Cross coupling math
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T1700229-v2

1. base equations
2. matrix description
3. system drawing
4. simplify products
5. work closed loop from drive to platform motion
6. extend to sensor readout

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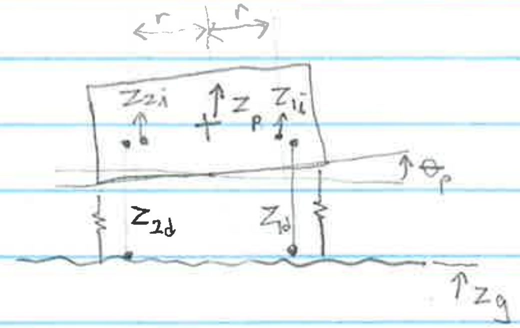


platform position Z_p, θ_p
ground position Z_g θ_g
inertial sensors Z_{1i}, Z_{2i}
displacement sensors Z_{1d} Z_{2d}

sensors have scale factors: S_{1i}, S_{2i}, S_{1d}, S_{2d}
let absolute performance be accurate, but with some small differential readback, so

$$Z_{1i} = (Z_p + r\theta_p) S_{1i} \qquad Z_{2i} = (Z_p - r\theta_p) S_{2i}$$

let $S_{1i} = 1 + \Delta_i$, $S_{2i} = 1 - \Delta_i$

$S_{1d} = 1 + \Delta_d$ $\qquad S_{2d} = 1 - \Delta_d$

$$Z_{1i} = Z_p + r\theta_p + \Delta_d Z_p + \Delta_d r\theta_p$$

$$Z_{2i} = Z_p - r\theta_p - \Delta_d Z_p + \Delta_d r\theta_p$$

$$Z_{1d} = \left[(Z_p + r\theta_p) - (Z_g + r\theta_g) \right] S_{1d}$$

let $\theta_g = 0$, $S_{1d} = 1 + \Delta_d$

$$= (Z_p - Z_g) + r\theta_p + (Z_p - Z_g)\Delta_d + r\theta_p \Delta_d$$

$$Z_{2d} = \left[(Z_p - r\theta_p) - (Z_g - r\theta_g) \right] \cdot S_{2d} \quad , \quad S_{2d} = 1 - \Delta d$$

$$= (Z_p - Z_g) - r\theta_p - (Z_p - Z_g)\Delta_d + r\theta_p \Delta_d$$

Cant Basis Signals

Z_{ci} Z motion, cart basis, inertial sensor

$$Z_{ci} = \tfrac{1}{2}(Z_{1i} + Z_{2i})$$
$$= Z_p + \Delta_i r\theta_p$$

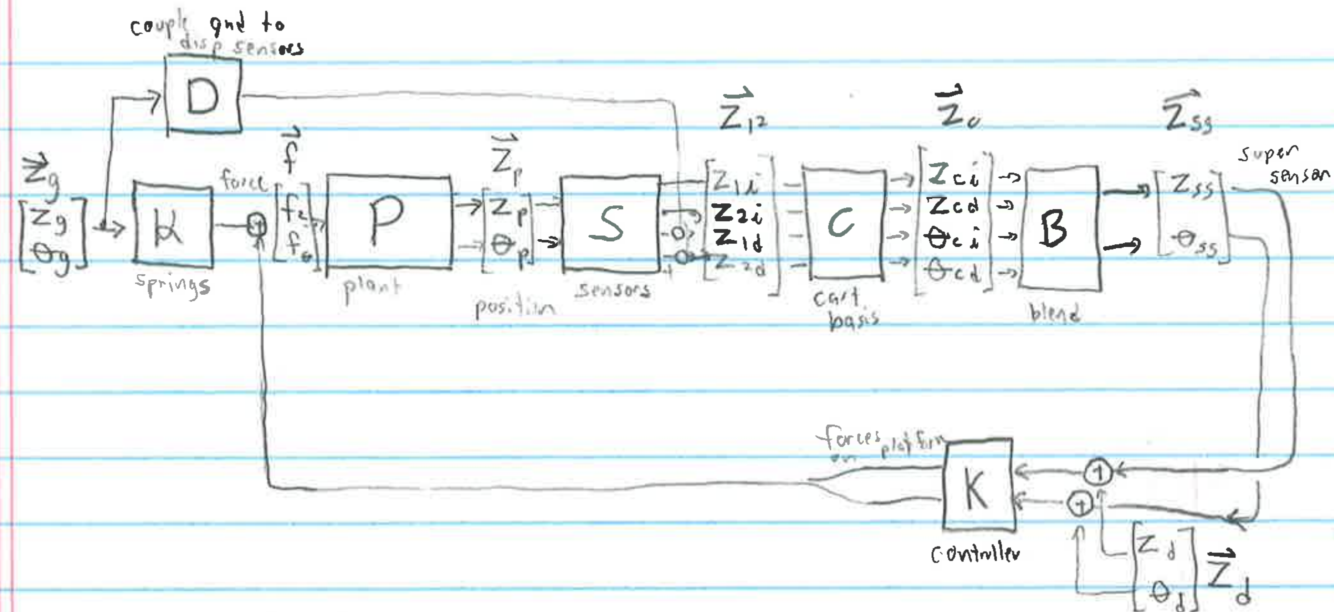
$$\theta_{ci} = \frac{1}{2r}(Z_{1i} - Z_{2i})$$
$$= \frac{1}{r}(r\theta_p + Z_p \Delta_i)$$
$$= \theta_p + \frac{\Delta_i}{r} Z_p$$

$$Z_{cd} = \tfrac{1}{2}(Z_{1d} + Z_{2d})$$
$$= (Z_p - Z_g) + r\Delta_d(\theta_p - \cancelto{0}{\theta_g})$$

$$\theta_{cd} = \frac{1}{2r}(Z_{1d} - Z_{2d})$$
$$= (\theta_p - \cancelto{0}{\theta_g}) + \frac{\Delta_d}{r}(Z_p - Z_g)$$

$$\vec{Z}_c = \begin{bmatrix} Z_{ci} \\ Z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} = \begin{bmatrix} 1 & r\Delta_i \\ 1 & r\Delta_d \\ \frac{\Delta_i}{r} & 1 \\ \frac{\Delta_d}{r} & 1 \end{bmatrix} \begin{bmatrix} Z_p \\ \theta_p \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ -1 & -r\Delta_d \\ 0 & 0 \\ -\frac{\Delta_d}{r} & -1 \end{bmatrix}}_{\text{ignore these}} \begin{bmatrix} Z_g \\ \theta_g \end{bmatrix}$$

② & ③



Vectors of signals composed of several individual signals, eg $\vec{z}_p = \begin{bmatrix} z_p \\ \theta_p \end{bmatrix}$

Only cross coupling from z to θ happens in the S & D matricies from scale factor mismatches

$P = \begin{bmatrix} P_{zz} & P_{z\theta} \\ P_{\theta z} & P_{\theta\theta} \end{bmatrix}$, since gains are large, and plant is decent, let $P_{z\theta}$, $P_{\theta z}$ be 0 for now.

$K = \begin{bmatrix} K_{zz} & 0 \\ 0 & K_{\theta\theta} \end{bmatrix}$ we run SISO controllers here

$$P \cdot K = \begin{bmatrix} P_{zz} & 0 \\ 0 & P_{\theta\theta} \end{bmatrix} \cdot \begin{bmatrix} K_{zz} & 0 \\ 0 & K_{\theta\theta} \end{bmatrix} \equiv \begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix}$$ for simplicity

$$C \cdot S = \begin{bmatrix} 1 & r\Delta_i \\ 1 & r\Delta_d \\ \Delta_i/r & 1 \\ \Delta_d/r & 1 \end{bmatrix}$$ and $$C \cdot D \approx \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -\Delta_d/r & 0 \end{bmatrix}$$ from before.

blend filters are just

$$\begin{bmatrix} z_{ss} \\ \theta_{ss} \end{bmatrix} = \underbrace{\begin{bmatrix} H_z & L_z & 0 & 0 \\ 0 & 0 & H_\theta & L_\theta \end{bmatrix}}_{B} \begin{bmatrix} z_{ci} \\ z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix}$$

H, L are High pass / Low pass complementary filters for z or θ

5) Solve Closed loop

We want to know $\vec{z}_c$ from drives and ground

We need to invert the loop matrix, so need to choose calc point with 2 element vector so G is 2×2, not 4×4. The 4×4 is not invertable, Also need to pick point which can be easily propogated forward to sensors, Plant is good point. Super sensor is bad, requires back-propogation (inversion) of blend filters, which doesn't work.

calc inputs to z_p, propogate $\vec{z}_p$ forward to $\vec{z}_c$

$$\vec{z}_p = PK\vec{z}_g + P \cdot K \cdot \vec{z}_d + P \cdot K \cdot B \cdot C\left(S \cdot \vec{z}_p + D \cdot \vec{z}_g\right)$$

$$\vec{z}_p = PK\vec{z}_g + P \cdot K \cdot \vec{z}_d + P \cdot K \cdot B \cdot C \cdot S \cdot \vec{z}_p + P \cdot K \cdot B \cdot C \cdot D \cdot \vec{z}_g$$

$$\vec{z}_p = PK\vec{z}_g + PKBCD \cdot \vec{z}_g + P \cdot K\vec{z}_d + G\vec{z}_p \quad , \; G \equiv P \cdot K \cdot B \cdot C \cdot S$$

$$(I - G)\vec{z}_p = (PK + PKBCD)\vec{z}_g + PK\vec{z}_d$$

assume $I - G \approx -G$

$$z_p \approx -G^{-1}\left[(PK + PKBCD)\vec{z}_g + PK\vec{z}_d\right]$$

next, find G^{-1}

$$\vec{z}_p \approx -G^{-1}\left(PK + PKBCD\right)\vec{z}_g - G^{-1} PK \vec{z}_d$$

$$G = P\cdot K \cdot B \cdot C \cdot S$$

$$\begin{array}{cccc} & P\cdot K & \cdot\ B & \cdot\ C\cdot S \\ = & \begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix} & \begin{bmatrix} H_z & L_z & 0 & 0 \\ 0 & 0 & H_\theta & L_\theta \end{bmatrix} & \begin{bmatrix} 1 & r\Delta_i \\ 1 & r\Delta_d \\ \frac{\Delta_i}{r} & 1 \\ \frac{\Delta_d}{r} & 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} PK_z & \\ & PK_\theta \end{bmatrix} \begin{bmatrix} H_z + L_z & \left(H_z r \Delta_i + L_z \cdot r \cdot \Delta_d\right) \\ \left(H_\theta \frac{\Delta_i}{r} + L_\theta \frac{\Delta_d}{r}\right) & H_\theta + L_\theta \end{bmatrix}$$

we know $L_z + H_z = 1$, $L_\theta + H_\theta = 1$,

these are interesting to us.

define $C_{2z} \equiv H_z r \Delta_i + L_z \cdot r \cdot \Delta_d$ (coupling to z)

$C_{2\theta} \equiv H_\theta \frac{\Delta_i}{r} + L_\theta \frac{\Delta_d}{r}$

$C_{2\theta}, C_{2z} \ll 1$

$$G = \begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix} \begin{bmatrix} 1 & C_{2z} \\ C_{2\theta} & 1 \end{bmatrix}$$

$$\text{inv}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\cdot\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} PK_z & PK_z C_{2z} \\ PK_\theta C_{2\theta} & PK_\theta \end{bmatrix}$$

$$\left(-G^{-1}\right) = \frac{-1}{PK_\theta PK_z\left(1 - C_{2\theta} \cdot C_{2z}\right)} \cdot \begin{bmatrix} PK_\theta & -PK_z C_{2z} \\ -PK_\theta C_{2\theta} & PK_z \end{bmatrix} \approx \begin{bmatrix} -1/PK_z & +C_{2z}/PK_\theta \\ +C_{2\theta}/PK_z & -1/PK_\theta \end{bmatrix}$$

$$\vec{z}_p \approx (-G^{-1})(PK + PKBCD)\vec{z}_g - G^{-1} \cdot P \cdot K\, \vec{z}_d$$

drive term

$$(-G^{-1})(PK) = \begin{bmatrix} -1/PK_z & C_{2z}/PK_\theta \\ C_{2\theta}/PK_z & -1/PK_\theta \end{bmatrix} \cdot \begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix}$$

$$= \begin{bmatrix} -1 & C_{2z} \\ C_{2\theta} & -1 \end{bmatrix} \equiv C_{pd}$$

Matrix coupling plant to drive

ground term

$PK \approx I$, platform follows ground passively at low frequency, cross couple terms are small. Assume smaller than $PK_z C_{2z}$ & $PK_\theta C_{2\theta}$

so $PK + PKBCD \approx PKBCD$

$$PK \cdot B \cdot CD$$

$$\begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix} \begin{bmatrix} H_z & L_z & 0 & 0 \\ 0 & 0 & H_\theta & L_\theta \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -\frac{\Delta d}{r} & 0 \end{bmatrix}$$

ignoring ground tilt.

$$= \begin{bmatrix} PK_z & 0 \\ 0 & PK_\theta \end{bmatrix} \begin{bmatrix} -L_z & 0 \\ -L_\theta \frac{\Delta d}{r} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -PK_z L_z & 0 \\ -PK_\theta L_\theta \frac{\Delta d}{r} & 0 \end{bmatrix}$$

Matrix coupling ground motion to the plant.

$$-G^{-1} \cdot PKBCD = \begin{bmatrix} -1/PK_z & C_{2z}/PK_\theta \\ C_{2\theta}/PK_z & -1/PK_\theta \end{bmatrix} \begin{bmatrix} -PK_z L_z & 0 \\ -PK_\theta L_\theta \frac{\Delta d}{r} & 0 \end{bmatrix} = \overbrace{\begin{bmatrix} L_z - L_\theta \frac{\Delta d}{r} C_{2z} & 0 \\ -L_z C_{2\theta} + L_\theta \frac{\Delta d}{r} & 0 \end{bmatrix}}^{C_{pg}}$$

6) $\vec{z}_p = C_{pg}\,\vec{z}_g + C_{pd}\,\vec{z}_d$

$$\vec{z}_c = C\cdot S\cdot \vec{z}_p + C\cdot D\cdot \vec{z}_g$$
$$= C\cdot S\cdot C_{pg}\, z_g + C\cdot S\cdot C_{pd}\,\vec{z}_d + C\cdot D\,\vec{z}_g$$

$\vec{z}_c$ from drive is

$$C\cdot S\cdot C_{pd}$$

$$\begin{bmatrix} 1 & r\Delta_i \\ 1 & r\Delta_d \\ \Delta_i/r & 1 \\ \Delta_d/r & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & C_{2z} \\ C_{2\theta} & -1 \end{bmatrix}$$

$$\begin{bmatrix} z_{ci} \\ z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} = \begin{bmatrix} -1 + r\Delta_i C_{2\theta} & C_{2z} - r\Delta_i \\ -1 + r\Delta_d C_{2\theta} & C_{2z} - r\Delta_d \\ C_{2\theta} - \Delta_i/r & -1 + \frac{\Delta_i}{r} C_{2z} \\ C_{2\theta} - \Delta_d/r & -1 + \frac{\Delta_d}{r} C_{2z} \end{bmatrix} \begin{bmatrix} z_{d,ss} \\ \theta_{d,ss} \end{bmatrix}$$

$$C_{2\theta} = H_\theta \frac{\Delta_i}{r} + L_\theta \frac{\Delta_d}{r}$$

$$C_{2\theta} - \frac{\Delta_i}{r} = H_\theta \frac{\Delta_i}{r} + L_\theta \frac{\Delta_d}{r} - \frac{\Delta_i}{r}\overbrace{(H_\theta + L_\theta)}^{=1}$$
$$= L_\theta \frac{\Delta_d}{r} - L_\theta \frac{\Delta_i}{r}$$

$$C_{2\theta} - \frac{\Delta_d}{r} = H_\theta \frac{\Delta_i}{r} + L_\theta \frac{\Delta_d}{r} - \frac{\Delta_d}{r}(H_\theta + L_\theta)$$
$$= H_\theta \frac{\Delta_i}{r} - H_\theta \frac{\Delta_d}{r}$$

$$\begin{bmatrix} z_{ci} \\ z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} \approx \begin{bmatrix} -1 \\ -1 \\ L_\theta \Delta_d/r - L_\theta \Delta_i/r \\ H_\theta \Delta_i/r - H_\theta \Delta_d/r \end{bmatrix} z_{d,ss}$$

$$\vec{Z}_{d,ss} = \overset{B}{\begin{bmatrix} H_z & L_z & 0 & 0 \\ 0 & 0 & H_\theta & L_\theta \end{bmatrix}} \cdot \overset{\vec{Z}_c}{\begin{bmatrix} z_{ci} \\ z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix}}$$

if we drive a signal in at z_{cd}

$$Z_{d,ss} = L_z \cdot z_{cd}$$

so

drive from cart disp to platform sensors looks like,

$$\begin{bmatrix} z_{ci} \\ z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} \approx \begin{bmatrix} -L_z \\ -L_z \\ -L_z\left(L_\theta \frac{\Delta_d}{r} - L_\theta \frac{\Delta_i}{r}\right) \\ -L_z\left(H_\theta \frac{\Delta_i}{r} - H_\theta \frac{\Delta_d}{r}\right) \end{bmatrix}$$

z drive at disp sensor

Z_c from ground

$$\left(\overset{C\cdot S\cdot C_{pg}}{\begin{bmatrix} 1 & r\Delta_i \\ 1 & r\Delta_d \\ \Delta_i/r & 1 \\ \Delta_d/r & 1 \end{bmatrix}} \cdot \overset{\vec{Z}_g}{\begin{bmatrix} L_z - L_\theta \frac{\Delta_d}{r} C_{2Z} & 0 \\ -L_z C_{2\theta} + L_\theta \frac{\Delta_d}{r} & 0 \end{bmatrix}} \overset{+}{+} \overset{C\cdot D\ \vec{Z}_g}{\begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ -\frac{\Delta_d}{r} & 0 \end{bmatrix}} \right) \vec{Z}_g$$

$$\begin{bmatrix} Z_{ci} \\ Z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} = \begin{bmatrix} L_z - L_\theta \frac{\Delta_d}{r} C_{2Z} - r\Delta_i L_z C_{2\theta} + L_\theta \frac{\Delta_d}{r} r\Delta_i \\ (L_z - 1) - L_\theta \frac{\Delta_d}{r} C_{2Z} - r\Delta_d L_z C_{2\theta} + r\Delta_d \Delta_d/r\, L_\theta \\ L_z \frac{\Delta_i}{r} - \frac{\Delta_i}{r} L_\theta \frac{\Delta_d}{r} C_{2Z} - L_z C_{2\theta} + L_\theta \frac{\Delta_d}{r} \\ L_z \frac{\Delta_d}{r} - \frac{\Delta_d}{r} L_\theta \frac{\Delta_d}{r} C_{2Z} - L_z C_{2\theta} + L_\theta \frac{\Delta_d}{r} - \frac{\Delta_d}{r} \end{bmatrix}$$

$$\approx \begin{bmatrix} L_z \\ L_z - 1 \\ L_z\left(\frac{\Delta_i}{r} - C_{2\theta}\right) + L_\theta \frac{\Delta_d}{r} \\ L_z\left(\frac{\Delta_d}{r} - C_{2\theta}\right) - H_\theta \frac{\Delta_d}{r} \end{bmatrix}$$

$$\begin{bmatrix} Z_{ci} \\ Z_{cd} \\ \theta_{ci} \\ \theta_{cd} \end{bmatrix} \approx \begin{bmatrix} L_z \\ -H_z \\ L_z\left(L_\theta \frac{\Delta_i}{r} - L_\theta \frac{\Delta_d}{r}\right) + L_\theta \frac{\Delta_d}{r} \\ L_z\left(H_\theta \frac{\Delta_d}{r} - H_\theta \frac{\Delta_i}{r}\right) - H_\theta \frac{\Delta_d}{r} \end{bmatrix} \cdot Z_g$$