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Analytic calculation of coupling coefficients on the bullseye photodiode

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1 Overview

The purpose of this document is to calculate the response function for the bulls-eye photodiode and use that to arrive at the calibration coefficients to be used in the filters used for the different aspects like pitch,yaw and beam size jitter.

2 Scope

The scope of this document is to do a little math for the Bulls Eye photodiode which currently resides on the $H1$ PSL table which has the geometry as shown in the figure below. The sum of the different segments of the bulls eye photo detector leads(PD) to rejection of intensity noise (ideally).

3 Setup

The photodiode signals are converted to the Degree of freedom(DoF) basis through a matrix implemented into the model. This matrix takes in the signal from each of the segments of the bullseye photodiode as an input and provides the pitch,yaw,beam size jitter and sum of the segments as an output. As discussed earlier, the sum of the four segments is used for normalising the intensity noise and a linear combination of the other segments yields information about the beam pointing in vertical and longitudinal. The pitch and yaw signals are normalised by the sum over the segments to reduce the dependance on power fluctuations

4 Bulls Eye Sensing Matrix

The signals are converted into DoF basis using the sensing matrix shown below. This matrix has been found to be most optimal in terms of sensing gain.

Seg	S1	S2	S3	S4
Yaw	0	1	0	-1
Pit	-2	1	0	1
Wid	1	1	-1	1
Sum	1	1	1	1

5 Signals : Pitch, Yaw, Beam size jitter

5.1 Intensity profile

The spatial profile of a Gaussian beam (propagating in the direction along z-axis) can be defined as :

$$E(x,y,z) = E_0 \exp\left[\frac{x^2+y^2}{\omega(z)^2}\right] \tag{1}$$

The intentity peofile can then be calculated to be :

$$I = I_0(\frac{2}{\pi\omega^2}) \exp\left[\frac{x^2+y^2}{\omega(z)^2}\right] \tag{2}$$

where , $I_0 = |E_0^2|$

5.2 Beam size jitter

The beam size jitter fluctuations can be calculated by the difference between the intensity of the Gaussian beam in the central segment and the the sum of the outer segments.(See line 3 in the sensing matrix under the parameter Wid).
For the central segment(S3), the intensity is integrated as shown below :

$$I_3 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\frac{2}{\pi\omega^2}) \exp\left[\frac{x^2+y^2}{\omega(z)^2}\right] dxdy \tag{3}$$

The cartesian coordinate system here is converted to spherical polar coordinates(SPC) in order to simplify the mathematics and to be able to exploit the properties of symmetry.The integration limits are between 0 to 2π for θ and 0 to R for the radial component. R here represents the radius of the central segment.

$$I_3 = \int_0^R\int_0^{2\pi} (\frac{2}{\pi\omega^2}) \exp\left[\frac{-2r^2}{\omega^2}\right] rdrd\theta \tag{4}$$

$$I_3 = (1 - \exp\left[\frac{-2R^2}{\omega^2}\right]) \tag{5}$$

For the sum of the three outer segments (S1, S2,S4), the integration limits for the radial component are different i.e. they lie between R and infinity and using the same SPC trick as above, the overall intensity of the gaussian beam in the outer segments of the bullseye PD can be written down as follows. One can however use the argument that the sum of all the segments is unity and hence can calculate the sum over the out segments to be :

$$I_1 + I_2 + I_4 = 1 - I_3 \tag{6}$$

This is then simplified into

$$I_1 + I_2 + I_4 = \exp(\frac{-2R^2}{\omega^2}) \tag{7}$$

The beam size jitter can be written now be written down as :

$$g(R,\omega) = I_3 - I_1 + I_2 + I_4 = (1 - \exp\left[\frac{-2R^2}{\omega^2}\right]) - \exp(\frac{-2R^2}{\omega^2}) \tag{8}$$

$$g(R,\omega) = \left[1 - 2\exp(\frac{-2R^2}{\omega^2})\right] \tag{9}$$

The response of the bulls eye PD is obtained by differentiating the response function wrt. to the beam size.

$$\frac{\partial g}{\partial \omega} = \frac{-8R^2}{\omega^3}\exp(\frac{-2R^2}{\omega^2}) \tag{10}$$

The variables can be rearranged and written as :

$$\Delta g = \frac{-8\sqrt{2}R^2}{\omega^2}\exp(\frac{-2R^2}{\omega^2}) \tag{11}$$

Owing to the fact that the beam size jitter arises from the beat between the $(2,0)$ mode and the $(0,0)$ mode, the coupling coefficient is indicated by $(\frac{\Delta\omega}{\sqrt{2}\omega})$ [1]. By using the inverse of the response function which is indicated by $\frac{-8\sqrt{2}R^2}{\omega^2}$ can be inverted and this serves as a good enough approximation for the gain to be implemented in the filters.

5.3 Yaw

In order to calculate the horizontal pointing jitter response function, we look at the spot being positioned on the individual segments and then use the sensing matrix to arrive at a complete response function. Yaw being the displacement in the horizontal direction can be obtained by the difference between th segments $S2$ and $S3$. We have used the first order pertubation to be able to do these calculations and to a large extent this is a good enough approximation. The calculations for the same are illustrated below .
For the intensity equation in section 5.1, a small pertubation in the x-direction can be written down as :

$$\frac{dI}{dx} = \frac{-8x}{\pi\omega^4}\exp\left[\frac{x^2+y^2}{\omega(z)^2}\right] \tag{12}$$

$$\frac{dI}{dx} = \frac{-4x}{\omega^2}I \tag{13}$$

Over the segment $S2$ the total change in the beam size is calculated to be :

$$\Delta I_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\frac{-4x}{\omega^2})Pdxdy \tag{14}$$

Using $x = r\cos\theta$ and $y = r\sin\theta$ the equation above can be written as

$$\Delta I_2 = \int_R^\infty \int_{-\pi/6}^{\pi/2} (\frac{-4r\cos\theta}{\omega^2})(\frac{2}{\pi\omega^2})\exp(\frac{-2r^2}{\omega^2}) r dr d\theta \tag{15}$$

$$\Delta I_2 = -3\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{2\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{16}$$

Similarly for segment $S4$:

$$\Delta I_4 = -3\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{3}{2\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{17}$$

From the sensing matrix,

$$\text{Yaw} = S_2 - S_4 \tag{18}$$

$$\text{Yaw} = -3\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{19}$$

where the response function is $-3\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]$

5.4 Pitch

Pitch or the vertical pointing jitter of the laser beam is obtained by this particular combination of segments : -2 * bottom(S1)+upper right($S2$) + upper left ($S4$). Upon using the first order pertubation similar to the derivation of the yaw signals, we see the following for the pitch signals.

$$\frac{dI}{dy} = \frac{-8y}{\pi\omega^4}\exp\left[\frac{x^2+y^2}{\omega(z)^2}\right] \tag{20}$$

$$\frac{dI}{dy} = \frac{-4y}{\omega^2} I \tag{21}$$

Over the segment $S2$ the total change in the beam size is calculated to be :

$$\Delta I_2 = \int_{-\infty}^\infty \int_{-\infty}^\infty (\frac{-4y}{\omega^2}) P dx dy \tag{22}$$

Using $x = r\cos\theta$ and $y = r\sin\theta$ the equation above can be written as

$$\Delta I_2 = \int_R^\infty \int_{\pi/6}^{\pi/2} (\frac{-4r\sin\theta}{\omega^2})(\frac{2}{\pi\omega^2})\exp(\frac{-2r^2}{\omega^2}) r dr d\theta \tag{23}$$

$$\Delta I_2 = -\frac{\sqrt{3}}{2}\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{24}$$

Similarly for segment $S4$:

$$\Delta I_4 = \int_R^\infty \int_{\pi/2}^{7\pi/6} (\frac{-4r\sin\theta}{\omega^2})(\frac{2}{\pi\omega^2})\exp(\frac{-2r^2}{\omega^2})rdrd\theta \tag{25}$$

$$\Delta I_4 = -\frac{\sqrt{3}}{2}\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{26}$$

For the final segment S1 :

$$\Delta I_1 = \int_R^\infty \int_{7\pi/6}^{11\pi/6} (\frac{-4r\sin\theta}{\omega^2})(\frac{2}{\pi\omega^2})\exp(\frac{-2r^2}{\omega^2})rdrd\theta \tag{27}$$

$$\Delta I_1 = \sqrt{3}\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{28}$$

From the sensing matrix,

$$\text{Pitch} = -2I_1 + I_2 + I_4 \tag{29}$$

$$\text{Pitch} = -3\sqrt{3}\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]\frac{\Delta\omega}{\omega} \tag{30}$$

where the response function is $-3\sqrt{3}\left[\frac{2R}{\pi\omega}\exp(\frac{-2R^2}{\omega^2}) + \frac{1}{\sqrt{2\pi}}\text{Erfc}(\sqrt{2}R\omega)\right]$

6 The actual values in the filters

Using the afore calculated response functions and tsdavg, the values for the calibration of these signals have been calculated. Although summarized in a table here, one can essentially 're-derive' these numbers to be used in the gain parameter of the filters by using the actual data from the segments averaged over time.

The exact methodology of the calculation of these numbers has been enumerated below :

- Using tdsavg obtain the time dependent average data for the signals in the individual segments S_1,S_2,S_3 and S_4. the syntax for the same is tdsavg (time of averaging) Ch1 Ch2 ..
- The sum of the outer segments S_1,S_2,S_4 is then divided by the sum over all the segments including S_3.

- This value is then used as $\exp(\frac{-2R^2}{\omega^2})$ and can be substituted for in the equation for the response function.
- The numerical value thus obtained is inverted and used as the gain term in the filters.
- The above obtained value is the claibration term required to convert the data from individual segments into DoFs of interest.

A sample calculation for beam size jitter is shown below :

- Sum of the outer segments $S_{outer} = S_1+S_2+S_4 = 26574.51$ counts.
- The sum over all segments, $RealSum = S_{outer}+S_{centre} = 47939.91$ counts.
- RealSum is then divided by S_{outer} and this value is then used as $\exp(\frac{-2R^2}{\omega^2})$ implying $(\frac{-2R^2}{\omega^2}) = 0.295$. If we were to name this exponential term Beam then the overall response equation can be visualised as :
  $Resp_{wid} = -8\sqrt{2}\left[\frac{S_{outer}}{RealSum}\right]\left[\frac{1}{Beam}\right]$.
- The inverse of this i.e. $\frac{1}{Resp_{wid}}$ is the calibration factor for beam size jitter which as seen from the table illustrated below is 0.54.

DoF	Value
Yaw	1.1
Pit	0.67
Wid	0.54

References

[1] Guido Mueller., et al. *Opt. Lett.* **25**, 266-268