LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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CALIFORNIA INSTITUTE OF TECHNOLOGY
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Testing General Relativity with Binary Black Hole Mergers

SURF 2017 Progress Report II

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1 Abstract

The several recent detections of gravitational waves (GWs) by the Laser Interferometer Gravitational-Wave Observatory (LIGO) has provided researchers with the first opportunities to test general relativity (GR) in the strong-field and highly-dynamical limit. Qualitative tests of the agreement between LIGO's GW observations and classical GR have already been done; we aim to carry out more quantitative tests in terms of controlled, parameterized deviations from GR. In this project, we run a matched-filter analysis on LIGOs prior detections using templates with known amplitude and frequency deviations from those predicted by GR to a construct a probability function on the deviations. We then simulate a number of BBH merger waveforms with similar deviations from GR, use Bayesian analysis to recover the deviation, and provide an estimate of the number of GW detections from BBH mergers necessary to establish a given deviation from classical GR.

2 Motivation

Gravitational waves, ripples in spacetime caused by rapidly changing gravitational fields, were predicted by Einstein's theory of general relativity. Since September 2015, the Advanced LIGO experiment has detected several gravitational wave signatures originating from binary black hole (BBH) systems [1], [2].

In classifying these mergers, LIGO ran a matched filter search across detector data with template BBH waveforms generated by numerical relativity. These waveforms were based off Einstein's formulation of general relativity and modeled BBH inspiral, merger, and ringdown phases parametrized by several characteristics of BBH mergers (such as masses and spins). These template waveforms provided a good fit to the signals that LIGO was detecting, leading to the successful detection and characterization of a number of merging events.

After detection, the post-newtonian coefficients to each BBH merger were computed, showing that general relativity was consistent with the observed signals. This provided further evidence that the signals were indeed from BBH mergers.

However, post-newtonian coefficients calculated in this manner are valid only for the early inspiral phase of a merge. As the black holes in a binary system spiral closer together, v/c approaches 1, leading to a probable breakdown of the post-newtonian expansion (see Figure 1). The post-newtonian coefficients become invalid as the BBH system pushes even further into the strong-field gravity limit. It is during the merger and ringdown phase that these BBH systems become perfect candidates for testing for deviations from general relativity.

3 Research Problem and Anticipated Outline

My goal in this project is to determine the extent to which deviations from general relativity can account for the BBH signals LIGO has detected. Any deviations from general relativity will be most evident when gravity approaches the strong field limit, when $\frac{v}{c}$ approaches 1 (which occurs in the merger and ringdown phases; see Figure 2). Note that $\frac{v^2}{c^2} = \frac{4\pi^2 r^2}{P^2 c^2} =$ (using Kepler's 3rd law) $\frac{GM_{tot}}{rc^2}$, where G is Newton's gravitation constant, M_{tot} is the sum of the two black hole masses for a given merger event, c is the speed of light in vacuum, v is the relative velocity between the two black holes, r is the orbital separation, and P is the orbital period. This makes the term $\frac{GM_{tot}}{rc^2}$ an apt one to characterize strong-field gravity. Therefore, for my analyses, I will assume that any such deviations will take the functional form $e^{\frac{\lambda GM}{rc^2}}$, and I will investigate what happens as the parameter λ deviates from 0 (the general relativistic limit).

Note that although the quantity $\frac{v}{c}$ was derived classically here, K (through Kepler's laws), Kepler's laws provide a reasonable approximation to a BBH system up until the merger. Further, it is a quantity measuring compactness of strong-field relativistic objects ([4], [5]) and therefore acceptable to use in testing general relativity.

I anticipate my analyses being broken up into four stages:

Stage 1 (preparation): I will familiarize myself with the parameters used to classify BBH mergers, then simulate a number of random mergers. I will then learn Maximum Likelihood Estimation and Bayesian analysis techniques to characterize the behavior of the parameters across the entirety of the mergers. Learning these techniques, specifically Bayesian analysis, will improve the sophistication of my analyses during later stages. Time: 3 weeks.

Stage 2: for each BBH signal that LIGO has detected and characterized, I will modify the template waveform that was the best fit for that signal by $e^{\frac{\lambda GM}{rc^2}}$, taking care to vary λ from 0 in a controlled fashion. Note that λ is complex; Re(λ) will act as a measure of amplitude modulation, and Im(λ) will act as a measure of frequency modulation. For different values of λ, I will check to what degree the modified waveform template proves a better fit to the BBH signal than the original waveforms (i.e which template, when subtracted from the detector signal, creates noise that is more Gaussian), taking care to ensure my methods and findings are statistically meaningful. The result (for each BBH signal) will be a graph modeling the signal-to- noise ratio for a range of λ's. Anticipated time: 2-3 weeks.

Stage 3: I will analyze the findings of stage 2 and use Bayesian analysis to calculate a probability distribution function for each BBH event, measuring the likelihood that a given value of λ correctly accounts for that BBH event. I will then compile all the analyzed BBH events into a single distribution function with the goal of finding the value of λ that best models general relativity deviations for all the BBH events. This graph will mark the completion of stage 3. Anticipated time: 2-3 weeks.

Stage 4: I will calculate the number of BBH detections needed to corroborate (or rule out) a deviation from general relativity at any given value of λ. Creating this graph will mark the successful completion of my analyses. Anticipated time: 2-3 weeks.

4 Work Completed

Week 1 (6/20-6/23) was spent familiarizing myself with the 15 parameters characterizing BBH mergers as well as their probability distribution functions, all of which are compiled in the table below.

BBH Parameters		
Parameter	pdf	Range
Right Ascension α	uniform in α	$[0, 2\pi)$
Declination δ	uniform in $\cos\delta$	$[0, \pi)$
Inclination angle ι	uniform in $\cos\iota$	$[0, \pi)$
Phase ψ	uniform in ψ	$[0, 2\pi)$
Distance r	quadratic	$[0, 1\text{Mpc}]$
M_{tot}	Salpeter stellar mass	$M_{tot} \geq .5M_{sol}$
symmetric mass ratio η	gaussian, $\mu = .25, \sigma = .05$	$[0, .25]$
$\lvert s_1 \rvert$	gaussian, $\mu = .7, \sigma = .1$	$[0, .25]$
$\lvert s_2 \rvert$	gaussian, $\mu = .7, \sigma = .1$	$[0, .25]$
s_1 azimuthal angle ϕ_1	uniform in ϕ_1	$[0, 2\pi)$
s_2 azimuthal angle ϕ_2	uniform in ϕ_2	$[0, 2\pi)$
s_1 polar angle θ_1	uniform in $\cos\theta_1$	$[0, \pi)$
s_2 polar angle θ_2	uniform in $\cos\theta_2$	$[0, \pi)$
Coalesence phase κ	uniform in κ	$[0, 2\pi)$
Coalesence time t	uniform in t	$[0, 1{*}\text{LIGO observing run})$

Note that the spins for the two black holes are uncorrelated in magnitude and in all angles, which may or may not be a good model for actual mergers. At the time of writing, too few mergers have been detected to prove a convincing argument for any type of spin correlation where the benefits of modeling correlated spins outweigh the complexities of actually implementing these correlations.

Using `python`, I randomly simulated 10,000 BBH mergers by taking 10,000 random draws from each nontrivial (i.e. each parameter that didn't have a uniform or uniform in cos distribution) parameter's pdf and comparing it to the expected distribution.

Week 2 (6/26-6/30) was spent on further analyses of the BBH simulations. Once again, I simulated 10,000 BBH events (this time simulating all 15 parameters), then used Maximum Likelihood Estimation (i.e. `scipy` curve fit) to fit each simulated parameter distribution and compare the fit to the expected pdf. The mass function proved to be a bit challenging to fit, mainly because the steep exponential represses events with high M_{tot}, so the fits for both masses consistently had the largest errors out of all 15 parameters.

I spent the next few days familiarizing myself with the general benefits of Bayesian over Maximum Likelihood analyses as well as learned `pymc`. I then simulated 20,000 random BBH mergers and used Bayesian techniques to fit all the parameters. For the trivial parameters, this analysis was costly in terms of time and effort spent learning the new software, but the effort was justified since the same techniques can be used to efficiently analyze higher-dimensional systems, which I will be doing in the last stages of my project as I test general relativity.

All images generated during weeks 2 and 3 of the project are in Appendix A. Note the differences between the Maximum Likehood fits and the Bayesian fits.

Week 3 (7/5-7/7) (the short week) was spent cleaning up the code to make the images as described above. It was also spent preparing for **Week 4**'s assignment, which involves using the formula for BBH signal SNR (slightly modified from formula 4.3 in the FINDCHIRP paper, [6]) to estimate some representative SNRs for future LIGO detectors. I also began to write code that would modify some of LIGO's template waveforms by the factors of $e^{\frac{\lambda GM}{rc^2}}$ with the intention of using them to compute the SNR as a function of λ during the following week.

Week 4 (7/10-7/14) was spent working on progress report 1 and the waveform modulation code. Working in the time domain, implementing amplitude modulation was easy: I computed f_{GW} by taking the gradient of the complex waveform (which I generated by using the matched-filter template for GW150914, using $h_+ + ih_x$). I then calculated the orbital separation from Kepler's laws, $r = (\frac{GM_{tot}}{4\pi^2 f_{GW}^2})^{1/3}$, then multiplied the waveform vector by the quantity $\exp(\lambda \frac{GM_{tot}}{rc^2})$.

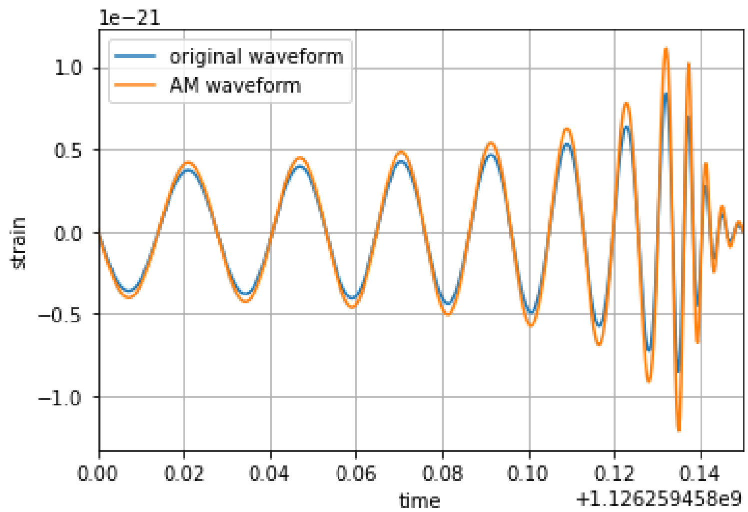

Figure 1: An amplitude-modulated waveform with $\lambda = .6$

However, after running into problems as to how to best execute the frequency modulation with a waveform in the time domain, my mentor suggested working in the frequency domain with the Fourier transform of the waveform. To modulate a waveform, I took its Fourier transform, then multiplied that by $e^{\lambda \frac{GM_{tot}}{rc^2}}$ with r defined as above and $\lambda \in C$. Although I wouldn't realize it until later, there was an error in this code: as I had defined it, r was dependent on t when it should have been dependent on f!

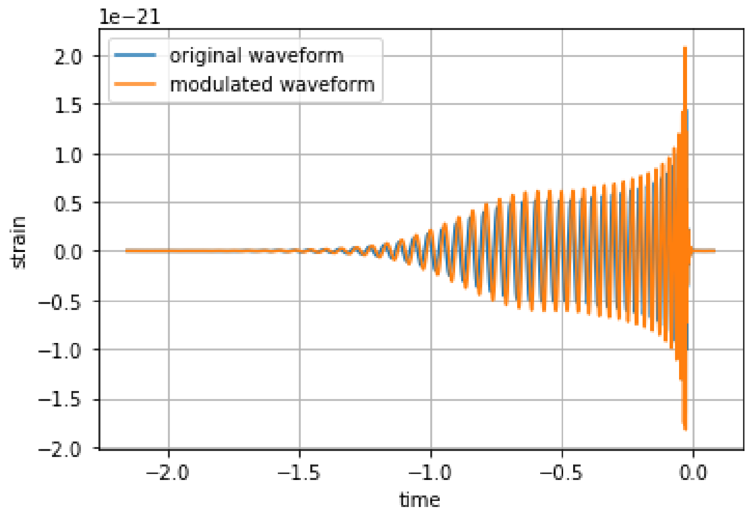

Figure 2: An modulated waveform with $\lambda = .3 + 1.6i$

Week 5 (7/17-7/21) was mostly spent visiting LIGO's Livingston detector in Louisiana. However, I was able to implement a rudimentary SNR calculation, heavily following the calculation done in the LIGO GW150914 tutorial [3]. Using a template waveform that I had modulated by a given λ as my data, $h_d(t)$, and the unmodified template waveform as m template, $h_t(t)$, the calculation of the SNR was simple:

I first calculated the unnormalized matched filter output o, where

$$\widetilde{o}(f) = \frac{\widetilde{h_d}(f)\widetilde{h_t}(f)*}{psd}$$

where psd is the power spectral density of the next-generation LIGO detectors. Since I needed an SNR in the time domain, I then used $\widetilde{o}(f)$ to calculate its inverse fourier transform, $o(t)$. I then normalized the matched filter output, to account for the possibility of the template having a disproportionately large amplitude (this would come in handy when I started using an amplitude-modulated waveform as the template in the following week), with the normalization

$$\sigma^2 = \sum \frac{\widetilde{h_t}(f)\widetilde{h_t}(f)*}{psd}.$$

I could then calculate the complex time-domain $SNR_c = \frac{o(t)}{\sigma}$. For my analyses, for each BBH event I analyzed, I would take the SNR to be $\max(|SNR_c|)$, maximized over the duration of the event.

Week 6 (7/24-7/28) was spent actually using the SNR calculation code. I generated some $h_d(t)$ for a few different trial values of λ, then used a grid method to calculate SNR values using a range of modulated template waveforms as a matched-filter. The results, shown below, indicate that for noiseless waveforms, a given value can be recovered by a gridding analysis reasonably well.

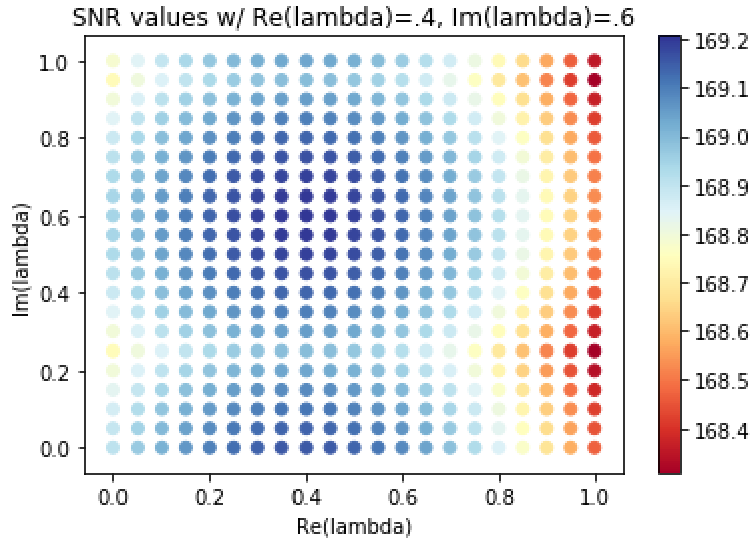

Figure 3: Grid method SNR results with $\lambda = .4 + .6i$. Note the true value of λ clearly has the highest SNR of all the grid values, although there is an unexpected peak at low values of Im(λ).

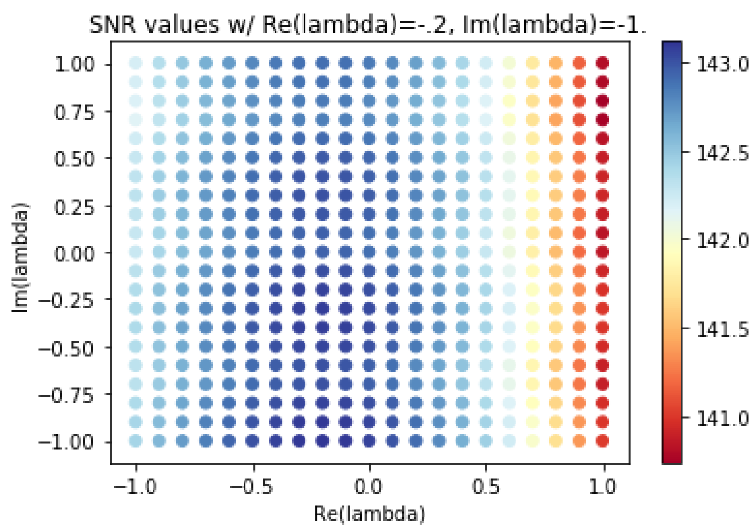

Figure 4: Grid method SNR results with $\lambda = -.2 - i$. The true value of λ appears to have the highest SNR, although there are some unexplained smaller peaks at other values of λ; perhaps this is an effect of frequency being periodic.

However, by the end of the week, my mentor and I decided to no longer work in the time domain for calculating SNR to save on computation time. I gained access to LIGO's `lalsimulation` software, which contained code I could use to calculate waveform $\widetilde{h}_d(f)$ data (at a given detector) given an arbitrary set of parameters such as right ascension, declination, BBH spin orientations, and masses. In making this switch, I finally realized that my original waveform modulation code was wrong, and I fixed it such that when it calculated r, it did so in terms of f, and not t.

Using the first event from the 10000 random BBH events I had generated a few weeks earlier (although scaling the masses by a factor of 20 to improve runtime), I generated a BBH frequency-domain waveform and tested my code's ability to modulate it. Graphs are shown below for this event, which has the following parameters:

BBH Parameters	
Parameter	value
Right Ascension α	4.1392109149
Declination δ	1.5685986802
Inclination angle ι	1.64755849433
Phase ψ	4.79400993236
Distance r	0.85598581004 (Mpc)
M_1	1.36376768923
M_2	1.14790856099
$\lvert s_1 \rvert$	0.521942706109
$\lvert s_2 \rvert$	0.607436944454
s_1 azimuthal angle ϕ_1	4.25197385672
s_2 azimuthal angle ϕ_2	2.50172610002
s_1 polar angle θ_1	0.846810925991
s_2 polar angle θ_2	1.47934712711
Coalesence phase κ	0.261997763323

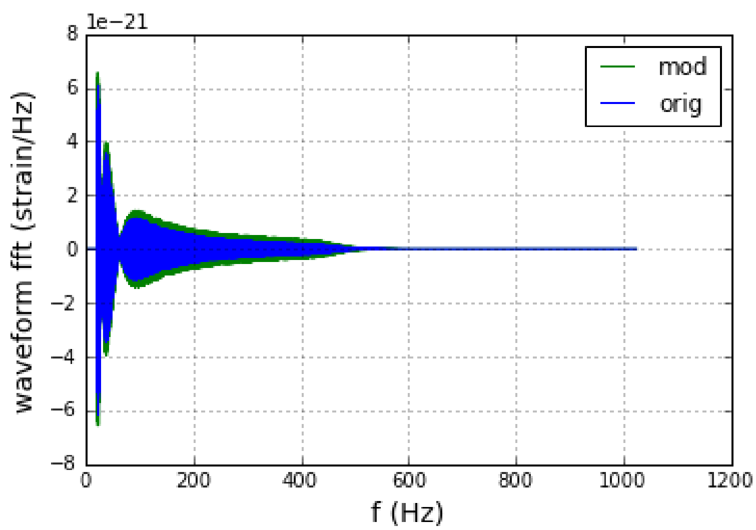

Figure 5: Re(Waveform fft) for simulated BBH event 1 with $\lambda = .8 + 3.8i$

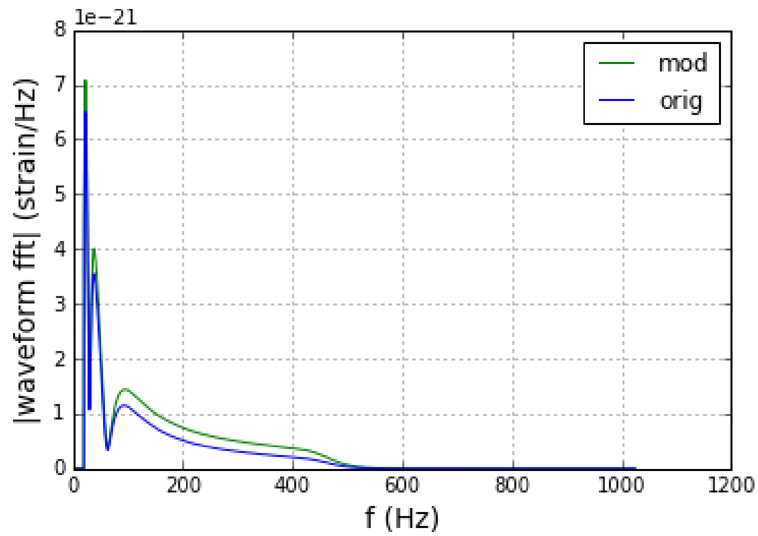

Figure 6: —Waveform fft— for simulated BBH event 1 with $\lambda = .8 + 3.8i$

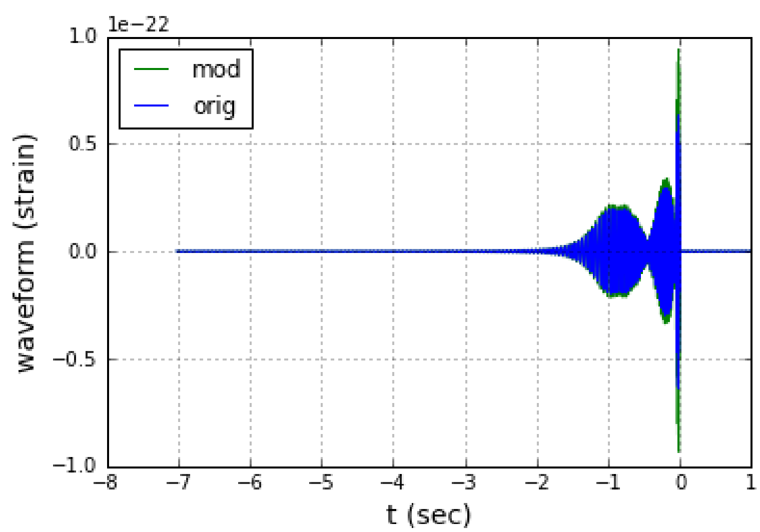

Figure 7: Waveform for simulated BBH event 1 with $\lambda = .8 + 3.8i$

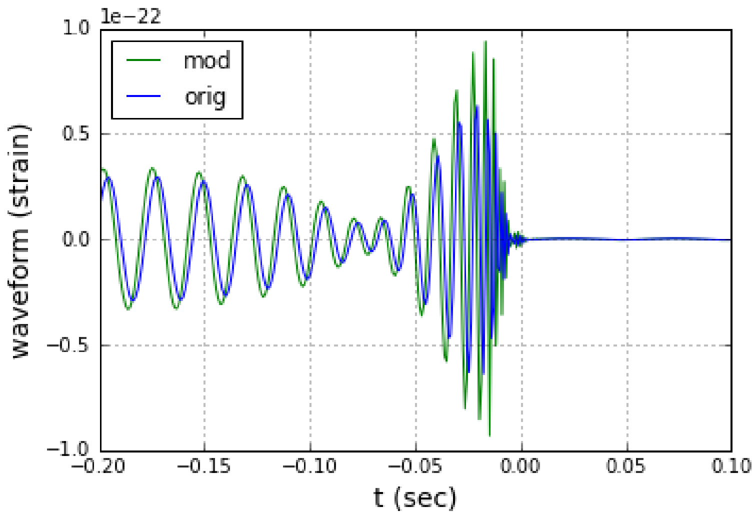

Figure 8: Close-up of waveform for simulated BBH event 1 with $\lambda = .8 + 3.8i$. The extreme amplitude modulation comes from the fact that this waveform was generated for a precessing BBH system as seen at the Livingston detector, and the precession causes the relative amplitudes of h_+ and h_x to vary with time.

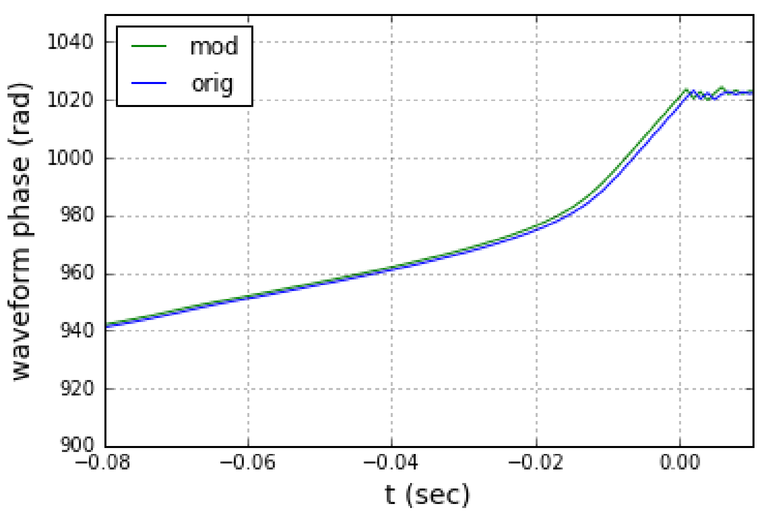

Figure 9: Waveform phase for simulated BBH event 1 with $\lambda = .8 + 3.8i$

I spent the rest of the week familiarizing myself with python's `emcee` package in preparation to write code that could, given a modulated waveform with a secret value of λ, run a Bayesian matched-filter analysis and calculate the secret value of λ.

Week 7 (7/31-8/4) was spent generating a single waveform with a known value of λ, then using `emcee` to use Bayesian parameter analysis to calculate the value of λ that maximized the SNR of the template with the data. So far, the posterior estimates I'm getting aren't great; I'm currently testing a number of values for number of iterations, number of `emcee` walkers, and amount of thinning necessary to get a good estimation of λ. All plots shown below are with 100 `walkers` each going for 1000 iterations with a thinning on every 10th iteration. Each analysis took around 40 minutes to complete. Note that ”amp” = Re(λ) and ”freq” = Im(λ).

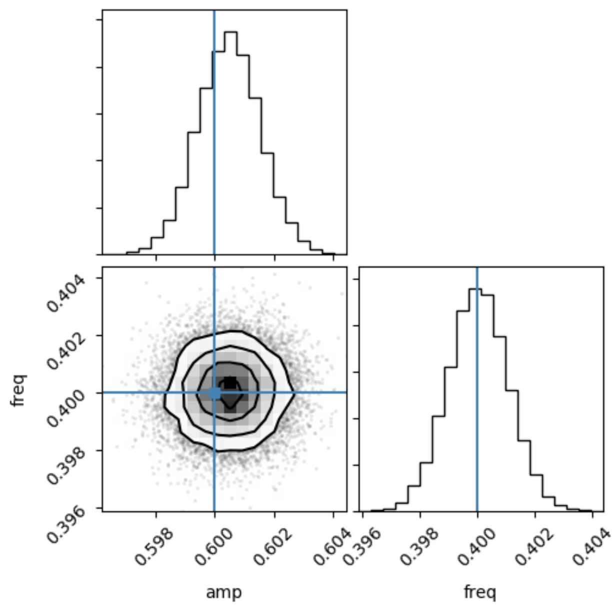

Figure 10: Bayesian estimation results for simulated BBH event 1 with $\lambda = .6 + .4i$. The results are both precise and accurate

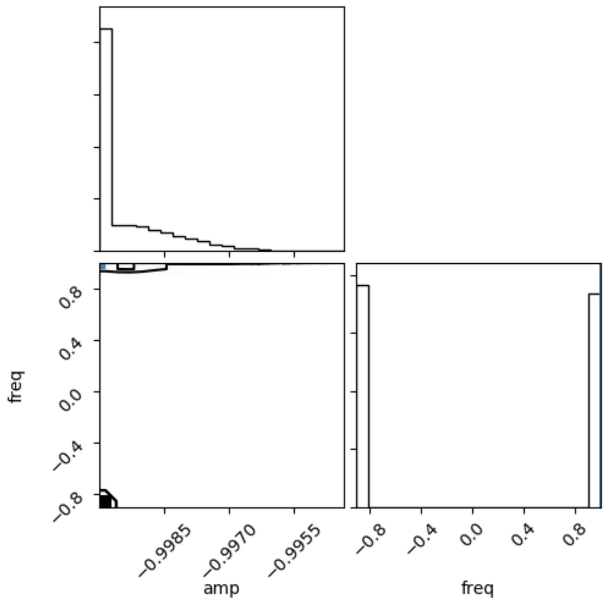

Figure 11: Bayesian estimation results for simulated BBH event 1 with $\lambda = -1 + i$. The Bayesian fit is clearly terrible (it can't seem to discern a negative frequency modulation from a positive one) and the error likely stems from a bad choice of prior.

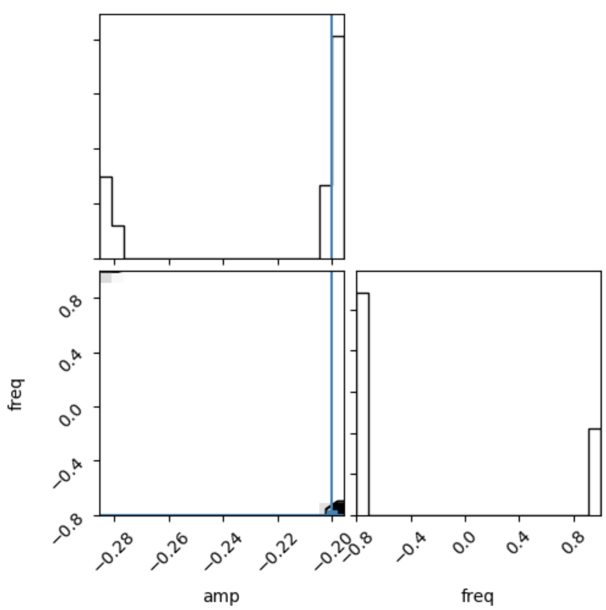

Figure 12: Bayesian estimation results for simulated BBH event 1 with $\lambda = -.2 - .8i$. The parameter estimate is decent, but compared to the one in Fig 10, it's evident that there wasn't enough sampling of the parameter space.

5 Challenges and Anticipated Issues

The most pressing issues I've encountered are with `emcee`: as I mentioned in a previous section, I don't yet have a good grasp on how to best set up `emcee` to get reasonable results; so far, I'm having trouble with my walker traces converging too quickly and not exploring all of the parameter space. Further, getting a good amount of parameter-space sampling in `emcee` requires runtimes on the order of an hour to analyse a single waveform. Luckily, I have found another SURF student and postdoc who are much more experienced in `emcee` than I am, so I've been asking them for advice on samplers, number of iterations and walkers, and convergence.

References

[1] B. P. Abbott et al., *GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence*. PHYSICAL REVIEW LETTERS. 116, 241103 (2016).

[2] B. P. Abbott et al., *Binary Black Hole Mergers in the First Advanced LIGO Observing Run*. PHYS. REV. X 6, 041015 (2016).

[3] https://losc.ligo.org/s/events/GW150914/LOSC_Event_tutorial_GW150914.html

[4] Blanchet, L. Living Rev. Relativ. (2014) 17: 2. doi:10.12942/lrr-2014-2

[5] Charalampos Markakis et al., *Neutron star equation of state via gravitational wave observations*. J. Phys.: Conf. Ser. 189 012024 (2009).

[6] Bruce Allen et al., *FINDCHIRP: an algorithm for detection of gravitational waves from inspiraling compact binaries*. Phys. Rev. D 85, 122006 (2012).