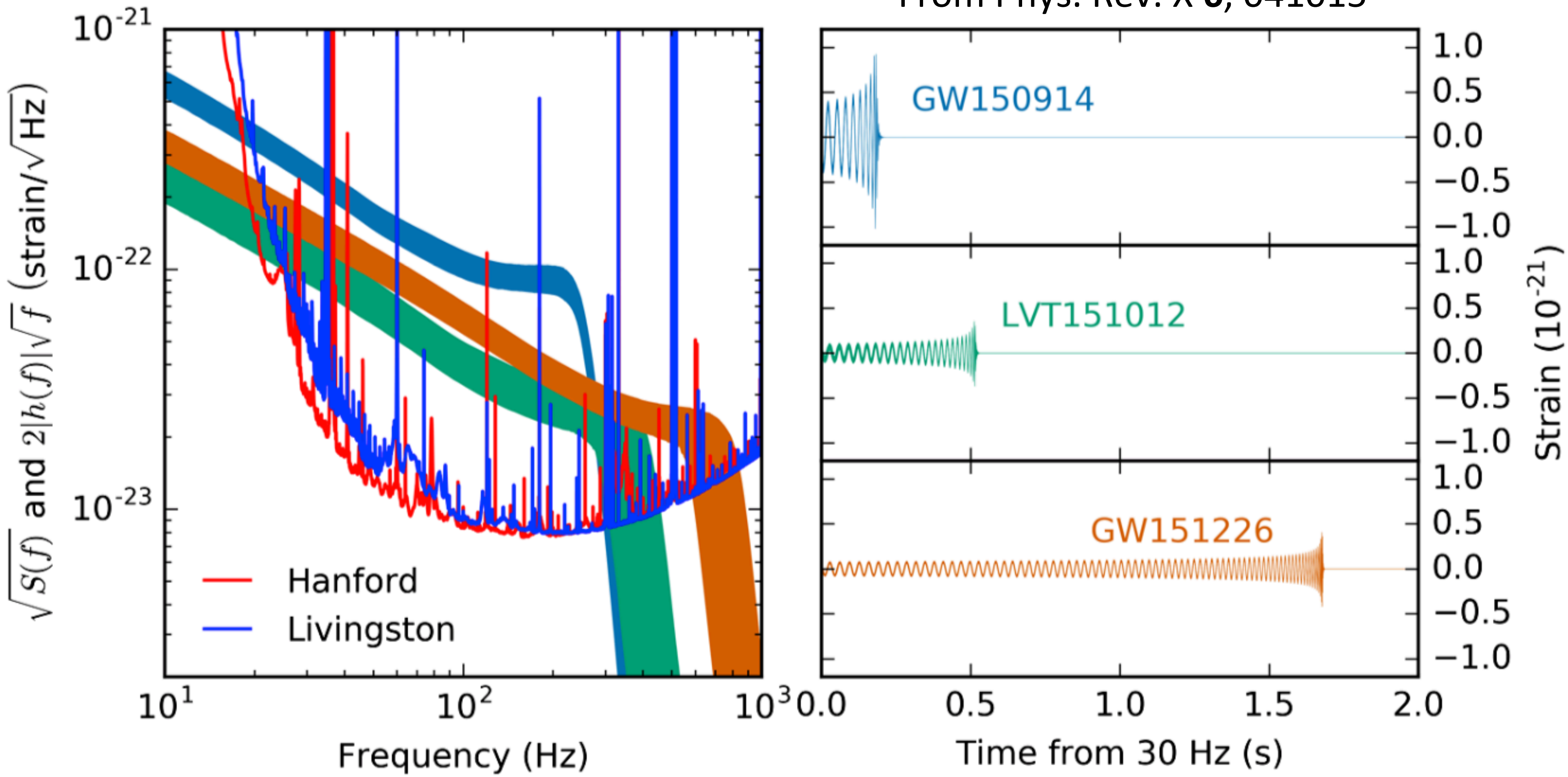
$$P(f_1 < f < f_2) = \int_{f_1}^{f_2} PSD(f)\, df$$

- We often prefer to show the square root, which is linearly proportional to the signal amplitude



GW150914 and friends

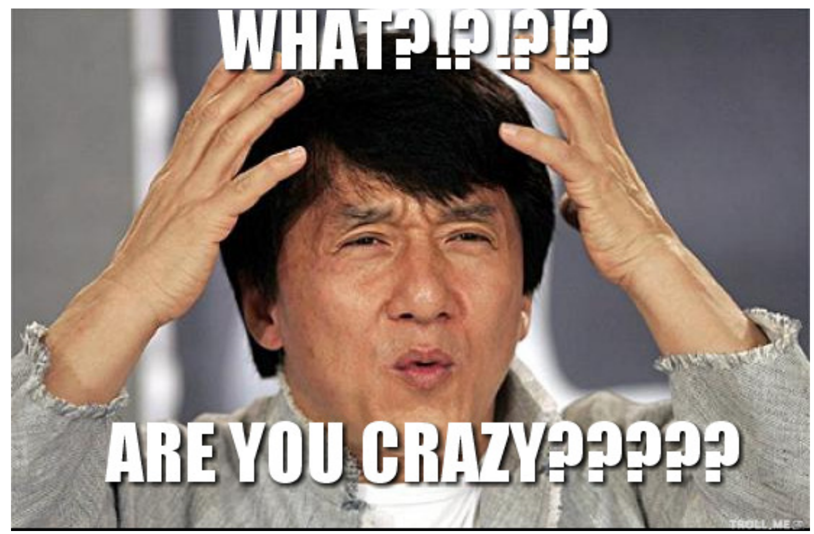
From Phys. Rev. X **6**, 041015



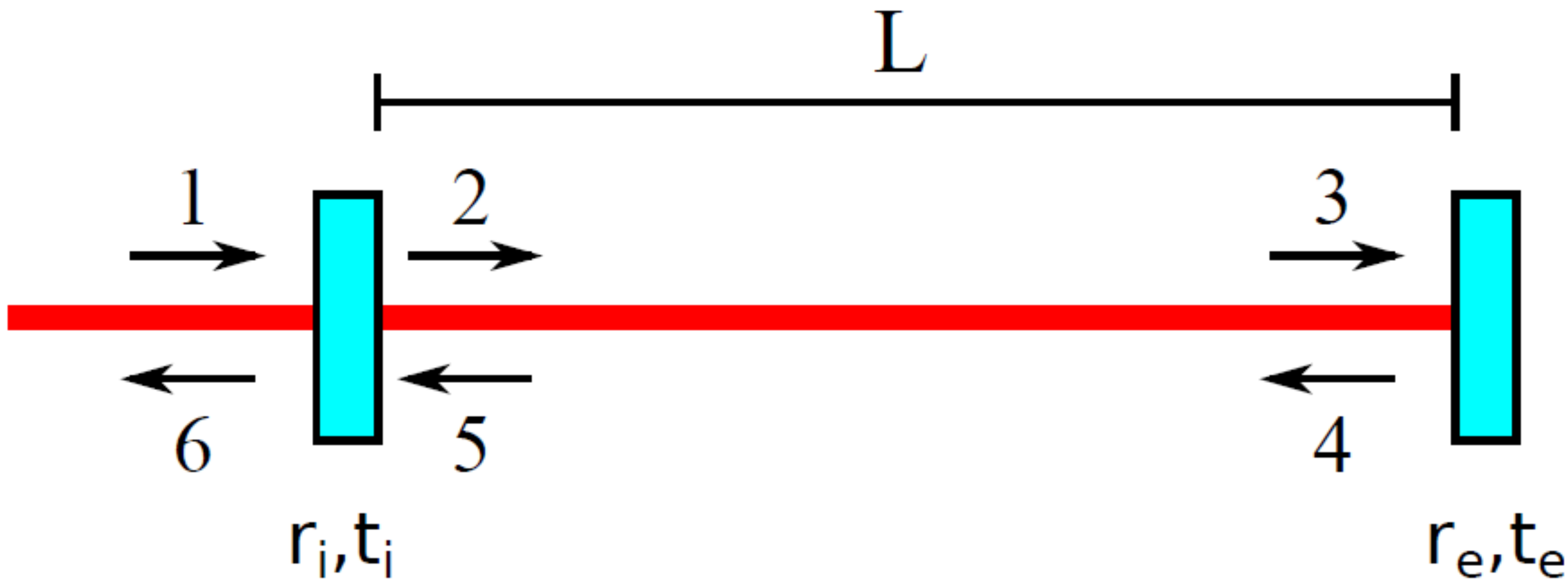
Best Michelson sensitivity:
h ≈ 1.3×10^{-20} m/√Hz

How to improve?

- If we want a SNR of at least 10, we need a sensitivity of about 10^{-23} m/√Hz: missing a factor 1000
- Sensitivity is linearly proportional to length
  - 4km > 4000 km
- Sensitivity is proportional to square root of power
  - 1 W > 1 MW
- Need more clever tricks….



Fabry-Perot cavities

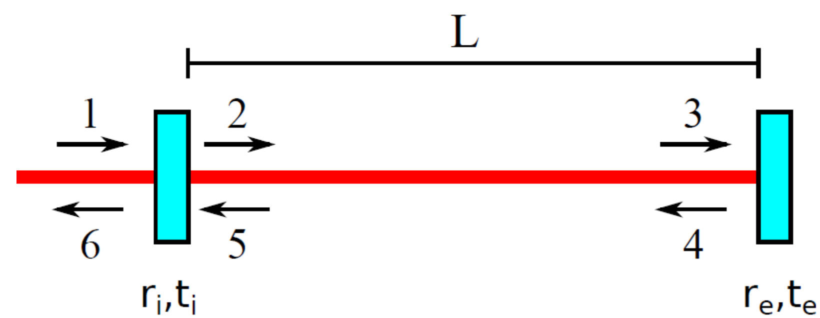


- Two semitransparent mirrors aligned on the optical axis
- Laser field undergoes multiple reflections and interferences

$$\begin{aligned}\Psi_2 &= t_i \Psi_1 - r_i r_e e^{-2ikL}\,\Psi_2 \\ \Psi_5 &= i r_e e^{-2ikL}\Psi_1 \\ \Psi_6 &= t_i \Psi_5 + i r_i \Psi_1\end{aligned}$$

$$\Psi_{cav} = \frac{t_i}{1 + r_i r_e e^{-2ikL}} \Psi_1$$

$$\Psi_{ref} = i \frac{r_i + r_e (t_i^2 + r_i^2) e^{-2ikL}}{1 + r_i r_e e^{-2ikL}} \Psi_1$$

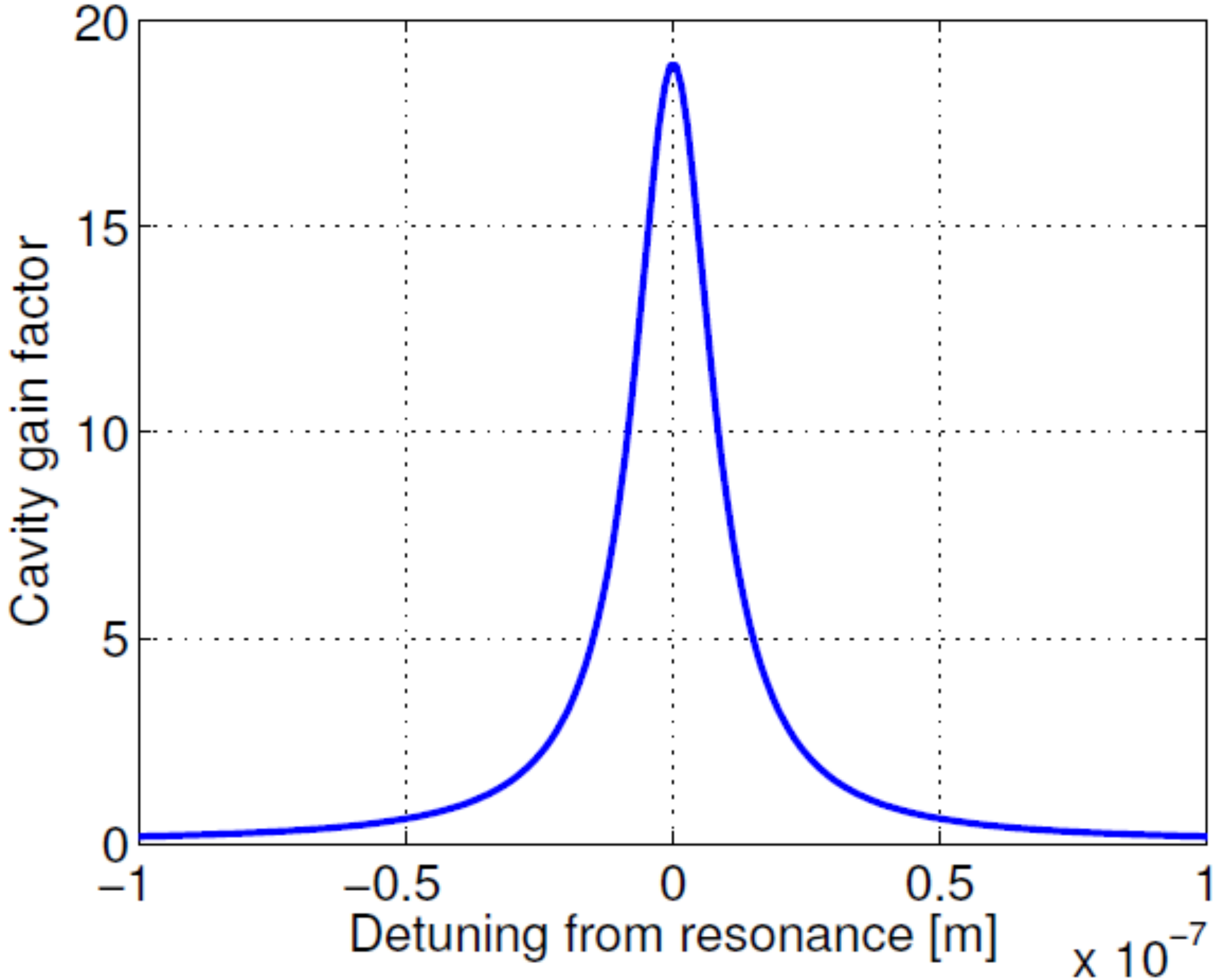


- For a proper tuning of the microscopic length we can get the resonance condition:

$$e^{-2ikL} = -1$$

- That makes the power inside the cavity maximum

$$P_{cav} = \frac{t_i^2}{(1 - r_i r_e)^2} P_{in} = G_{cav} \, P_{in}$$


20
15
10
5
0
Cavity gain factor
−1
−0.5
0
0.5
1
Detuning from resonance [m]
x 10^−7

LIGO

- Resonance happens every *free spectral range*

half wave-length or $$\delta f_{FSR} = \frac{c}{2L}$$

- Considering the detuning from resonance

$$P_{cav} = \frac{t_i^2}{(1 - r_i r_e)^2 + 4 r_i r_e \sin^2 k\delta L} P_{in} = G_{cav} \frac{1}{1 + \left[\frac{2\sqrt{r_i r_e}}{1 - r_i r_e} \sin \frac{2\pi\delta L}{\lambda}\right]^2} P_{in}$$

- We get the *half width at half maximum*

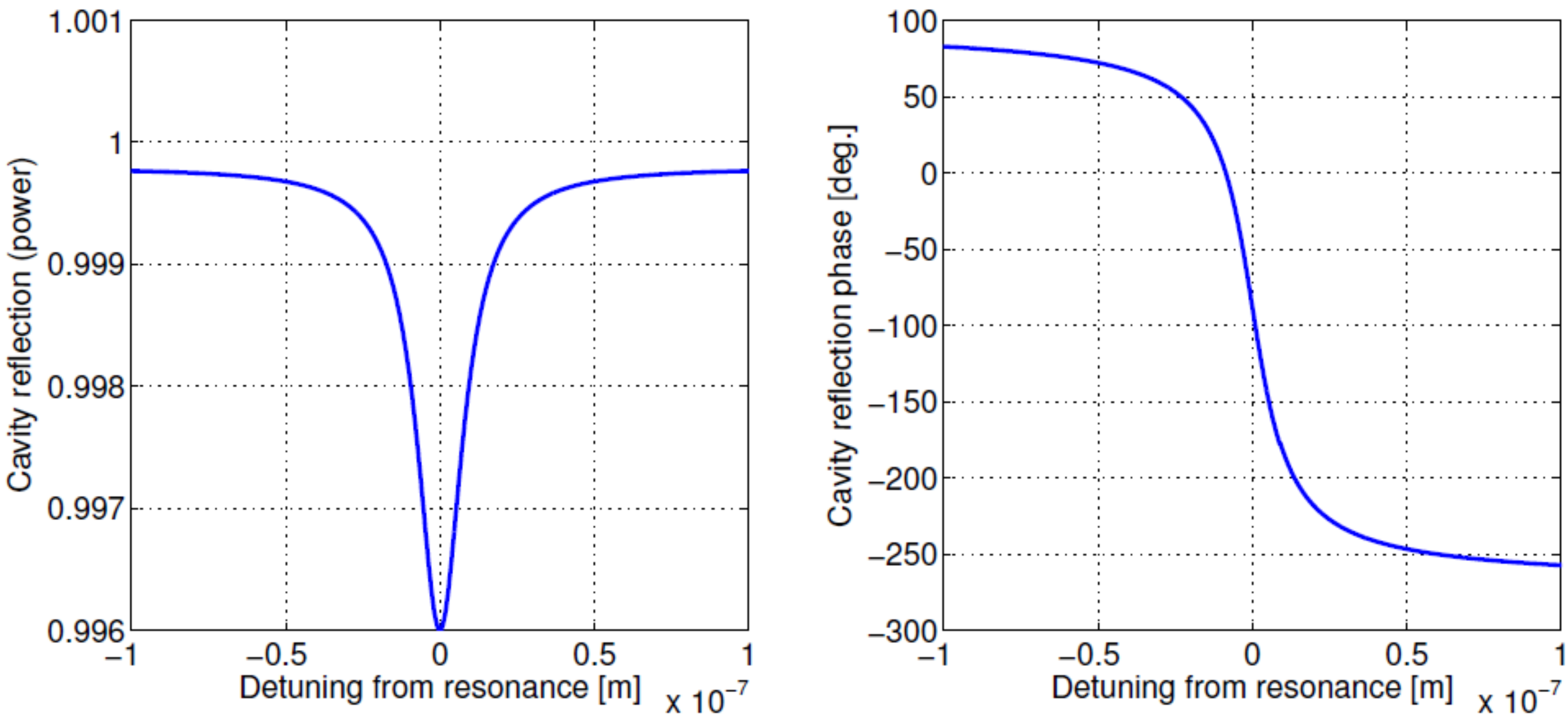
$$\delta L_{HWHM} = \frac{\lambda}{4\mathcal{F}} \qquad \mathcal{F} = \frac{\pi\sqrt{r_i r_e}}{1 - r_i r_e}$$

$$\boxed{P_{cav} = \frac{G_{cav}}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2 \frac{2\pi\delta L}{\lambda}} P_{in}}$$

Fabry-Pérot reflection

$$\Psi_{ref} = i\frac{r_i + r_e(t_i^2 + r_i^2)e^{-2ikL}}{1 + r_i r_e e^{-2ikL}}\Psi_1$$

- Cavity power losses determines the reflectivity



- The phase of the reflected field has a large slope

$$\frac{d\phi}{dx} = \frac{8\mathcal{F}}{\lambda}$$

- To be compared with the same response for a simple propagation over the free distance (as in the Michelson)

$$\frac{d\phi}{dx} = \frac{4\pi}{\lambda}$$

- Using a Fabry-Pérot cavity the **optical gain** is increased proportionally to the finesse

$$\frac{G_{FP}}{G_{space}} = \frac{2\mathcal{F}}{\pi}$$

- We would like to use very high finesse, but everything has a cost
- The largest the finesse, the narrow the resonance is, the more complex it is to control the cavity
- In Advanced LIGO F = 450, so we gain a factor x286 in sensitivity

Frequency response of a FP cavity

- Let's consider one mirror moving.

$$x(t) = x_0 \cos(2\pi f_s t) = x_0 \cos \omega_s t$$

- It creates signal sidebands

$$\begin{aligned}\Psi_4 &= i r_e e^{-2ikx(t)} \Psi_3 = i r_e \Psi_3 + 2k r_e x(t) \Psi_3 \\ &= i r_e \Psi_3 + 2\pi \frac{x_0}{\lambda} \left(e^{-i\omega_s t} + e^{i\omega_s t}\right) \Psi_3\end{aligned}$$

- Focusing on the signal sidebands:

$$\Psi_4(\omega_s) = i r_e i r_i e^{-2i\left(k+\frac{\omega_s}{c}\right)}\Psi_4(\omega_s) + 2 r_e \pi \frac{x_0}{\lambda}\Psi_3(0)$$

$$\Psi_4(\omega_s) = \frac{2\pi\Psi_3(0)}{1 - r_i r_e e^{-2i\frac{\omega_s}{c}L}}\frac{x_0}{\lambda}$$

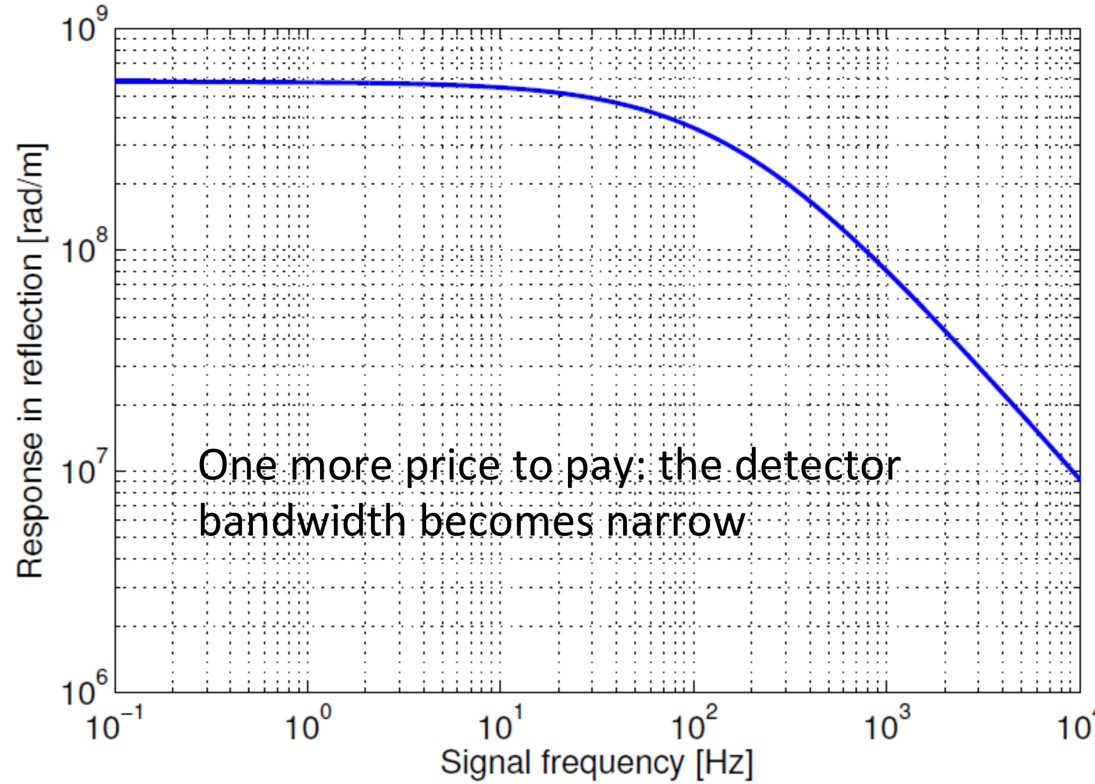
$$\Psi_R(\omega_s) = \left[\frac{2\pi}{\lambda} r_e \frac{i t_i e^{-i\frac{\omega_s}{c}L}}{1 - r_i r_e e^{-2i\frac{\omega_s}{c}L}} \frac{i t_i}{1 - r_i r_e}\Psi_1\right] x_0$$

$$\Psi_R(\omega_s) = -\frac{2\pi}{\lambda} r_e \frac{1}{1 + i\frac{f_s}{\frac{c}{4L\mathcal{F}}}} \frac{t_i^2}{(1 - r_i r_e)^2}\Psi_1 \, x_0$$

Using the high finesse approximation

LIGO

- The signal sideband amplitude is proportional to the field stored inside the arm
- It is then filtered with a low pass at the cavity pole frequency (created by the interference conditions of the sidebands)



$$\Psi_R(\omega_s) = -\frac{2\pi}{\lambda} r_e \frac{1}{1 + i\frac{f_s}{\frac{c}{4L\mathcal{F}}}} \frac{t_i^2}{(1 - r_i r_e)^2} \Psi_1 \, x_0$$

How to improve?

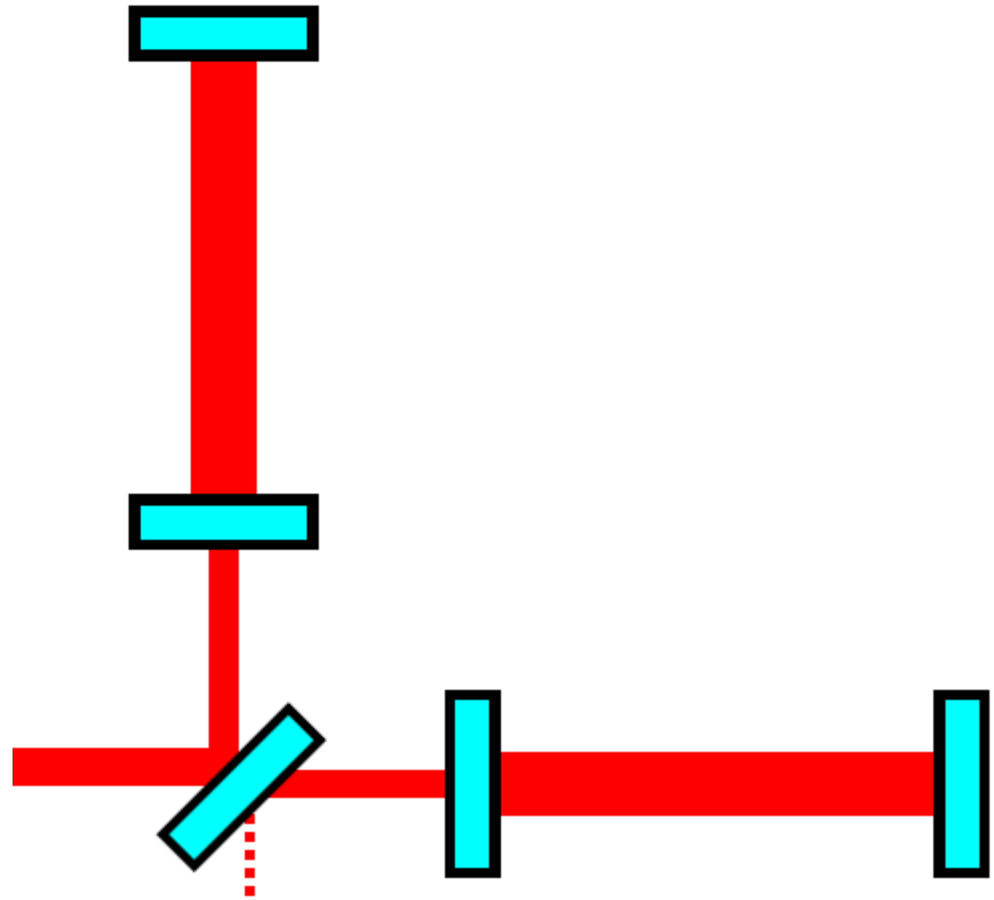
- If we want a SNR of at least 10, we need a sensitivity of about 10^{-23} m/√Hz: missing a factor 1000
- Sensitivity is linearly proportional to length
  - ~~4km ... km~~

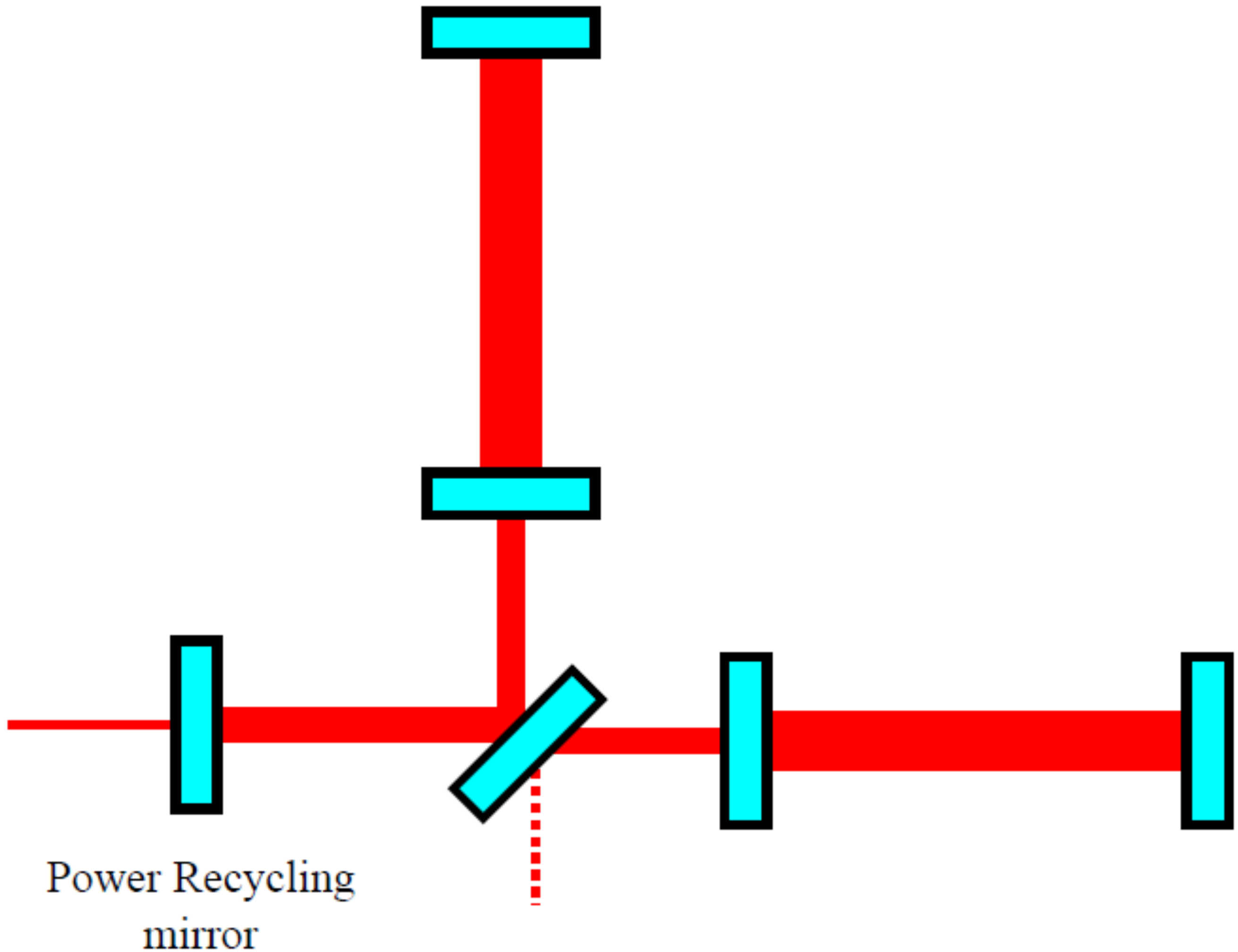
Fabry-Perot cavities, finesse 450, we gain a factor 290 in “effective length”
(paying the price of reduced bandwidth)

- Sensitivity is proportional to square root of power
  - 1 W > 1 MW
- Need more clever tricks….

Power recycling

- End mirrors are almost completely reflective
- Power lost inside the arms is not too large
- Antisymmetric port is at dark fringe (no power exits through it)
- Where does the laser power go?




Power Recycling
mirror

- Michelson interferometer acts like a "effective" mirror with high reflectivity determined by the Fabry-Perot cavities

$$R = -i(1 - \frac{\mathcal{F}}{\pi} L)$$

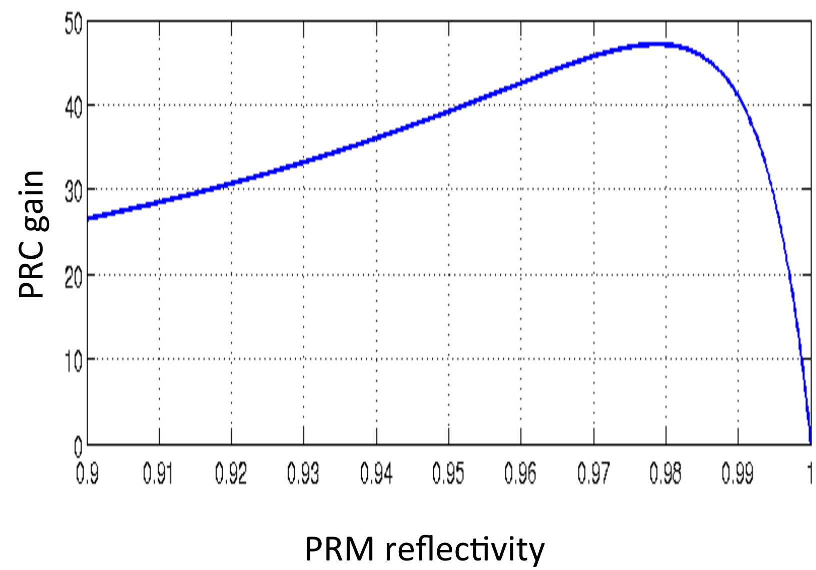
L = relative power loss in the Fabry-Perot cavities (i.e. scattering, absorption, transmission of end mirrors, ...)

- And form a resonant cavity with the PR mirror with gain:

$$G_{rec} = \frac{t_R^2}{\left[1 - r_R\left(1 - \frac{\mathcal{F}}{\pi} L\right)\right]^2}$$

- Which is maximum when the PR reflectivity matches the FP reflectivity:

$$G_{max} = \frac{\pi}{2\mathcal{F}L}$$



- The IFO response to differential signals (like GW) is unchanged by the presence of the power recycling cavity
- All differential signal sidebands have sign difference when coming back to the beam splitter
- Destructive and constructive interference is reversed
- Differential signal sidebands exits only through the anti-symmetric port
- They don't see the power recycling cavity

- The only effect of the power recycling cavity is to increase the power that is injected inside the arms

- In Advanced LIGO $G_{rec} \approx 30$

How to improve?

- If we want a SNR of at least 10, we need a sensitivity of about 10^{-23} m/√Hz: missing a factor 1000
- Sensitivity is linearly proportional to length
  - 4km [illegible] km

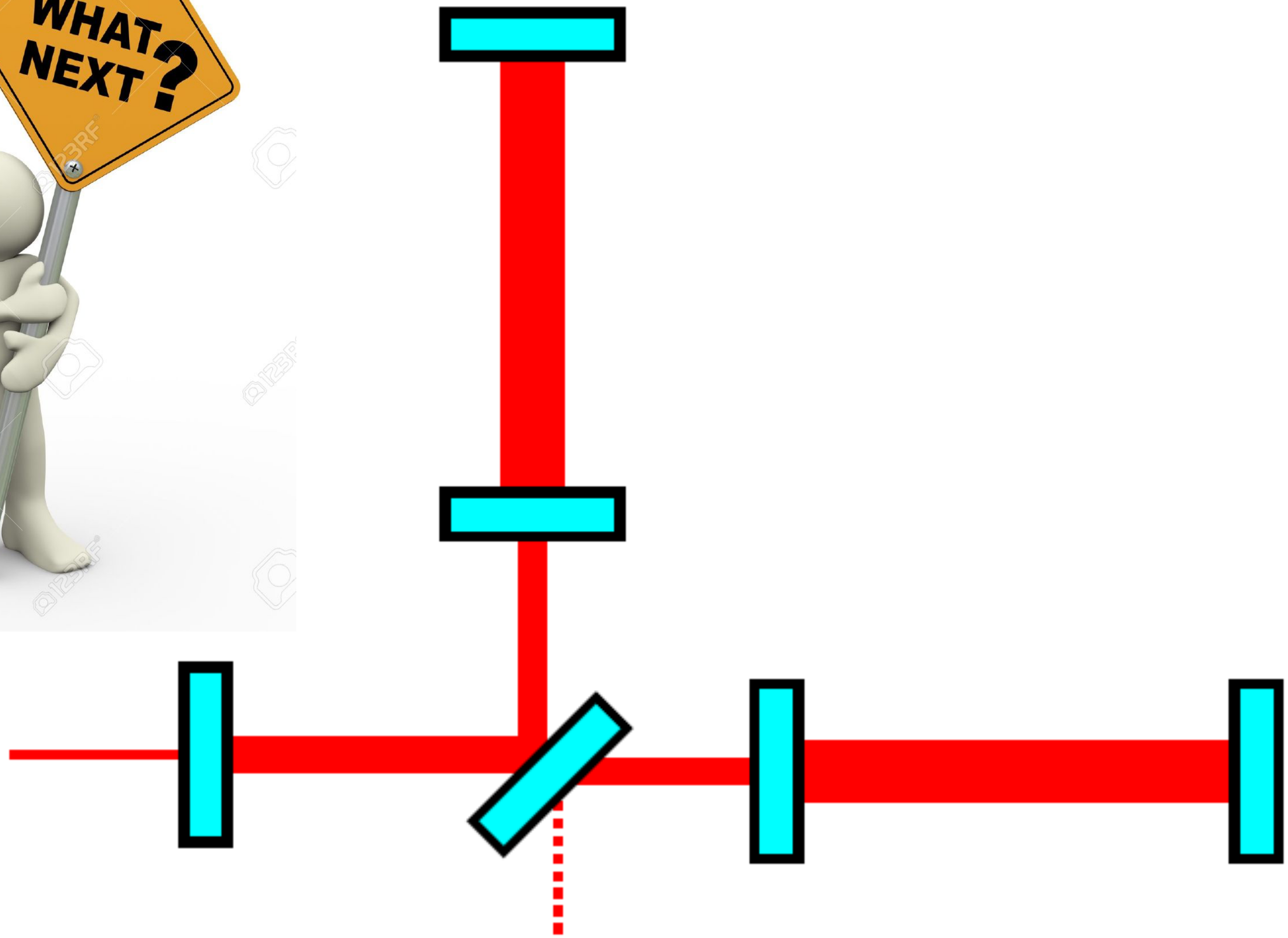
Fabry-Perot cavities, finesse 450, we gain a factor 290 in "effective length"
(paying the price of reduced bandwidth)

- Sensitivity is proportional to square root of power
  - 1 [illegible] W

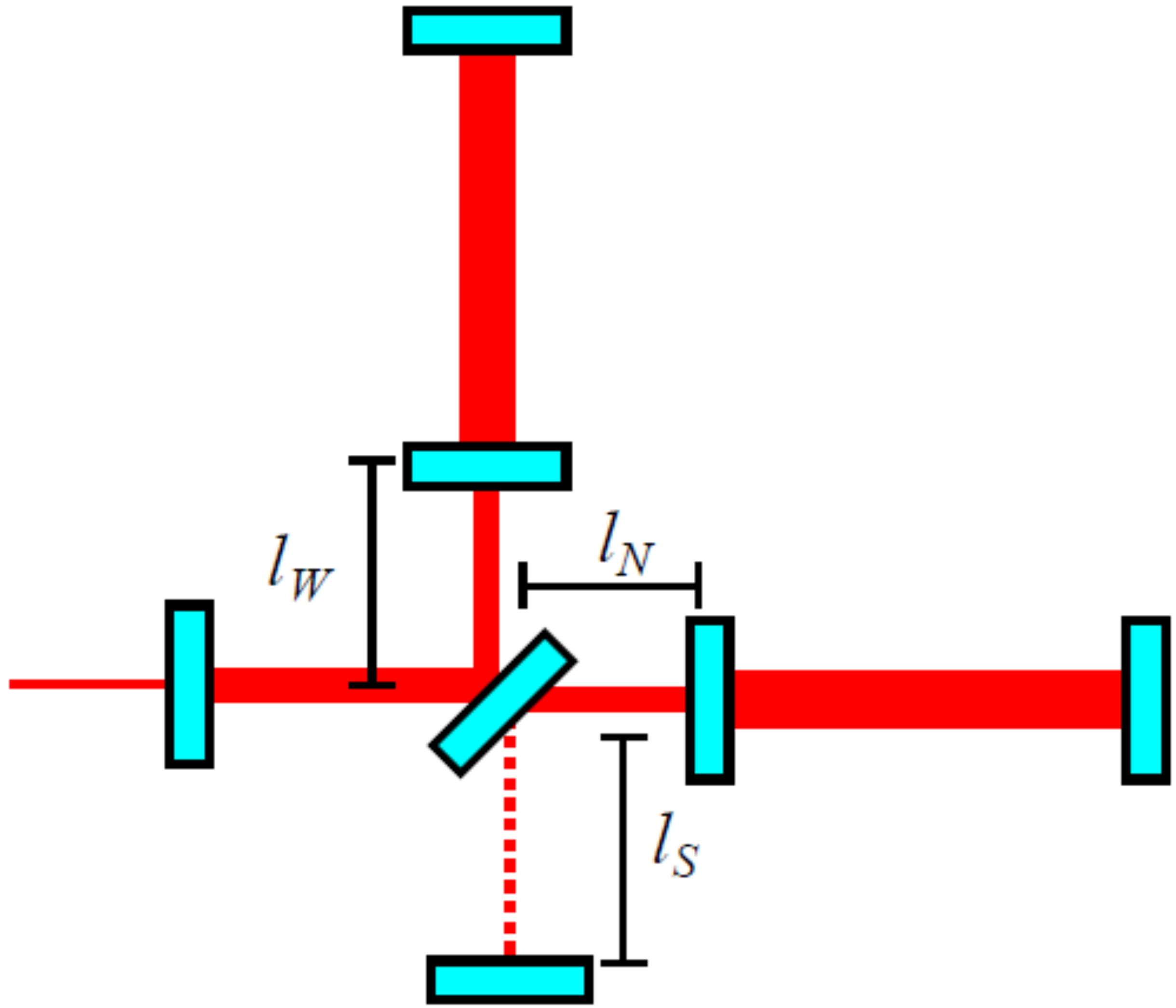
Input power ~20 W, recycling gain ~30, effectively having 600 W on the beam splitter. Sensitivity improved by x25

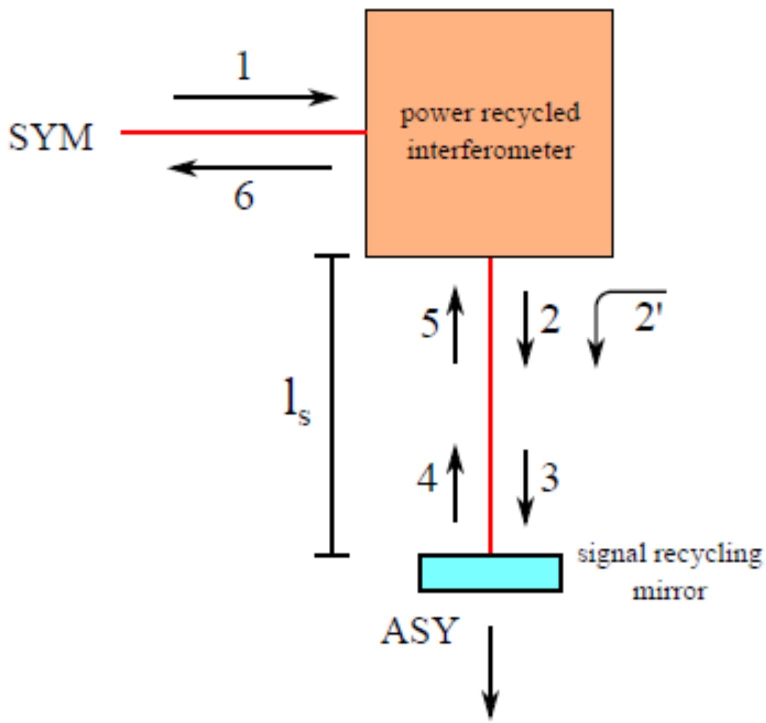
- Need more clever tricks….

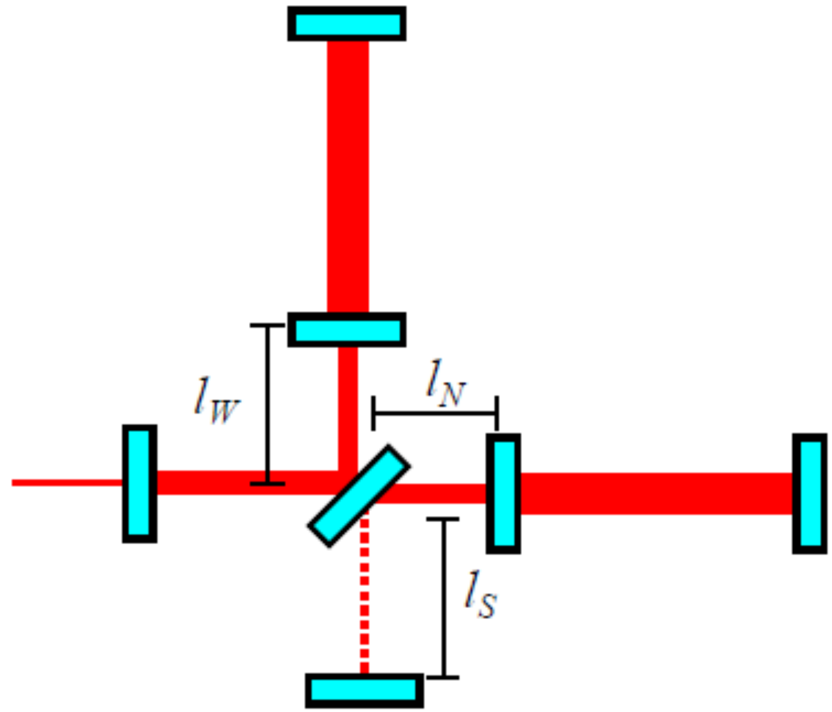
This was the iLIGO and eLIGO configuration



Signal recycling



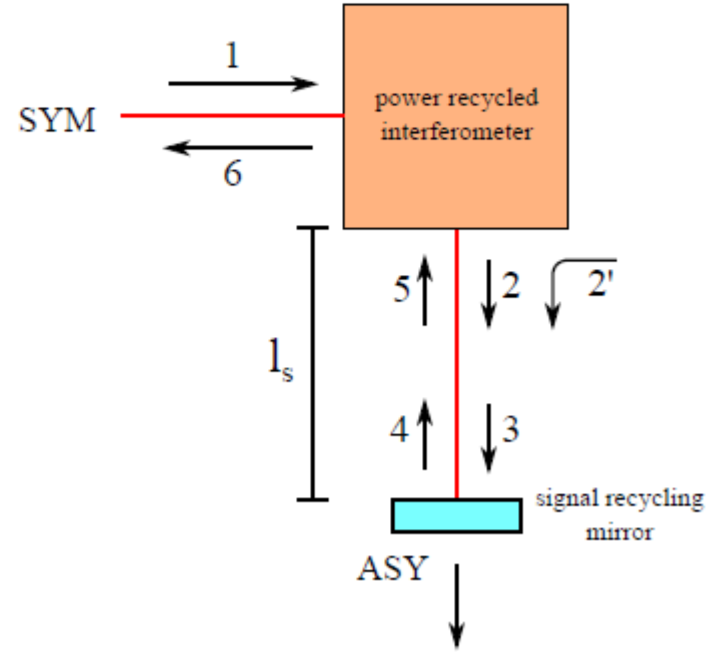




- The signal recycling cavity is seen only by the differential signal sidebands
- The signal recycling mirror does not affect the power stored inside the IFO

LIGO

- Power recycled IFO can be considered as a box with known reflectivity
- The signal sidebands are also "injected" at the anti-symmetric port



$$\Psi_2'(\omega_s) = -e^{i\pi/4 - ikl_+}\frac{2t_R}{1-r_R R}\Psi_{in}\frac{t_i^2 r_e}{1-r_i r_e}\frac{e^{-i\frac{\omega_s}{c}L}}{1-r_i r_e e^{-2i\frac{\omega_s}{c}L}}$$

$$\begin{aligned}\Psi_2(\omega_s) &= \Psi_2'(\omega_s) + i r_S R_{ASY}(\omega_s) e^{-2i\left(k+\frac{\omega_s}{c}\right)l_s}\Psi_2(\omega_s)\\ \Psi_4(\omega_s) &= t_s e^{-i\left(k+\frac{\omega_s}{c}\right)l_s}\Psi_2(\omega_s)\end{aligned}$$

$$\Psi_4(\omega_s) = t_s \frac{e^{-i\left(k+\frac{\omega_s}{c}\right)l_S}}{1 - i r_S r_{FP}(\omega_s) e^{-i\left(k+\frac{\omega_s}{c}\right)l_{SRC}}} \Psi_2'(\omega_s)$$

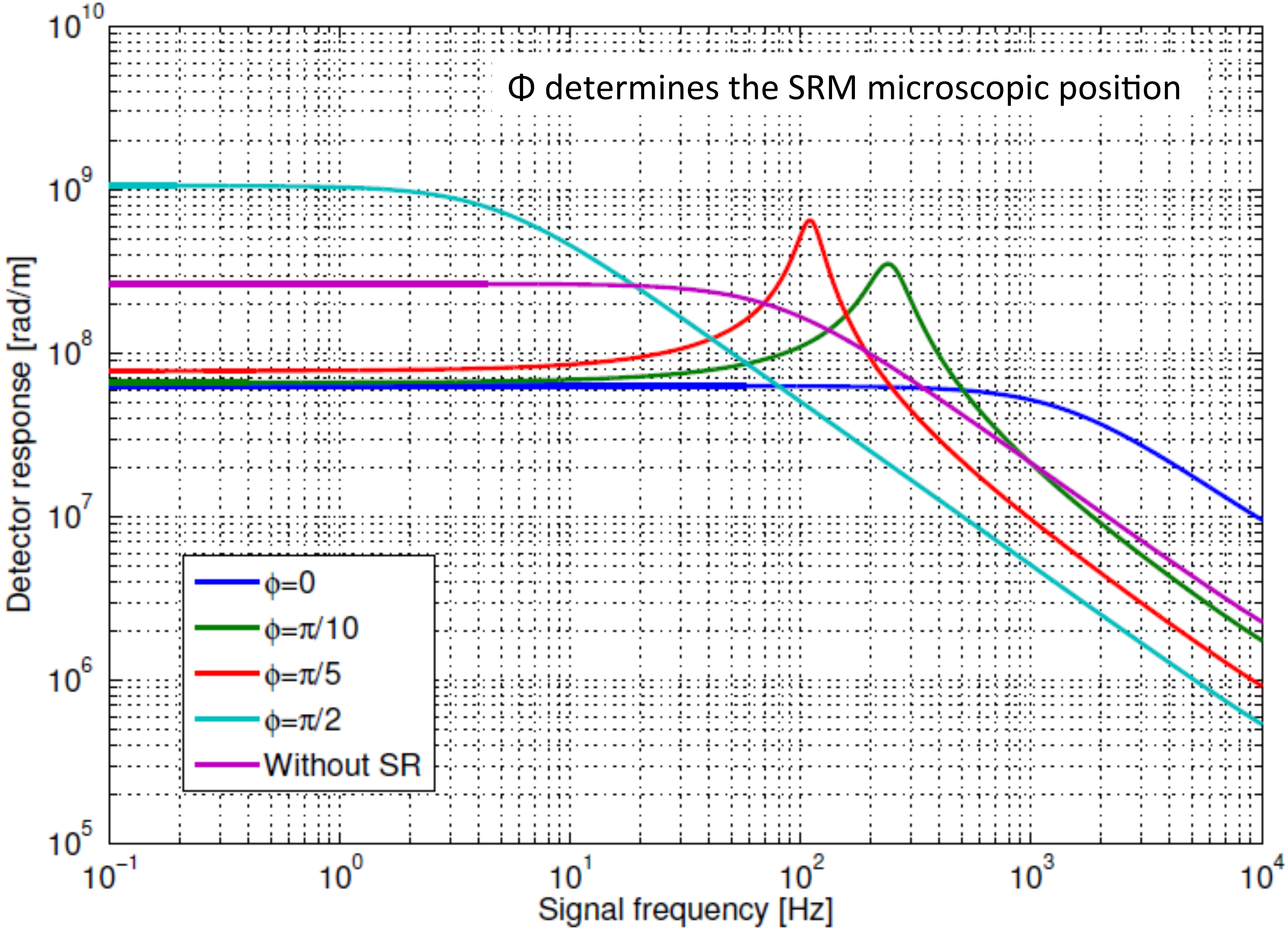
Introducing the conventional *signal recycling tuning*

$$\phi = k l_{SRC} + \frac{\pi}{4}$$

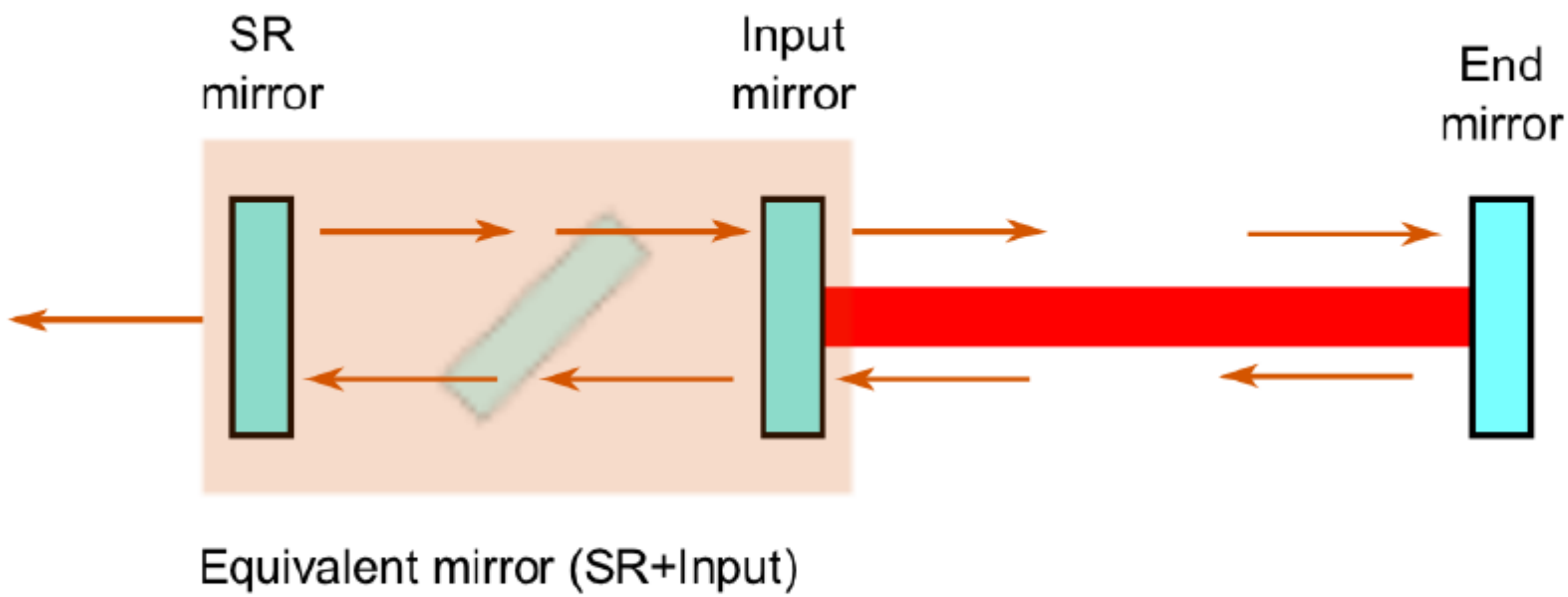
$$\Psi_{ASY}(\omega_s) = -\frac{2 t_S t_R}{1 - r_R R} \Psi_{in} \frac{t_i^2 r_e}{1 - r_i r_e} \frac{e^{i\phi}}{1 - r_S r_{FP}(\omega_s) e^{2i\phi}} \cdot \cdot \frac{e^{-i\frac{\omega_s}{c}L}}{1 - r_i r_e e^{-2i\frac{\omega_s}{c}L}} \frac{2\pi}{\lambda} x_0$$

$$r_{FP}(\omega_s) = \frac{r_i - r_e(1-L) e^{-2i\frac{\omega_s}{c}L}}{1 - r_i r_e e^{-2i\frac{\omega_s}{c}L}}$$

$$R = \frac{r_i - r_e(1-L)}{1 - r_i r_e}$$


Φ determines the SRM microscopic position
Detector response [rad/m]
Signal frequency [Hz]
φ=0
φ=π/10
φ=π/5
φ=π/2
Without SR
10^{10}
10^{9}
10^{8}
10^{7}
10^{6}
10^{5}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}

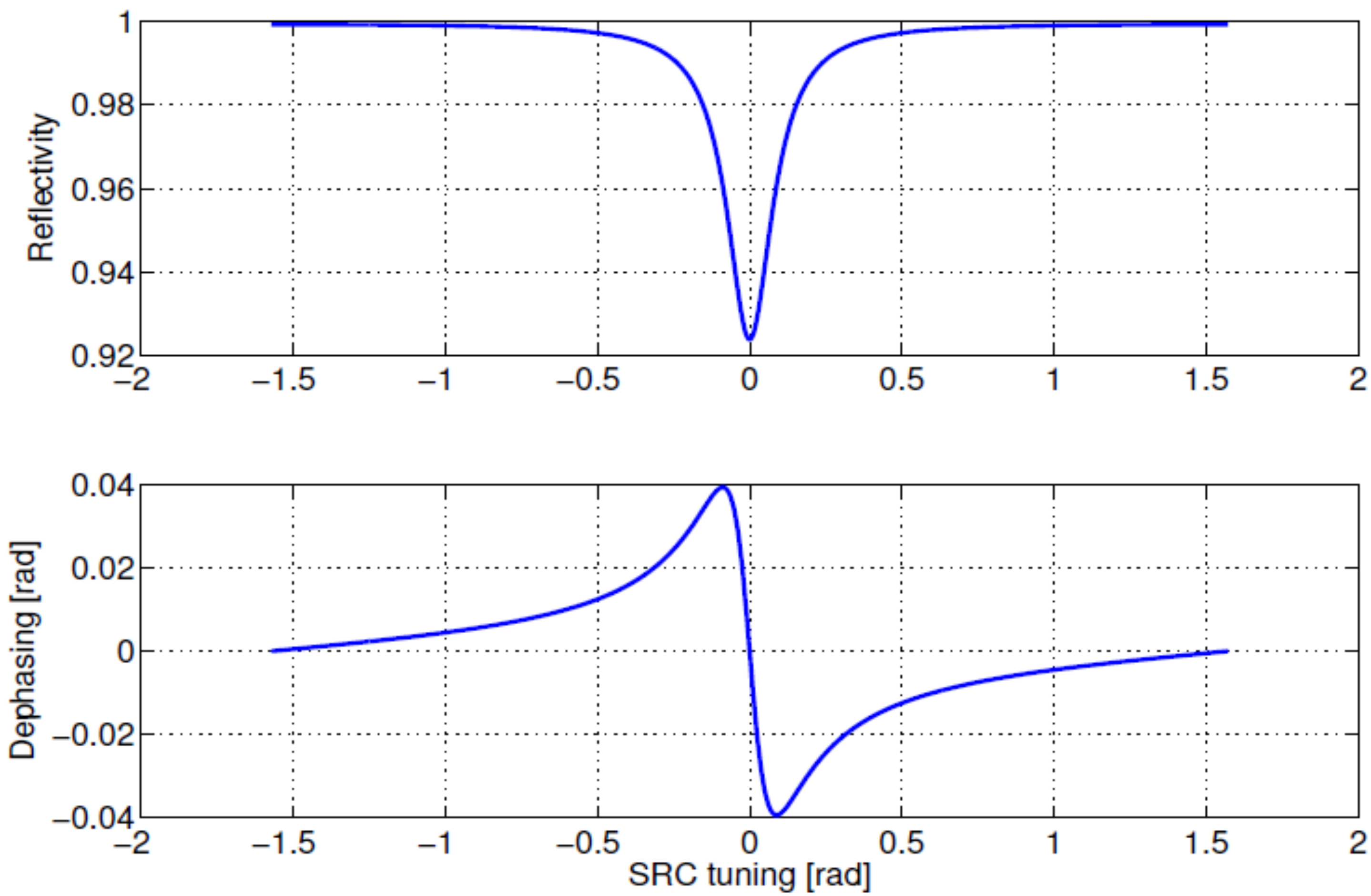
Intuitive understanding of SRC



- Input mirror plus signal recycling makes an “equivalent input mirror” for the Fabry-Perot cavity (seen only by differential sidebands)

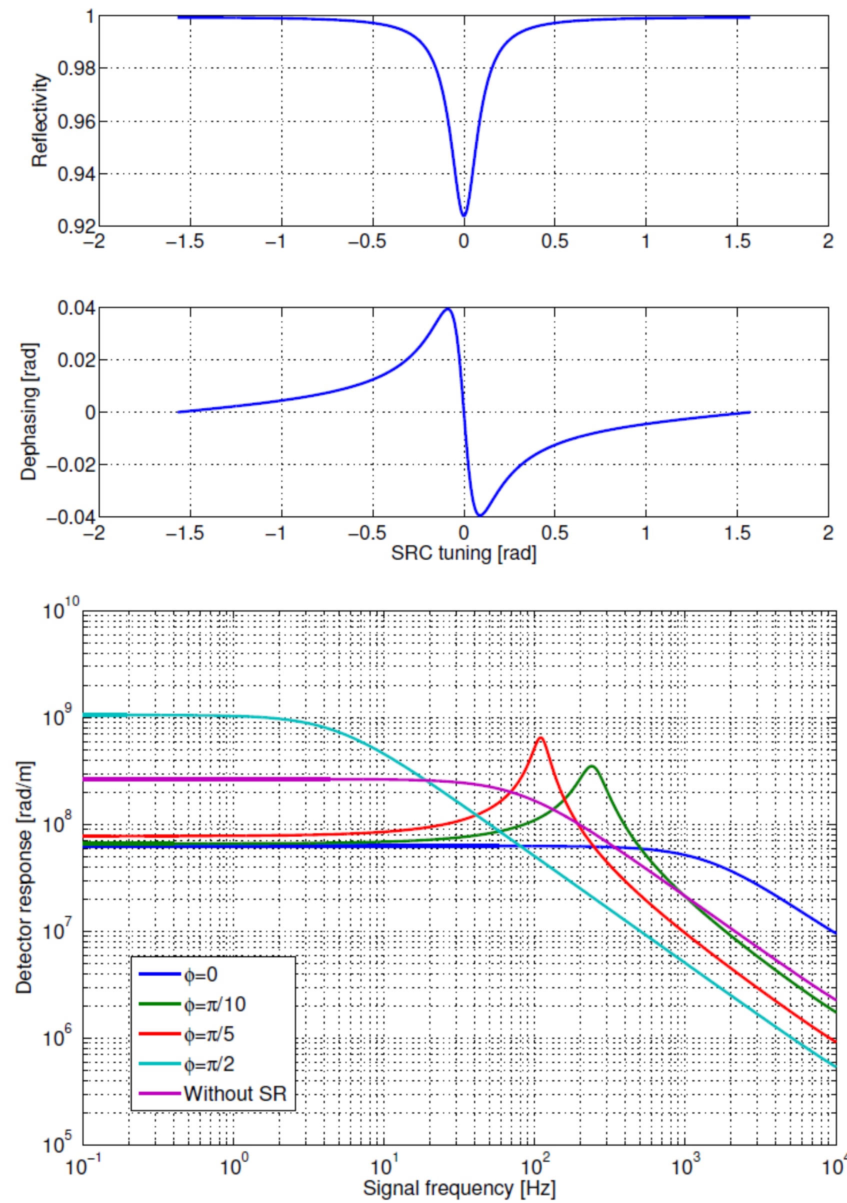
$$R = i\frac{r_i - r_s e^{2i\phi}}{1 - r_i r_s e^{2i\phi}} \qquad T = \frac{t_i t_s e^{i\phi}}{1 - r_i r_s e^{2i\phi}}$$

- Depending on the SRC tuning the "equivalent" mirror has varying reflectivity and additional de-phasing (independent of signal frequency because the cavity is short)



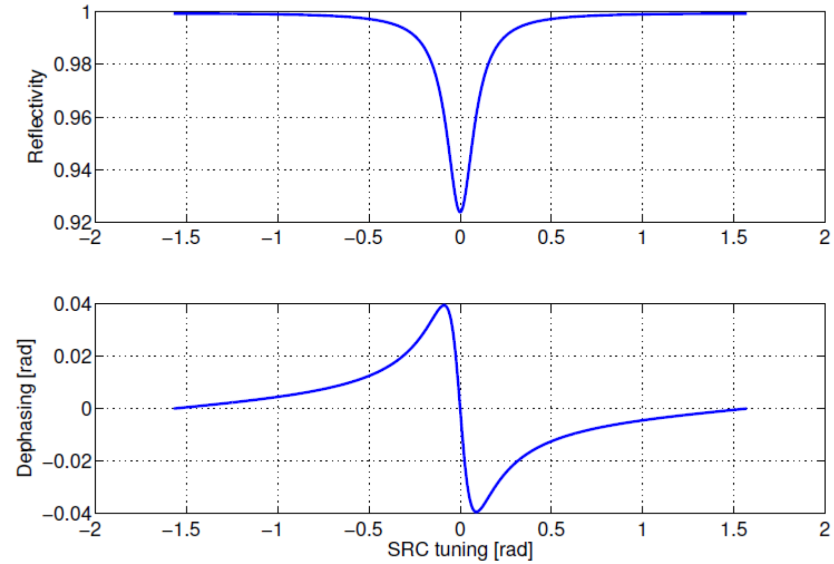
$\phi = 0$

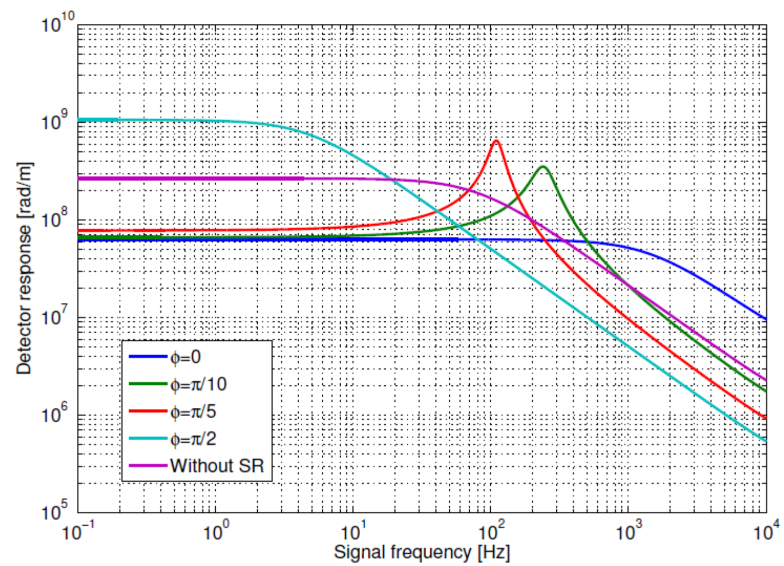
- The reflectivity is the lowest possible and the phase is zero
- Differential signal sidebands see a mirror with reduced reflectivity
- They are still resonant
- The Fabry-Perot cavity bandwidth is increased (broad-band configuration **blue curve**)



$\phi = \pi/2$

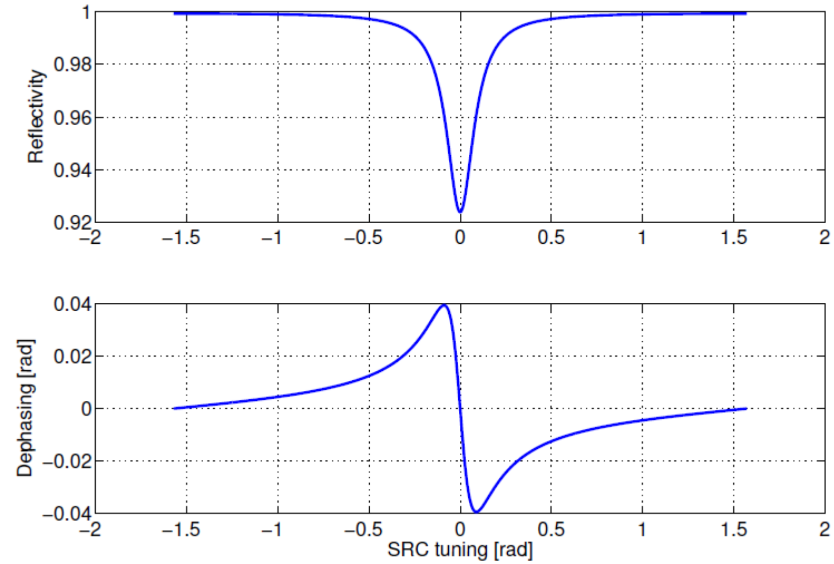
- Equivalent mirror reflectivity is high (SRC out of resonance)
- De-phasing is again zero
- Differential sidebands see a Fabry-Perot cavity at resonance, with very high finesse, reducing the band-width
- (Narrow-band configuration, **light blue curve**)

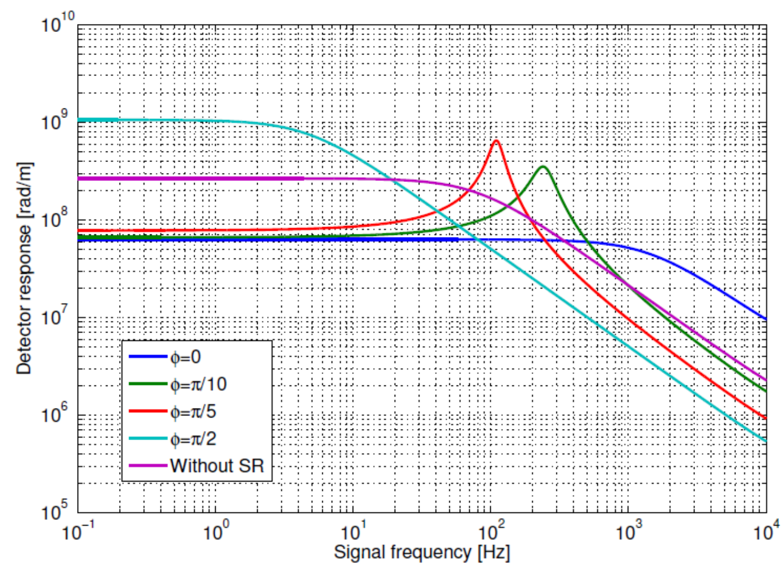




Other detunings

- The reflectivity is intermediate, but there is an additional de-phasing which moves the differential sidebands at zero frequency out of resonance
- But signal sidebands get resonant for a non zero frequency (only one of the two sidebands for a given tuning)
- That frequency is amplified (detuned configuration, **red** and **green** curves)





How to improve?

- If we want a SNR of at least 10, we need a sensitivity of about 10^{-23} m/√Hz: missing a factor 1000
- Sensitivity is linearly proportional to length
  - 4km … km
- Sensitivity is proportional to square root of power
  - 1 … W
- Need more clever tricks….

Fabry-Perot cavities, finesse 450, we gain a factor 290 in "effective length"
(p … n)

Signal recycling

Input power ~20 W, recycling gain ~30, effectively having 600 W on the beam splitter. Sensitivity improved by x25

Putting it all together: Advanced LIGO

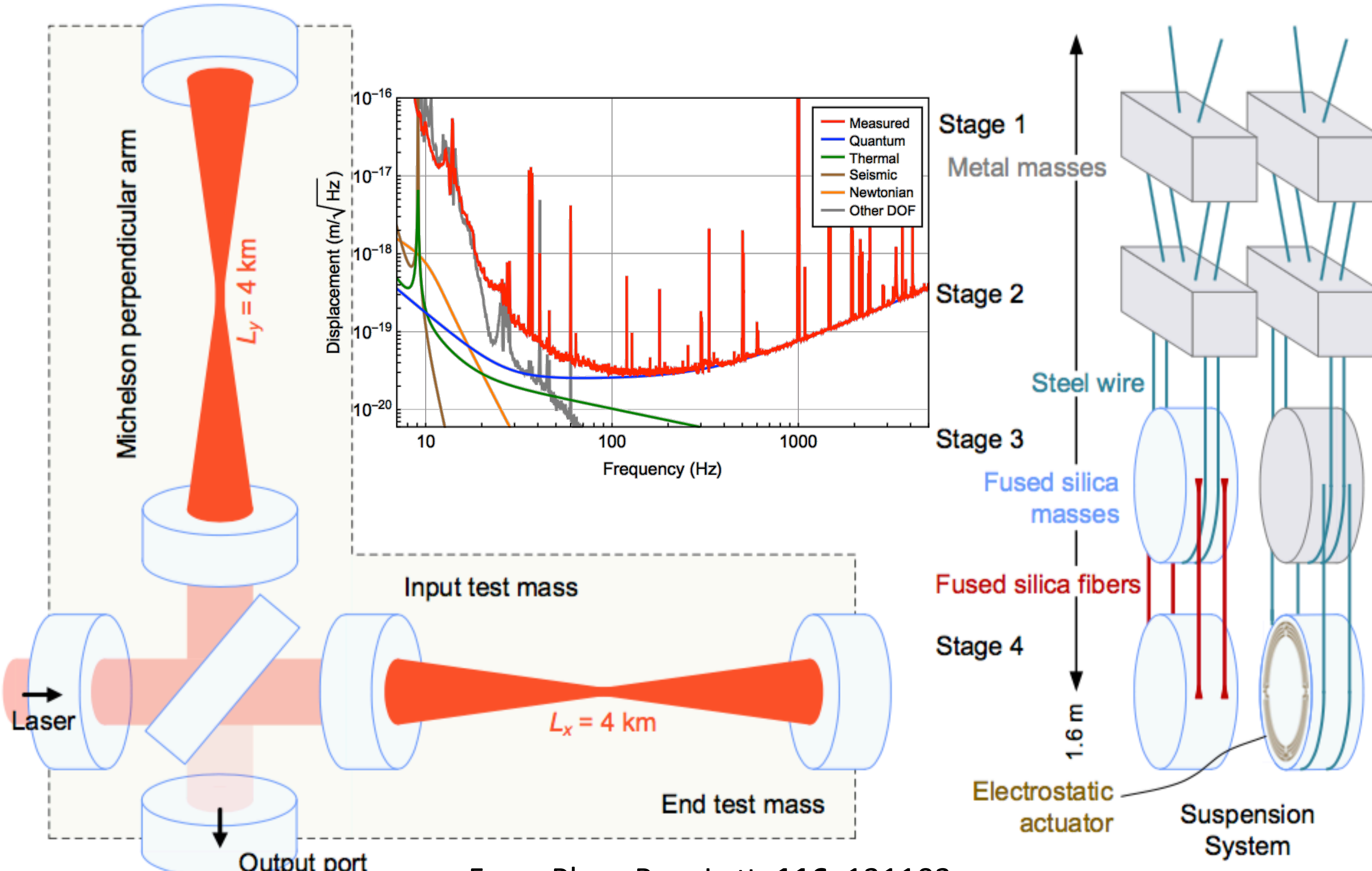
High power laser

Fabry-Perot cavities

Signal recycling

Power recycling

Advanced LIGO



From Phys. Rev. Lett. **116**, 131103

Advanced LIGO design sensitivity

- 125 W input laser power
- Fabry-Perot cavities Finesse 450
- Power and signal recycling cavities
- Quadruple suspension and active seismic isolation for test masses
- 40 km fused silica test masses with state of the art coatings