All of O1 Calibration Uncertainty Review

August 15, 2016

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O1 Uncertainty Pipeline Preview

- Measurement
  - Frequency: Sensing and Actuation
  - Time: Kappas
- DARM Model
  - Kappa Corrected Models
  - Residuals = Measurement / DARM Model
- Residual Uncertainty
  - Covariance
  - 2D Uncertainty Ellipse
- Final Response Function Uncertainty
  - Relative to Absolute Uncertainty
  - Final Response Uncertainty = 2 x 2 Covariance Matrix
- All of O1 Uncertainty
  - Three Events: GW150914, GW151226, and LVT151012
  - All of O1 Spectrograms
- All of O1 Publication: in progress

Transfer Function Measurement Technique

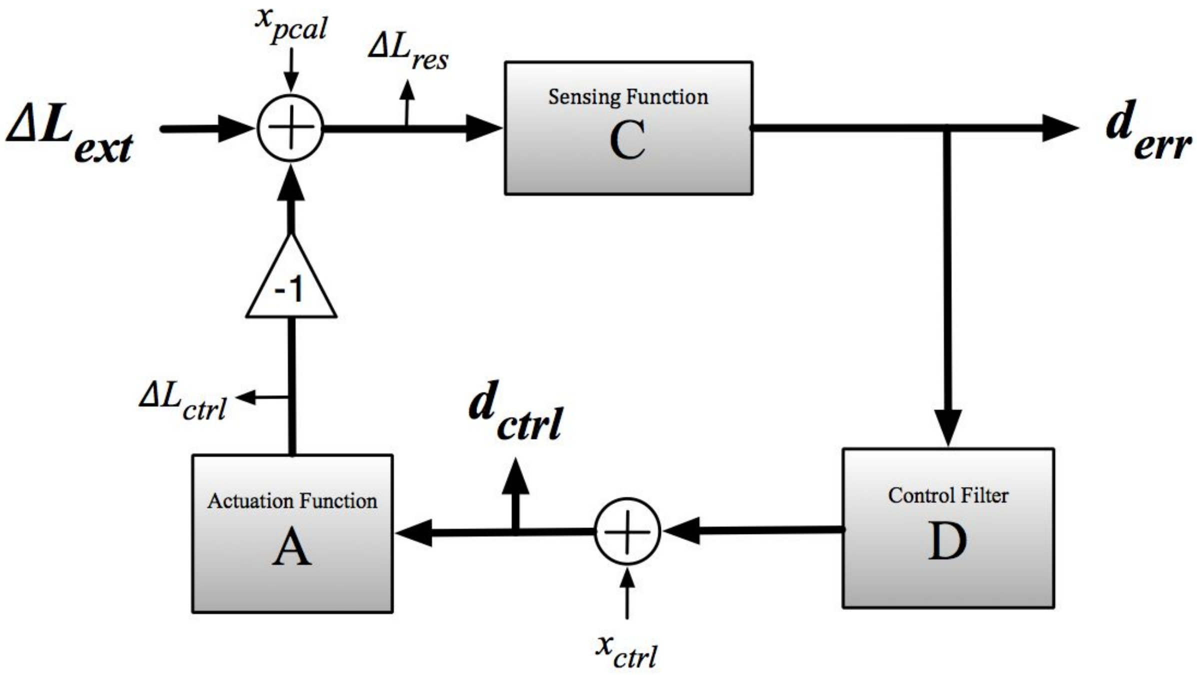
Measurements are specific transfer functions taken of our DARM loop.

We isolate the sensing plant *C(f)* and actuation plant *A(f)* since they are the part of the DARM loop we don't know well.

We must also measure the open loop gain *G(f)=C(f)D(f)A(f)* because we must measure when the IFO is in full lock with the DARM loop closed. This allows us to decouple the loop suppression from the measurement of *C(f)* and *A(f)*.

IFO Response to Gravitational Waves

$$R(f) = \frac{1}{C(f)} + D(f)\,A(f) = \frac{1 + i\,f/f_{CC}(t)}{\kappa_C(t) C_R(f)} + D(f)\;[\kappa_{tst}(t) A_{tst}(f) + \kappa_{pu}(t) A_{pum}(f) + \kappa_{pu}(t) A_{uim}(f)]$$



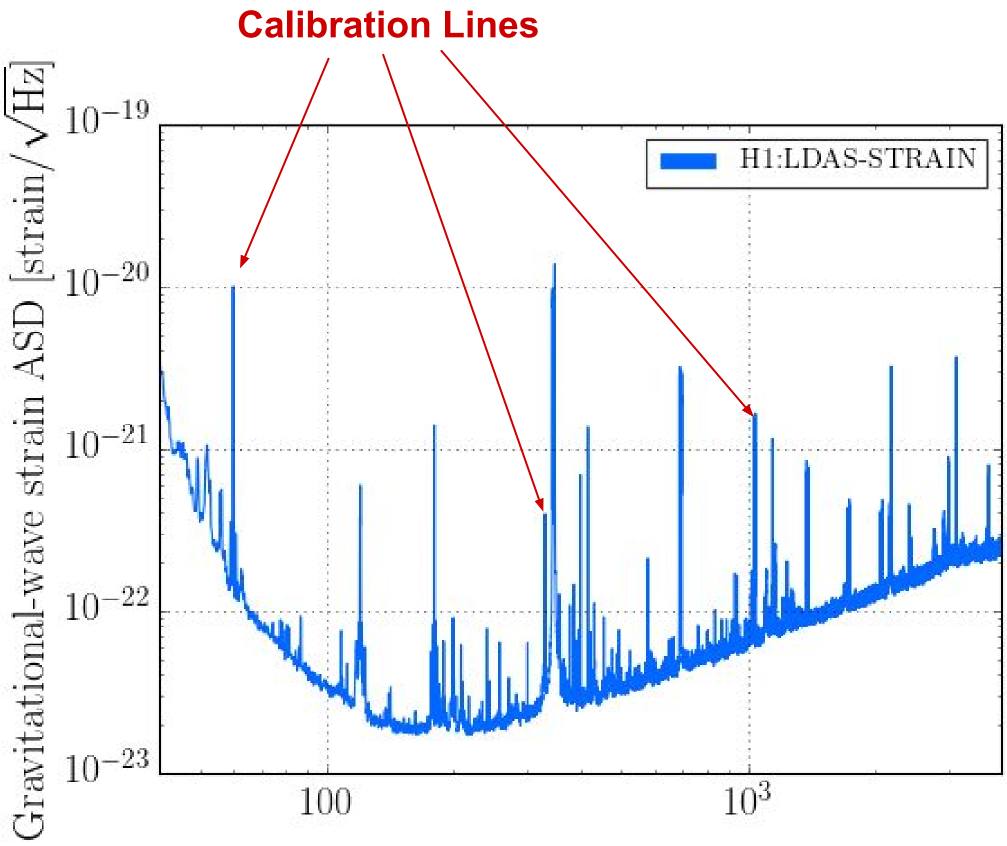
Kappa Measurement Technique

Calibration line excitations monitor our plant continuously. Specific combinations of these calibration lines feed us information on our changing interferometer response.

They track overall changes in our frequency dependent functions $C_R(f)$, $A_{TST}(f)$, $A_{PUM}(f)$, and $A_{UIM}(f)$.

These time dependent factors are called “kappas”. These volatile measurements are smoothed by Maddie, and the uncertainties are quantified by me.

More information is available in the time dependence paper by Darkhan and Sudarshan.



Calibration Parameters

$$R(f) = \frac{1}{C(f)} + D(f)\,A(f) = \frac{1 + i\,f/f_{CC}(t)}{\kappa_C(t)C_R(f)} + D(f)\,\left[\kappa_{tst}(t)A_{tst}(f) + \kappa_{pu}(t)A_{pum}(f) + \kappa_{pu}(t)A_{uim}(f)\right]$$

Frequency Dependent Functions

$C_R(f)$: *Sensing Residual*

$A_{tst}(f)$: *Actuation Test Mass*

$A_{pum}(f)$: *Actuation Penultimate Mass*

$A_{uim}(f)$: *Actuation Upper Intermediate Mass*

Time Dependent Functions

$\kappa_{tst}(t)$: *Electrostatic Drive Strength*

$\kappa_{pu}(t)$: *Electromagnet Actuator Strength*

$\kappa_C(t)$: *Optical Gain*

$f_{CC}(t)$: *Cavity Pole*

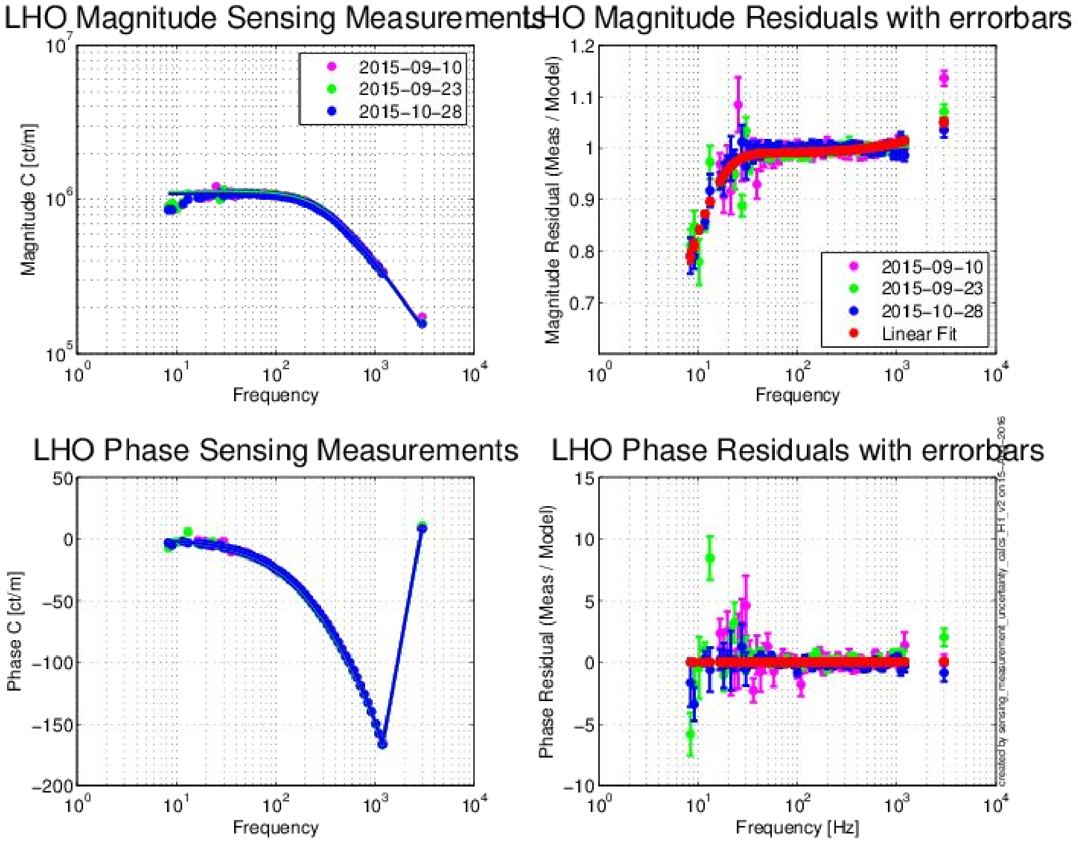
LHO O1 Sensing Measurements

LHO Sensing Measurements:
- Sept 10, 2015,
- Sept 23, 2015
- Oct 28, 2015

Left Plots: C(f) Mag and Phase
Dots = Measurement
Lines = DARM Model

Right Plots: C(f) Residuals
Non-Red = Measurement / Model
Red = Sensing Residuals Fit

Note: The sensing magnitude has a distinct downturn at ~20 Hz. This is probably detuning.



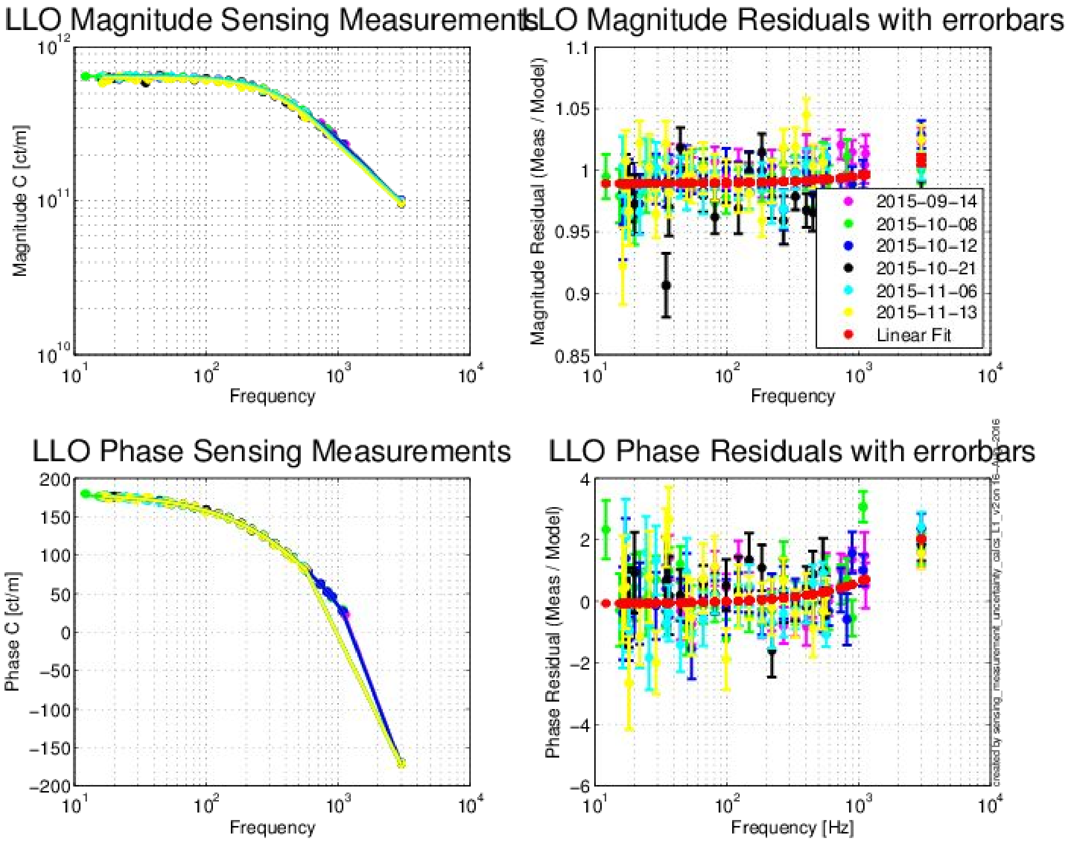
LLO O1 Sensing Measurements

LLO Sensing Measurements:
- Sept 14, Oct 8, Oct 12, Oct 21, Nov 6, Nov 13

Left Plots: C(f) Mag and Phase
Dots = Measurement
Lines = DARM Model

Right Plots: C(f) Residuals
Non-Red = Measurement / Model
Red = Sensing Residuals Fit

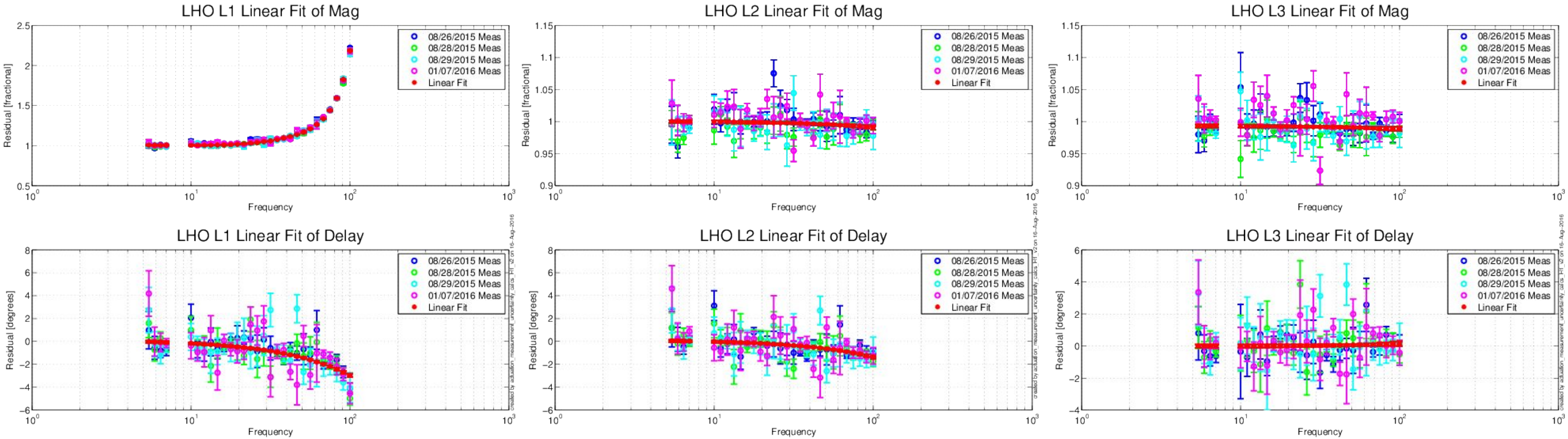
Note: We do not see detuning at Livingston. This warrants some study comparing the two SRCs.



LHO O1 Actuation Measurements

The 4 LHO Actuation Function Measurements: Aug 26, Aug 28, Aug 29, 2015, and Jan 7, 2016.
Only the residuals and their linear fits are shown in these plots.

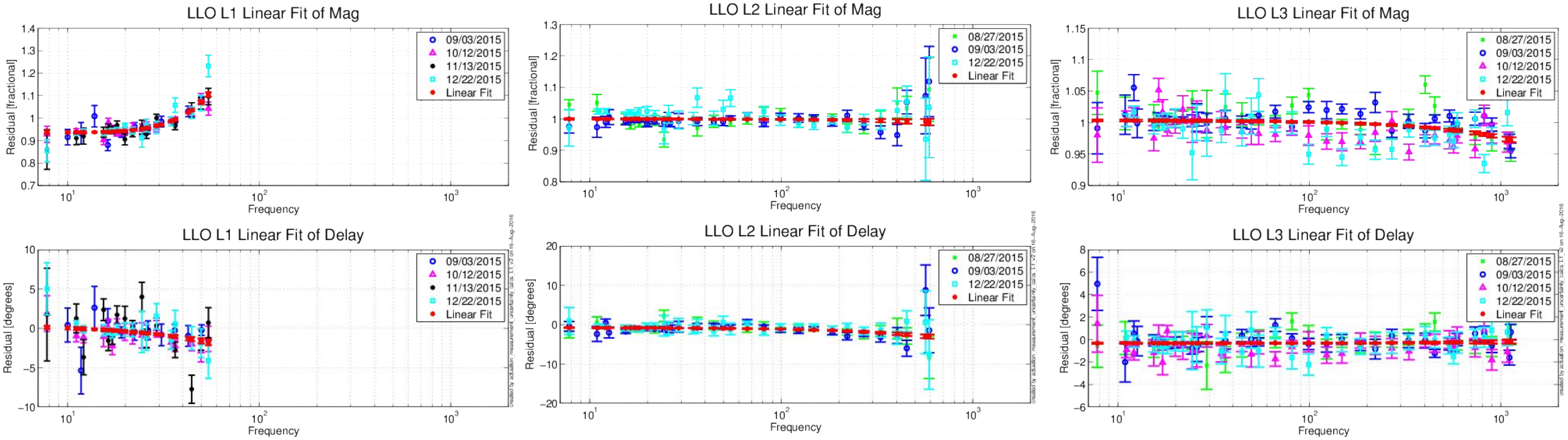
The UIM stage magnitude is not fully understood at higher frequencies, but this is relatively unimportant as the UIM response goes as $1/f^6$ and has negligible impact on *A(f)* above ~15 Hz.
Otherwise the DARM model and measurement agree.



LLO O1 Actuation Measurements

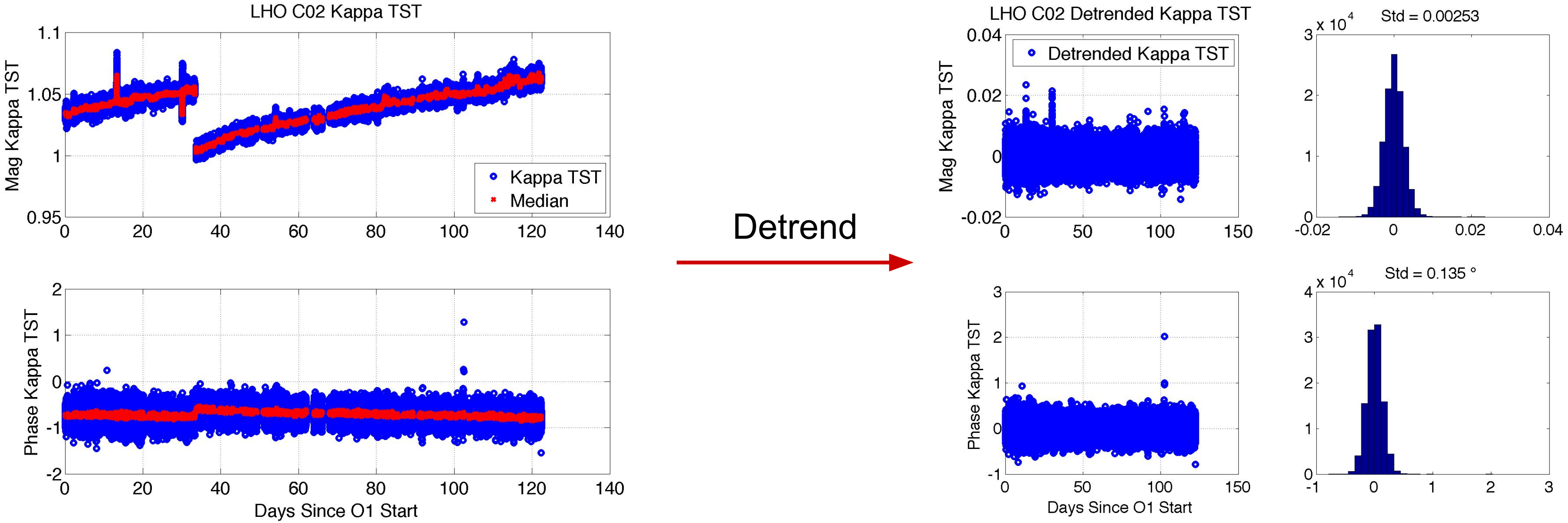
The 5 LLO Actuation Function Measurements: Aug 8, Sept 3, Oct 12, Nov 13, and Dec 22, 2015.
Only the residuals and their linear fits are shown in these plots.
LLO was slightly problematic since they did not measure every actuation stage each measurement day.

The UIM stage magnitude is not fully understood at higher frequencies, but this is relatively unimportant as the UIM response goes as $1/f^6$ and has negligible impact on *A(f)* above ~15 Hz.
Otherwise the DARM model and measurement agree.

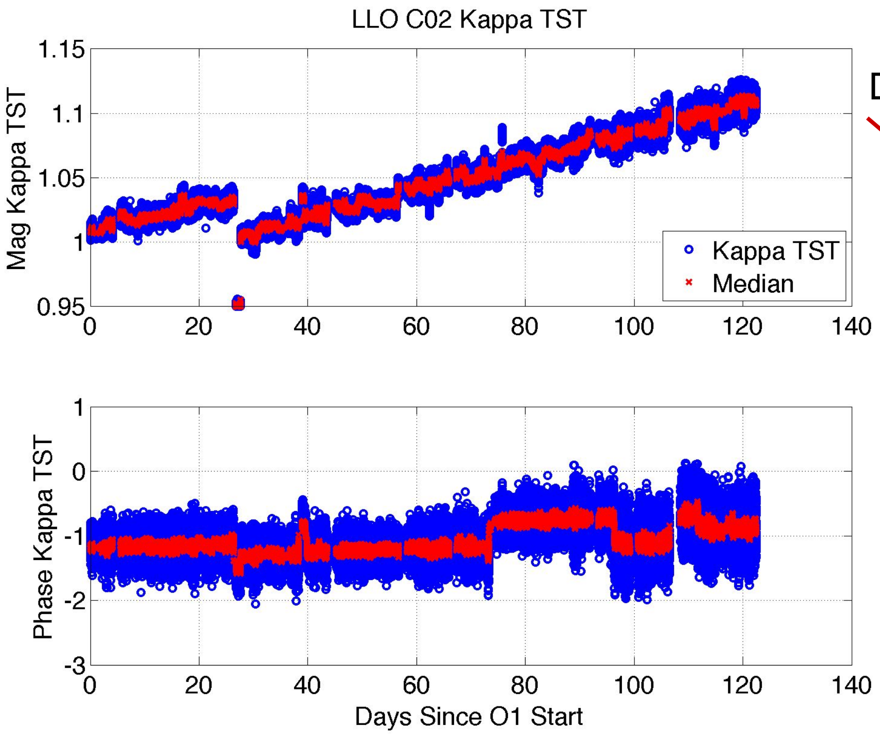


LHO O1 Kappa Measurements

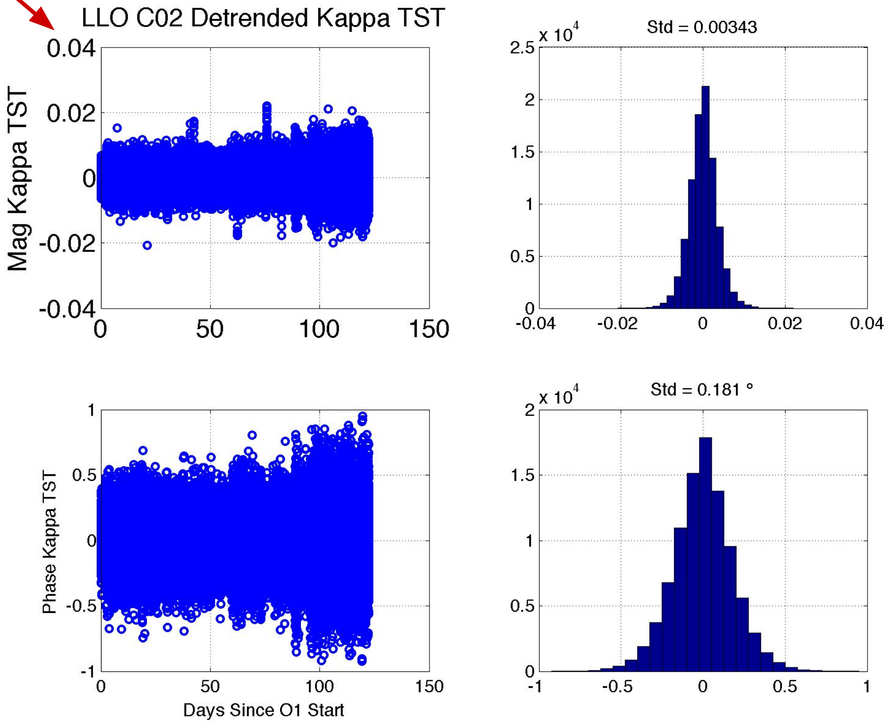
- “Kappas” refers to the IFO time-dependent parameters monitored via calibration lines during IFO operation.
- O1 60 second kappas were provided by Darkhan.
- Data quality vetoes were supplied by Alan.
- We “detrended” the kappas to get a statistical estimate of the kappa values.
    - I found the median kappa value from the 100 values surrounding, but not including, each kappa



LLO O1 Kappa Measurements



Detrend



Uncertainty Quantification

- Now we have our frequency dependent function residuals: $C_{res}(f) = \frac{C_{meas}(f)}{C_{model}(f)}$
-
- We then divide out the fit from the residual: $C_{stat} = \frac{C_{res}(f)}{C_{fit}(f)} - 1$
-
- ====> $C_{meas}(f) = C_{model}(f)\, C_{fit}(f)\; [1 + C_{stat}]$
- We assume this eliminates all frequency dependence from our function, and we are left only with noise
- We quantify this noise by taking the standard deviation of C_{stat}, $\sigma_{C_{stat}}$
- The standard deviation of C_{stat} is taken to be the total relative uncertainty in the sensing function $C(f)$:
-
- $$\frac{\sigma_C(f)}{C(f)} = \sigma_{C_{stat}}$$
-
- What value do we use for $C(f)$?
- I use the DARM model value, $C_{model}(f)$

The above method scales the uncertainty appropriately over its large dynamic range. 1% uncertainty in the bucket is a smaller absolute uncertainty than 1% uncertainty at low or high frequency.

Uncertainty Quantification

- The previous slide is an oversimplification for two reasons:
  - $C(f)$ is a complex function and there is covariance between our real and imaginary function parts.
  - There is covariance between the actuation stages and kappa values.
- How do we quantify covariance?
- For frequency dependent functions, we take the real and imaginary parts of our statistical noise frequency vectors and compare all of them according to the covariance equation:

$$\sigma_{xy} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$

- Similarly, for the time dependent kappas we take the real and imaginary components at each GPS time and get covariance.
- Once we have the covariance values between all of our component parameters, I construct the **Covariance Matrix** Σ from those values.

Complex Uncertainty Overview

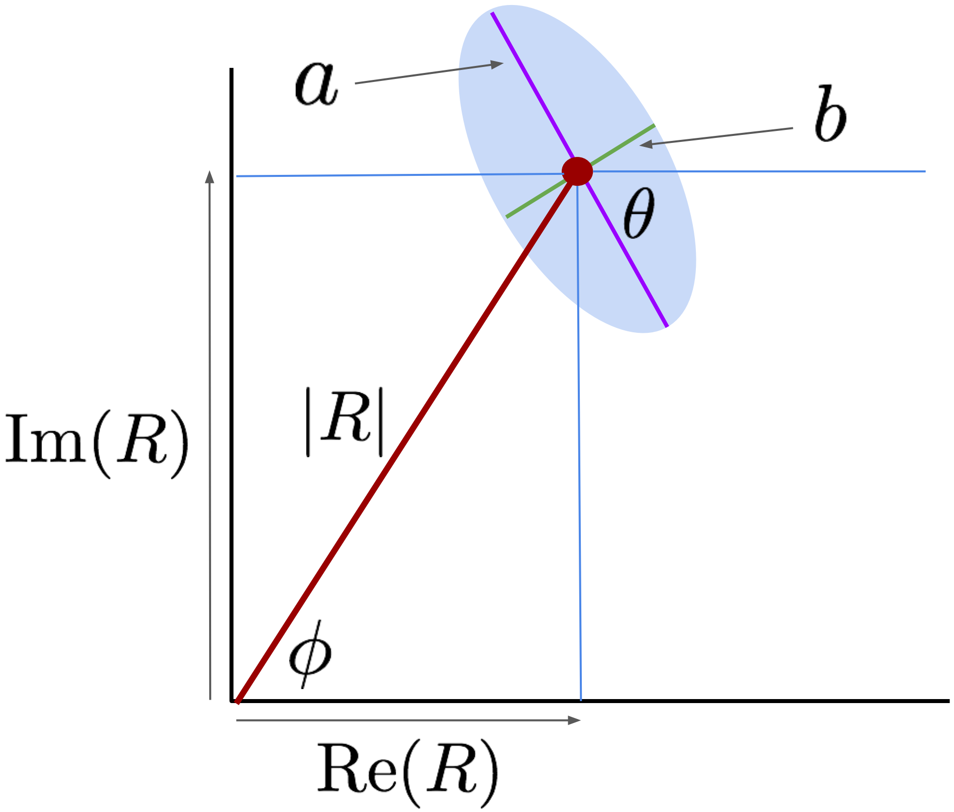
"Stick and ball" uncertainty picture:
The complex number *R* is shown as a vector in the complex plane.

$$R = |R|e^{i\phi}$$
$$R = \mathrm{Re}(R) + \mathrm{Im}(R)$$

Uncertainty in *R* can be represented as an ellipse in the complex plane.

Three parameters define an ellipse:

$$a = \text{Semimajor axis}$$
$$b = \text{Semiminor axis}$$
$$\theta = \text{Rotation angle}$$



Frequency Dependent Calibration Parameters

$$A_{meas}(f) = A_{model}(f)\, A_{fit}(f)\ [1 + A_{stat}]$$

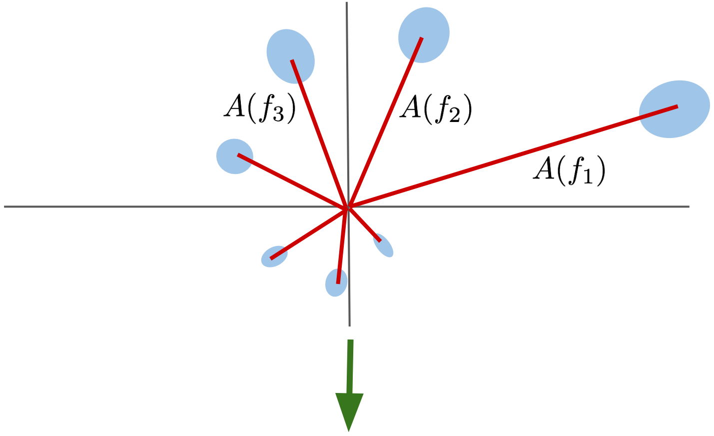
$A_{meas}(f)$ = complex transfer function measurement of the interferometer. Uncertainties are relative.

$A_{model}(f)$ = DARM loop model of the transfer function

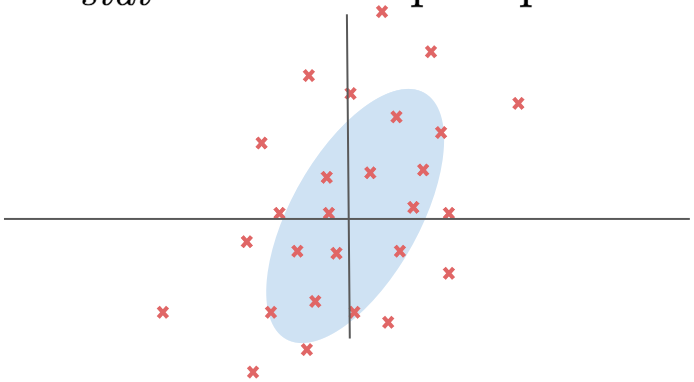
$A_{fit}(f)$ = remaining frequency dependence of $A_{meas}(f)$ not accounted for in $A_{model}(f)$.

A_{stat} = statistical fluctuation of our measurement. This quantifies our relative uncertainty

A_{meas} in the complex plane



A_{stat} in the complex plane

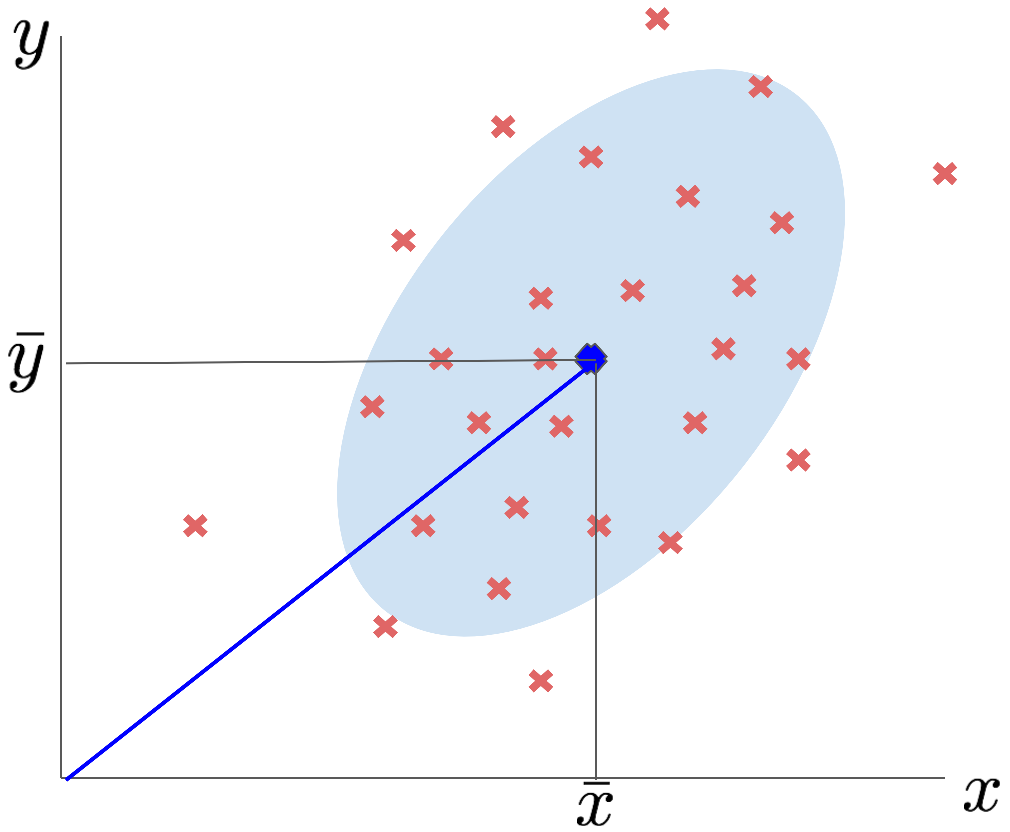


Data ⇔ Covariance Matrix

$$\sigma_{xy} = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$

Covariance between independent variables x and y.

If $x = y$, then the above calculates variance σ_x^2



Covariance Matrix ⇔ Uncertainty Ellipse

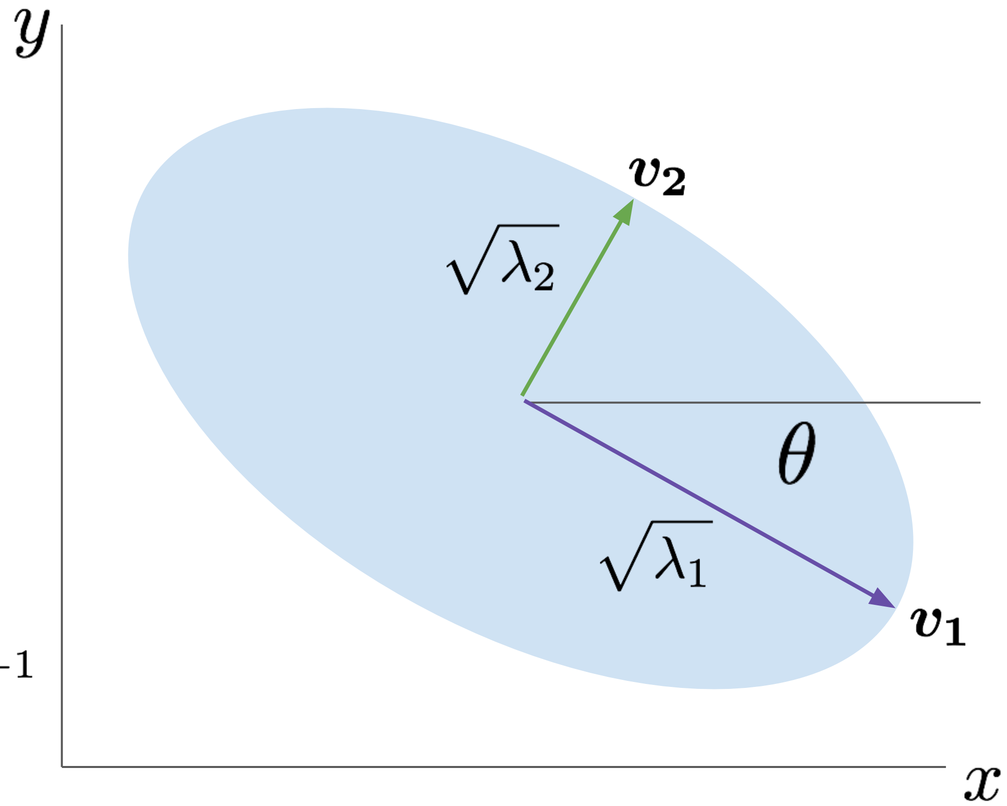
Given a covariance matrix σ_z^2 for some complex quantity z:

$$z = x + iy$$

$$\sigma_z^2 = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$$

We diagonalize σ_z^2 to get the three ellipse parameters:

$$\sigma_z^2 = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} \end{bmatrix}^{-1}$$



Covariance Matrix

The covariance matrix is the key to our new method of uncertainty propagation.

A covariance matrix is a square, symmetric, positive definite matrix whose entries characterize the variance of individual parameters with themselves and covariance of parameters with each other.

The matter is further complicated by the fact that our entries are the real and imaginary components of complex functions, AND are relative uncertainties rather than absolute uncertainties.

Our output should be absolute uncertainty in the complex response as a function of frequency: $\sigma_R(f)$

Total Covariance Matrix

$$\Sigma = \begin{bmatrix} \Sigma_A & 0 & 0 \\ 0 & \Sigma_C & 0 \\ 0 & 0 & \Sigma_\kappa \end{bmatrix}$$

$\Sigma_A =$ Actuation Covariance Matrix

$\Sigma_C =$ Sensing Covariance Matrix

$\Sigma_\kappa =$ Kappa Covariance Matrix

Covariance Matrix

We construct the total covariance matrix out of its components: $\Sigma_A, \Sigma_C, \Sigma_\kappa$

The covariance between $A(f), C(f), \kappa(t)$ is assumed to be zero.

This is because we cannot compute covariance between frequency dependent and time dependent parameters, and actuation and sensing were not always measured using the same frequency vectors or at the same time.

$$\Sigma = \begin{bmatrix} \Sigma_A & 0 & 0 \\ 0 & \Sigma_C & 0 \\ 0 & 0 & \Sigma_\kappa \end{bmatrix}$$

Covariance Matrix Components

$$\Sigma_A = \begin{bmatrix} \sigma^2_{\mathrm{Re}(A_U)} & \sigma_{\mathrm{Re}(A_U)\,\mathrm{Im}(A_U)} & \sigma_{\mathrm{Re}(A_U)\,\mathrm{Re}(A_P)} & \sigma_{\mathrm{Re}(A_U)\,\mathrm{Im}(A_P)} & \sigma_{\mathrm{Re}(A_U)\,\mathrm{Re}(A_T)} & \sigma_{\mathrm{Re}(A_U)\,\mathrm{Im}(A_T)} \\ \sigma_{\mathrm{Im}(A_U)\,\mathrm{Re}(A_U)} & \sigma^2_{\mathrm{Im}(A_U)} & \sigma_{\mathrm{Im}(A_U)\,\mathrm{Re}(A_P)} & \sigma_{\mathrm{Im}(A_U)\,\mathrm{Im}(A_P)} & \sigma_{\mathrm{Im}(A_U)\,\mathrm{Re}(A_T)} & \sigma_{\mathrm{Im}(A_U)\,\mathrm{Im}(A_T)} \\ \sigma_{\mathrm{Re}(A_P)\,\mathrm{Re}(A_U)} & \sigma_{\mathrm{Re}(A_P)\,\mathrm{Im}(A_U)} & \sigma^2_{\mathrm{Re}(A_P)} & \sigma_{\mathrm{Re}(A_P)\,\mathrm{Im}(A_P)} & \sigma_{\mathrm{Re}(A_P)\,\mathrm{Re}(A_T)} & \sigma_{\mathrm{Re}(A_P)\,\mathrm{Im}(A_T)} \\ \sigma_{\mathrm{Im}(A_P)\,\mathrm{Re}(A_U)} & \sigma_{\mathrm{Im}(A_P)\,\mathrm{Im}(A_U)} & \sigma_{\mathrm{Im}(A_P)\,\mathrm{Re}(A_P)} & \sigma^2_{\mathrm{Im}(A_P)} & \sigma_{\mathrm{Im}(A_P)\,\mathrm{Re}(A_T)} & \sigma_{\mathrm{Im}(A_P)\,\mathrm{Im}(A_T)} \\ \sigma_{\mathrm{Re}(A_T)\,\mathrm{Re}(A_U)} & \sigma_{\mathrm{Re}(A_T)\,\mathrm{Im}(A_U)} & \sigma_{\mathrm{Re}(A_T)\,\mathrm{Re}(A_P)} & \sigma_{\mathrm{Re}(A_T)\,\mathrm{Im}(A_P)} & \sigma^2_{\mathrm{Re}(A_T)} & \sigma_{\mathrm{Re}(A_T)\,\mathrm{Im}(A_T)} \\ \sigma_{\mathrm{Im}(A_T)\,\mathrm{Re}(A_U)} & \sigma_{\mathrm{Im}(A_T)\,\mathrm{Im}(A_U)} & \sigma_{\mathrm{Im}(A_T)\,\mathrm{Re}(A_P)} & \sigma_{\mathrm{Im}(A_T)\,\mathrm{Im}(A_P)} & \sigma_{\mathrm{Im}(A_T)\,\mathrm{Re}(A_T)} & \sigma^2_{\mathrm{Im}(A_T)} \end{bmatrix}$$

$$\Sigma_C = \begin{bmatrix} \sigma^2_{\mathrm{Re}(C_R)} & \sigma_{\mathrm{Re}(C_R)\,\mathrm{Im}(C_R)} \\ \sigma_{\mathrm{Im}(C_R)\,\mathrm{Re}(C_R)} & \sigma^2_{\mathrm{Im}(C_R)} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_A & 0 & 0 \\ 0 & \Sigma_C & 0 \\ 0 & 0 & \Sigma_\kappa \end{bmatrix}$$

$$\Sigma_\kappa = \begin{bmatrix} \sigma^2_{\mathrm{Re}(\kappa_T)} & \sigma_{\mathrm{Re}(\kappa_T)\,\mathrm{Im}(\kappa_T)} & \sigma_{\mathrm{Re}(\kappa_T)\,\mathrm{Re}(\kappa_P)} & \sigma_{\mathrm{Re}(\kappa_T)\,\mathrm{Im}(\kappa_P)} & \sigma_{\mathrm{Re}(\kappa_T)\,\kappa_C} & \sigma_{\mathrm{Re}(\kappa_T)\,f_{CC}} \\ \sigma_{\mathrm{Im}(\kappa_T)\,\mathrm{Re}(\kappa_T)} & \sigma^2_{\mathrm{Im}(\kappa_T)} & \sigma_{\mathrm{Im}(\kappa_T)\,\mathrm{Re}(\kappa_P)} & \sigma_{\mathrm{Im}(\kappa_T)\,\mathrm{Im}(\kappa_P)} & \sigma_{\mathrm{Im}(\kappa_T)\,\kappa_C} & \sigma_{\mathrm{Im}(\kappa_T)\,f_{CC}} \\ \sigma_{\mathrm{Re}(\kappa_P)\,\mathrm{Re}(\kappa_T)} & \sigma_{\mathrm{Re}(\kappa_P)\,\mathrm{Im}(\kappa_T)} & \sigma^2_{\mathrm{Re}(\kappa_P)} & \sigma_{\mathrm{Re}(\kappa_P)\,\mathrm{Im}(\kappa_P)} & \sigma_{\mathrm{Re}(\kappa_P)\,\kappa_C} & \sigma_{\mathrm{Re}(\kappa_P)\,f_{CC}} \\ \sigma_{\mathrm{Im}(\kappa_P)\,\mathrm{Re}(\kappa_T)} & \sigma_{\mathrm{Im}(\kappa_P)\,\mathrm{Im}(\kappa_T)} & \sigma_{\mathrm{Im}(\kappa_P)\,\mathrm{Re}(\kappa_P)} & \sigma^2_{\mathrm{Im}(\kappa_P)} & \sigma_{\mathrm{Im}(\kappa_P)\,\kappa_C} & \sigma_{\mathrm{Im}(\kappa_P)\,f_{CC}} \\ \sigma_{\kappa_C\,\mathrm{Re}(\kappa_T)} & \sigma_{\kappa_C\,\mathrm{Im}(\kappa_T)} & \sigma_{\kappa_C\,\mathrm{Re}(\kappa_P)} & \sigma_{\kappa_C\,\mathrm{Im}(\kappa_P)} & \sigma^2_{\kappa_C} & \sigma_{\kappa_C\,f_{CC}} \\ \sigma_{f_{CC}\,\mathrm{Re}(\kappa_T)} & \sigma_{f_{CC}\,\mathrm{Im}(\kappa_T)} & \sigma_{f_{CC}\,\mathrm{Re}(\kappa_P)} & \sigma_{f_{CC}\,\mathrm{Im}(\kappa_P)} & \sigma_{f_{CC}\,\kappa_C} & \sigma^2_{f_{CC}} \end{bmatrix}$$

Uncertainty From Covariance Matrix

If we have a function $y(x_1, x_2, \ldots, x_N)$, its uncertainty can be calculated by taking the first-order Taylor expansion w.r.t its parameters $x_1, x_2, \ldots, x_N$:

$$\sigma_y^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial y}{\partial x_i} \right) \left(\frac{\partial y}{\partial x_j} \right) \sigma_{x_i x_j}$$

This is basic uncertainty propagation. The above can be expressed as a matrix multiplication:

$$\sigma_y^2 = \boldsymbol{J}^T \boldsymbol{\Sigma} \boldsymbol{J}$$

where

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial y}{\partial x_1} \\ \frac{\partial y}{\partial x_2} \\ \vdots \\ \frac{\partial y}{\partial x_N} \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1 x_2} & \cdots & \sigma_{x_1 x_N} \\ \sigma_{x_2 x_1} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2 x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_N x_1} & \sigma_{x_N x_2} & \cdots & \sigma_{x_N}^2 \end{bmatrix}$$

Key Equation

Calibration Uncertainty

Matrix Uncertainty Propagation: $$\sigma_y^2 = \boldsymbol{J}^T \boldsymbol{\Sigma} \boldsymbol{J}$$

In calibration our final result is a *complex* response function $R(f)$ with arbitrary N frequency points.

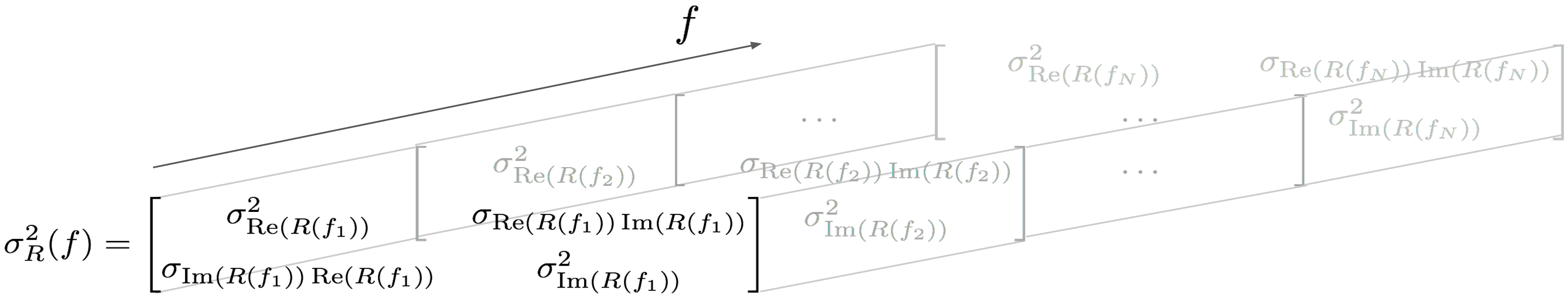
This means our uncertainty in $R(f)$, $\sigma_R(f)$, must be a **2 x 2 x N** matrix composed of its real and imaginary parts for **N** frequency points:

$$\sigma_R^2(f) = \begin{bmatrix} \sigma^2_{\mathrm{Re}(R)} & \sigma_{\mathrm{Re}(R)\,\mathrm{Im}(R)} \\ \sigma_{\mathrm{Im}(R)\,\mathrm{Re}(R)} & \sigma^2_{\mathrm{Im}(R)} \end{bmatrix}$$

We can get $\sigma_R(f)$ directly from our covariance matrix already constructed from the real and imaginary parts of our calibration parameters. We need to do two things:

- Expand our complex vector $\boldsymbol{J}$ into real and imaginary parts.
- Factor the frequency dependence back into our covariance matrix. (Recall that we formed a relative covariance matrix from C_{stat}, A_{stat}, κ_{stat} with no frequency or time dependence.)

2 x 2 x N Response Uncertainty Matrix



Expanding $\boldsymbol{J}$ into Real and Imaginary Parts

There are two ways to express a complex number.

Vector form: $z = x + i\,y$

Matrix form: $z = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$

The elements of the vector $\boldsymbol{J}$ are complex: $\frac{\partial R}{\partial p_i}$, where p_i are the calibration parameters.

We essentially take each $\frac{\partial R}{\partial p_i}$ and expand it into its complex matrix form.

$$\frac{\partial R}{\partial p_i} = \begin{bmatrix} \mathrm{Re}\left(\frac{\partial R}{\partial p_i}\right) & -\mathrm{Im}\left(\frac{\partial R}{\partial p_i}\right) \\ \mathrm{Im}\left(\frac{\partial R}{\partial p_i}\right) & \mathrm{Re}\left(\frac{\partial R}{\partial p_i}\right) \end{bmatrix}$$

But does this give us what we need?

Yes, thanks to the Cauchy-Riemann Equations

Expanding $\boldsymbol{J}$ into Real and Imaginary Parts

Cauchy Riemann Equations for complex derivatives:

For $f(z) = u(z) + i\,v(z)$ and $z = x + i\,y$, if f is analytic in the complex plane then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

And
$$\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y}$$

This means we can take complex derivatives and use their real and imaginary parts in $\boldsymbol{J}$

$$\mathrm{Re}\left(\frac{\partial R}{\partial p_i}\right) = \frac{\partial\,\mathrm{Re}(R)}{\partial\,\mathrm{Re}(p_i)} = \frac{\partial\,\mathrm{Im}(R)}{\partial\,\mathrm{Im}(p_i)}$$

$$\mathrm{Im}\left(\frac{\partial R}{\partial p_i}\right) = \frac{\partial\,\mathrm{Im}(R)}{\partial\,\mathrm{Re}(p_i)} = -\frac{\partial\,\mathrm{Re}(R)}{\partial\,\mathrm{Im}(p_i)}$$

Frequency Dependent Covariance Matrix $\boldsymbol{\Sigma}$

End Goal:

$$\sigma_y^2 = \boldsymbol{J}^T \boldsymbol{\Sigma} \boldsymbol{J}$$

Next we need to condition the covariance matrix $\boldsymbol{\Sigma}$ for use over our entire frequency vector.

Recall that all of our matrix elements in $\boldsymbol{\Sigma}$ are real and imaginary ***relative*** uncertainties of our statistical frequency vectors:

$$C_{R\,stat} = \frac{C_{R\,meas}}{C_{R\,model}\,C_{R\,fit}} - 1$$

$$\mathrm{Re}(C_{R\,stat}) \qquad \mathrm{Im}(C_{R\,stat})$$

$$\sigma_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

$$\sigma^2_{C_{R\,stat}} = \begin{bmatrix} \sigma^2_{\mathrm{Re}(C_{R\,stat})} & \sigma_{\mathrm{Re}(C_{R\,stat})\,\mathrm{Im}(C_{R\,stat})} \\ \sigma_{\mathrm{Im}(C_{R\,stat})\,\mathrm{Re}(C_{R\,stat})} & \sigma^2_{\mathrm{Im}(C_{R\,stat})} \end{bmatrix}$$

How do we get back to ***absolute*** uncertainties?

Frequency Dependent Covariance Matrix Σ

Question: How do we get $\sigma_{C_{R\,stat}}$ back to absolute uncertainties?

Answer: Scale everything according to $|C_{R\,model}|$:

$$\sigma^2_{C_R}(f) = |C_R(f)|^2 \begin{bmatrix} \sigma^2_{\mathrm{Re}(C_{R\,stat})} & \sigma_{\mathrm{Re}(C_{R\,stat})\,\mathrm{Im}(C_{R\,stat})} \\ \sigma_{\mathrm{Im}(C_{R\,stat})\,\mathrm{Re}(C_{R\,stat})} & \sigma^2_{\mathrm{Im}(C_{R\,stat})} \end{bmatrix}$$

Why $|C_{R\,model}|$?

- Elements of the covariance matrix must be real.
- The real and imaginary components of $C_{R\,model}$ are sometimes close to zero, but $|C_{R\,model}|$ always maintains the appropriate dynamic range without blowing up.

Frequency Dependent Covariance Matrix $\boldsymbol{\Sigma}$

Question: What about covariance between actuation stages or kappas?

Answer: Again, we scale according to the relevant actuation functions:

$$|A_{U\,model}(f)|$$
$$|A_{P\,model}(f)|$$
$$|A_{T\,model}(f)|$$

$$\sigma_{\mathrm{Re}(A_P)\,\mathrm{Re}(A_T)}(f) = |A_P(f)|\,|A_T(f)|\,\sigma_{\mathrm{Re}(A_{P\,stat})\,\mathrm{Re}(A_{T\,stat})}$$

$$\sigma_{\mathrm{Re}(A_P)\,\mathrm{Im}(A_T)}(f) = |A_P(f)|\,|A_T(f)|\,\sigma_{\mathrm{Re}(A_{P\,stat})\,\mathrm{Im}(A_{T\,stat})}$$

$$\sigma_{\mathrm{Im}(A_P)\,\mathrm{Re}(A_T)}(f) = |A_P(f)|\,|A_T(f)|\,\sigma_{\mathrm{Im}(A_{P\,stat})\,\mathrm{Re}(A_{T\,stat})}$$

$$\sigma_{\mathrm{Im}(A_P)\,\mathrm{Im}(A_T)}(f) = |A_P(f)|\,|A_T(f)|\,\sigma_{\mathrm{Im}(A_{P\,stat})\,\mathrm{Im}(A_{T\,stat})}$$

In this way we get a **14 x 14 x N** frequency dependent covariance matrix $\boldsymbol{\Sigma}(f)$ from our relative **14 x 14** covariance matrix $\boldsymbol{\Sigma}$.

Final Uncertainty Propagation

Now we can finally perform the calculation of the key equation:

$$\sigma_R^2(f) = \boldsymbol{J}^T(f)\,\boldsymbol{\Sigma}(f)\,\boldsymbol{J}(f)$$

Where $\boldsymbol{J}(f)$ is a **14 x 2 x N** matrix, and $\boldsymbol{\Sigma}(f)$ is a **14 x 14 x N** matrix. This gives $\boldsymbol{\sigma}_R^2(f)$ as a **2 x 2 x N** matrix, which is what we set out to get in slide 22.

Woo!
Uncertainty balloons!

(...look like uncertainty ellipses)

Post Processing

Now we would like to look at the uncertainty results with the usual magnitude and phase vs. frequency plots, but our pipeline output is now quite different:

$$\boldsymbol{\sigma}^2_R(f) = \begin{bmatrix} \sigma^2_{\mathrm{Re}(R)}(f) & \sigma_{\mathrm{Re}(R)\,\mathrm{Im}(R)}(f) \\ \sigma_{\mathrm{Im}(R)\,\mathrm{Re}(R)}(f) & \sigma^2_{\mathrm{Im}(R)}(f) \end{bmatrix}$$

We get the real and imaginary component variance as well as their covariance. We can make a simplification to get back to the plots we are familiar with. This is okay as long as the covariant terms of $\boldsymbol{\sigma}^2_R(f)$ are small.

Using the value of $R(f)$, we can use a rotation matrix to get the magnitude and phase uncertainty values from $\boldsymbol{\sigma}^2_R(f)$.

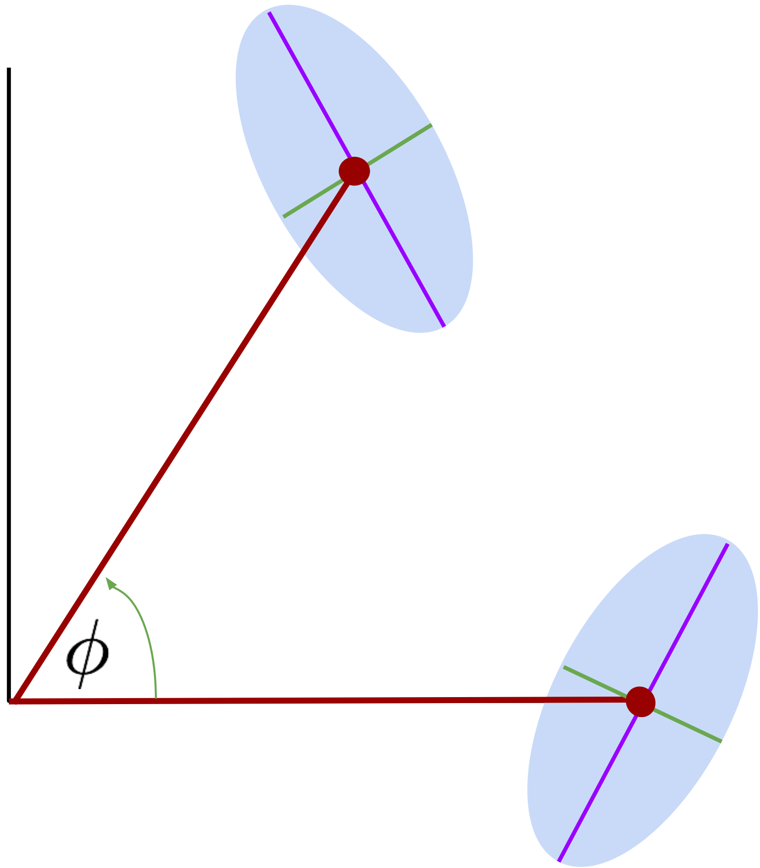
First we find the phase $\phi(f) = \arg(R(f))$, and relative response uncertainty $\frac{\boldsymbol{\sigma}^2_R(f)}{|R(f)|^2}$

Rotation Matrix ⇔ Uncertainty Ellipse

$$\boldsymbol{R}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

To convert from real and imaginary to magnitude and phase coordinate basis, we apply a rotation matrix $\boldsymbol{R}$ to our covariance matrix $\boldsymbol{\Sigma}$:

$$\boldsymbol{\Sigma}_{rot} = \boldsymbol{R}\,\boldsymbol{\Sigma}\,\boldsymbol{R}^T$$



Why Rotation Matrix and Not Jacobian Matrix

Typically when transforming between coordinate systems we use a Jacobian matrix:

Cartesian coordinates → Polar coordinates

$$z = x + iy = |z|e^{i\phi}$$

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial |z|}{\partial x} & \frac{\partial |z|}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{bmatrix}$$

$$\frac{\partial |z|}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \cos\phi$$

$$\frac{\partial |z|}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \sin\phi$$

$$\frac{\partial \phi}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{\sin\phi}{|z|}$$

$$\frac{\partial \phi}{\partial y} = \frac{x}{x^2 + y^2} = \frac{\cos\phi}{|z|}$$

$$\boldsymbol{J} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\frac{\sin\phi}{|z|} & \frac{\cos\phi}{|z|} \end{bmatrix}$$

Looks sorta like a Rotation Matrix...

Polar matrix = J * Cartesian matrix * J^T

$$\boldsymbol{A'} = \boldsymbol{J}\,\boldsymbol{A}\,\boldsymbol{J}^T$$

Why Rotation Matrix and Not Jacobian Matrix

Only difference between $\boldsymbol{J}$ and $\boldsymbol{R}(\phi)$ is sign of the rotation and scaling phase by $|z|$

$$\boldsymbol{J} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\dfrac{\sin\phi}{|z|} & \dfrac{\cos\phi}{|z|} \end{bmatrix}$$

$$\boldsymbol{R}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Typically report uncertainty as Relative in Magnitude: $\dfrac{\sigma_{|z|}}{|z|}$

And Absolute in Phase: σ_ϕ

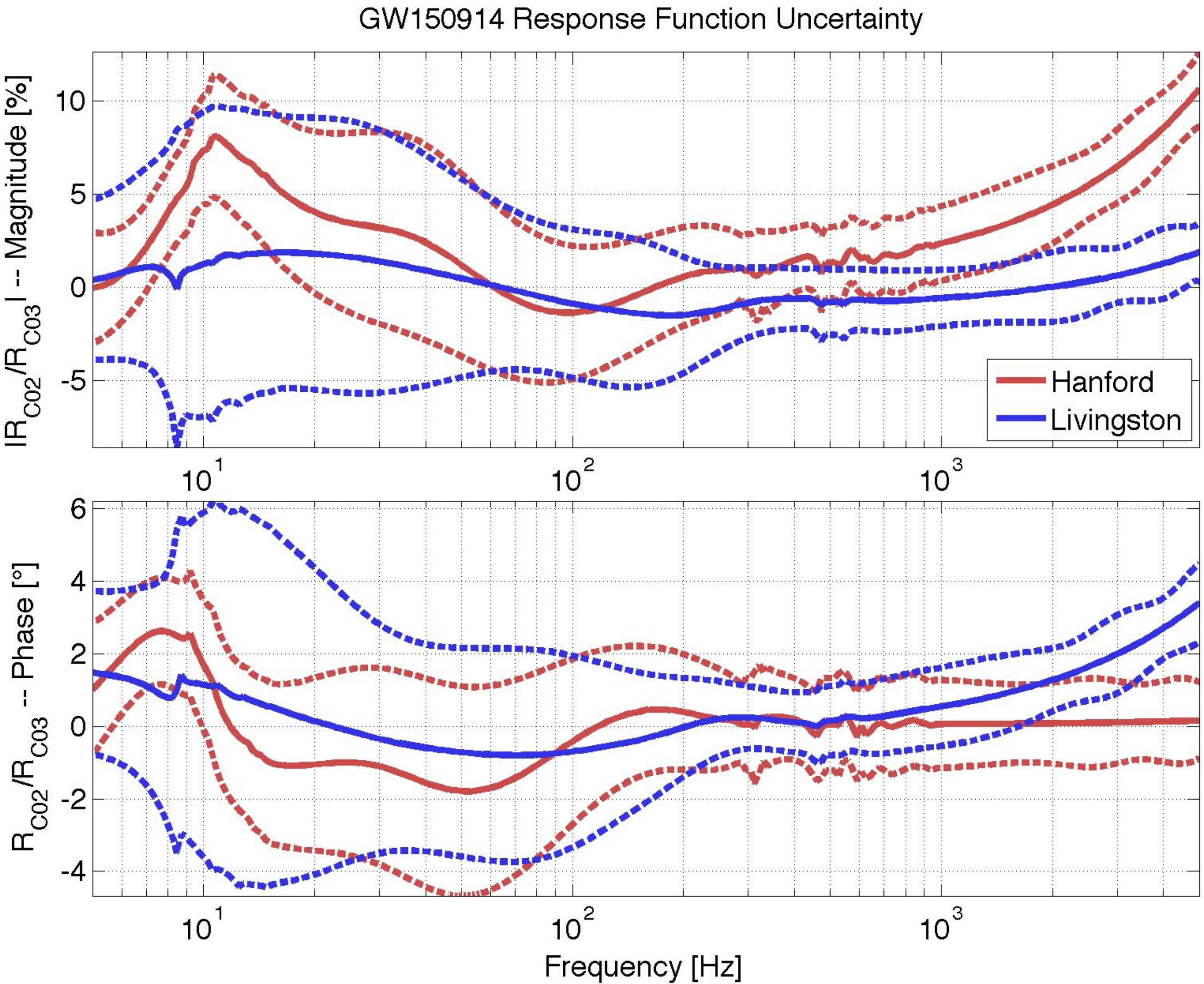
I find uncertainty in Real and Imaginary components in relative terms:

$$\frac{\sigma_{\mathrm{Re}(z)}}{|z|} \qquad \frac{\sigma_{\mathrm{Im}(z)}}{|z|}$$

Using the rotation matrix automatically gives me relative magnitude and absolute phase from relative real and imaginary uncertainty.

C02 Uncertainty Results

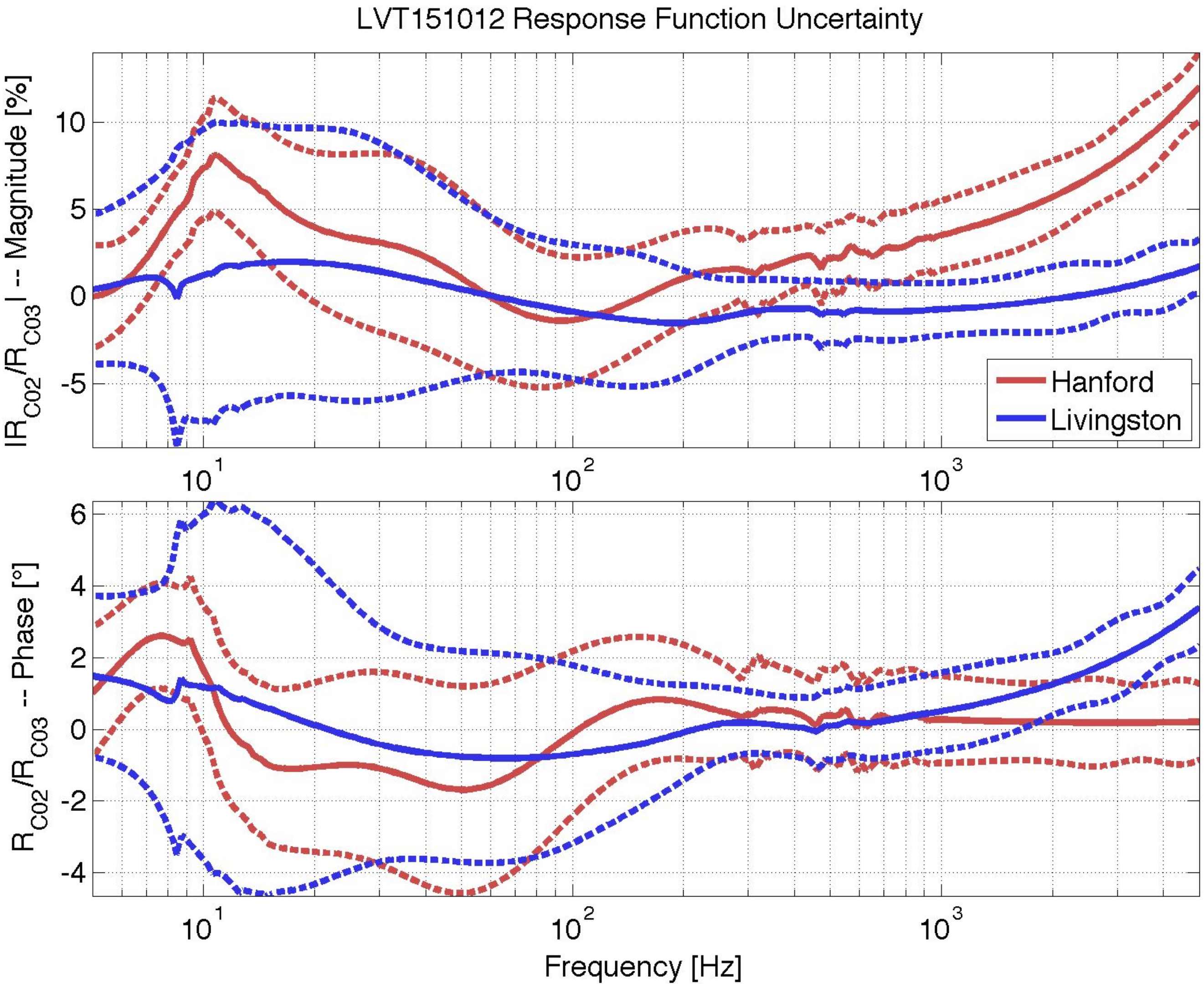
Recall that the C02 results already factor in most of the kappas (except the cavity pole) and should not change too much from GPS time to GPS time.



Freq Range:
5 - 5000 Hz

H1 Cavity Pole:
332.7 Hz

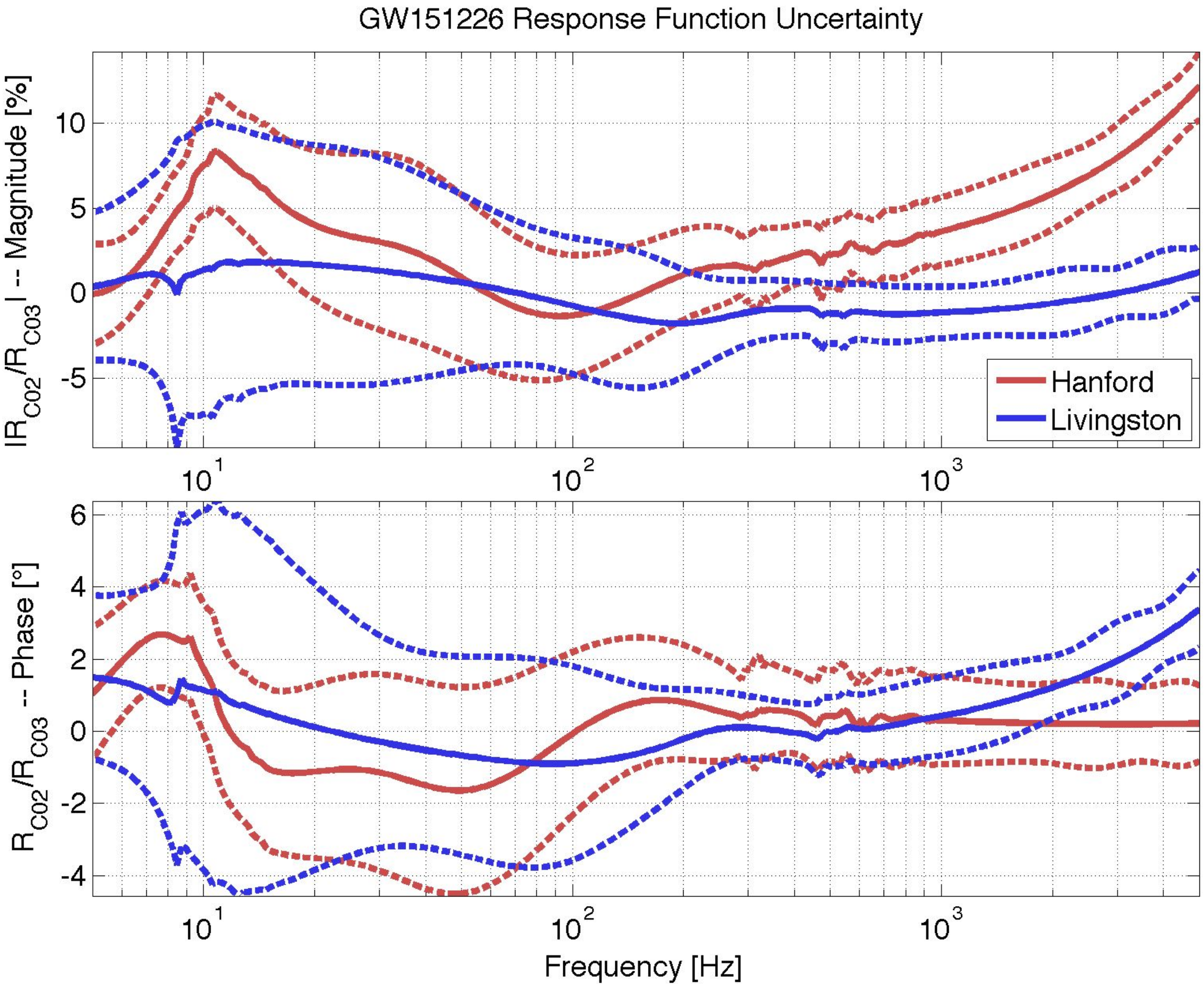
L1 Cavity Pole:
387.5 Hz



Freq Range: 5 - 5000 Hz

H1 Cavity Pole: 336.8 Hz

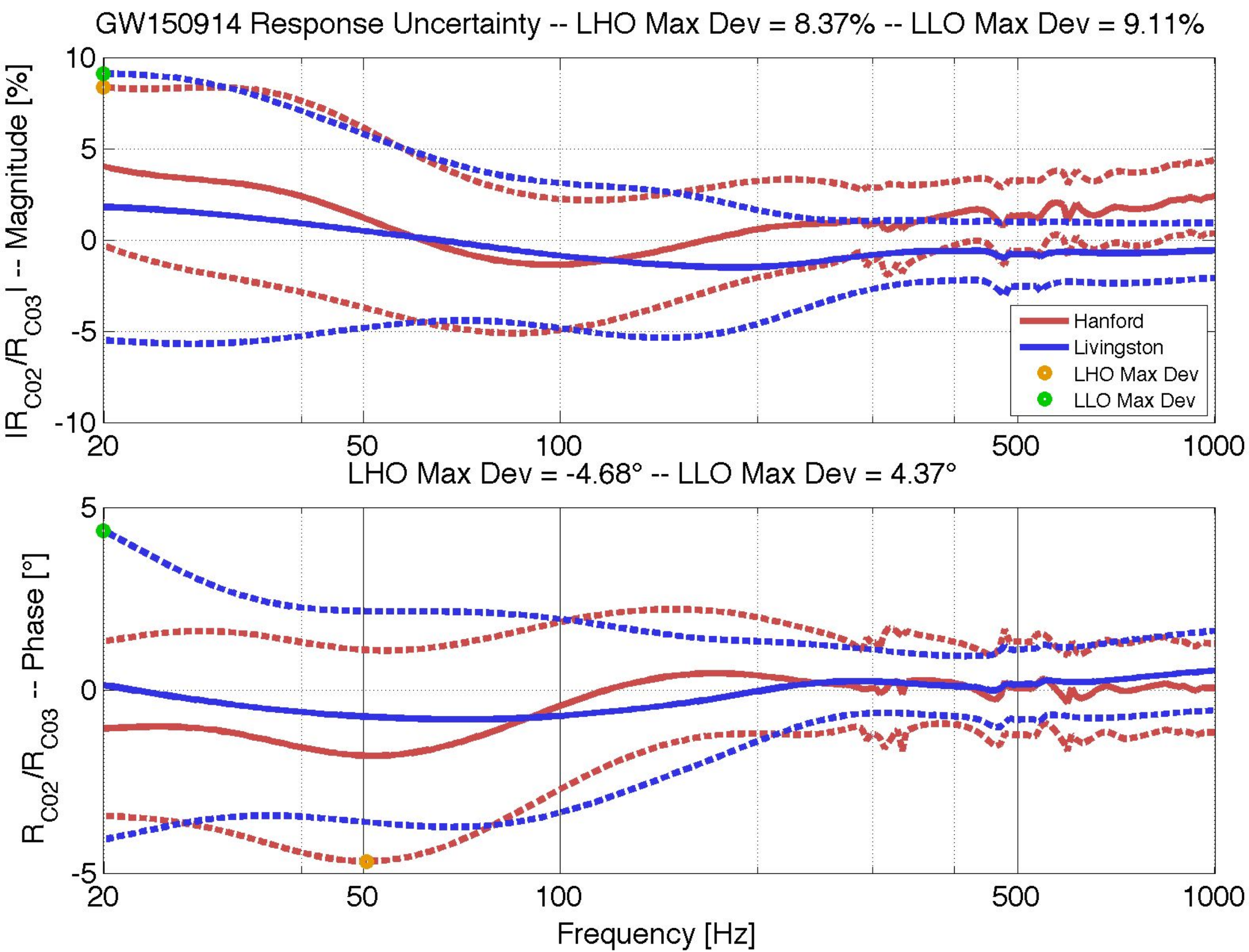
L1 Cavity Pole: 386.7 Hz



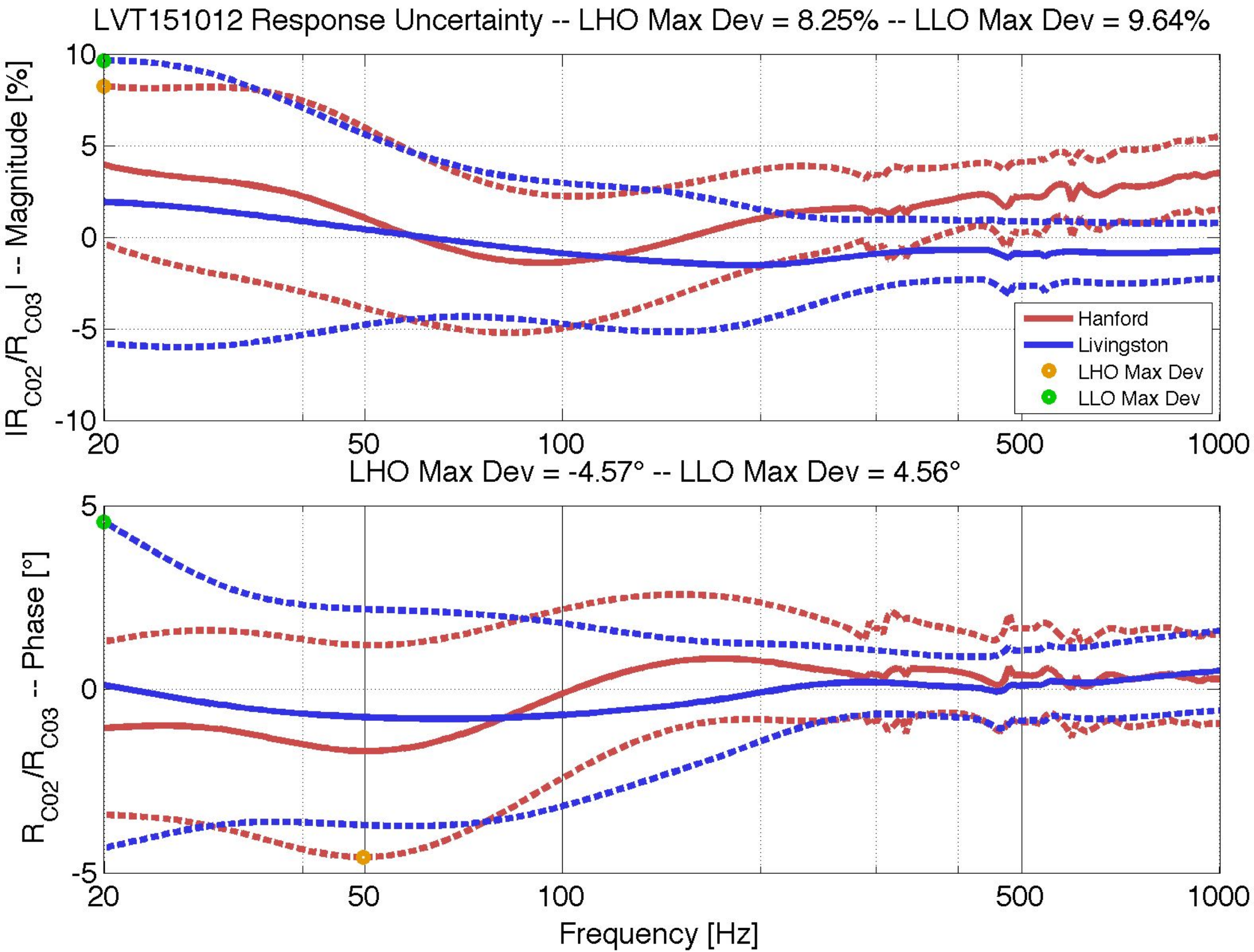
Freq Range:
5 - 5000 Hz

H1 Cavity Pole:
337.3 Hz

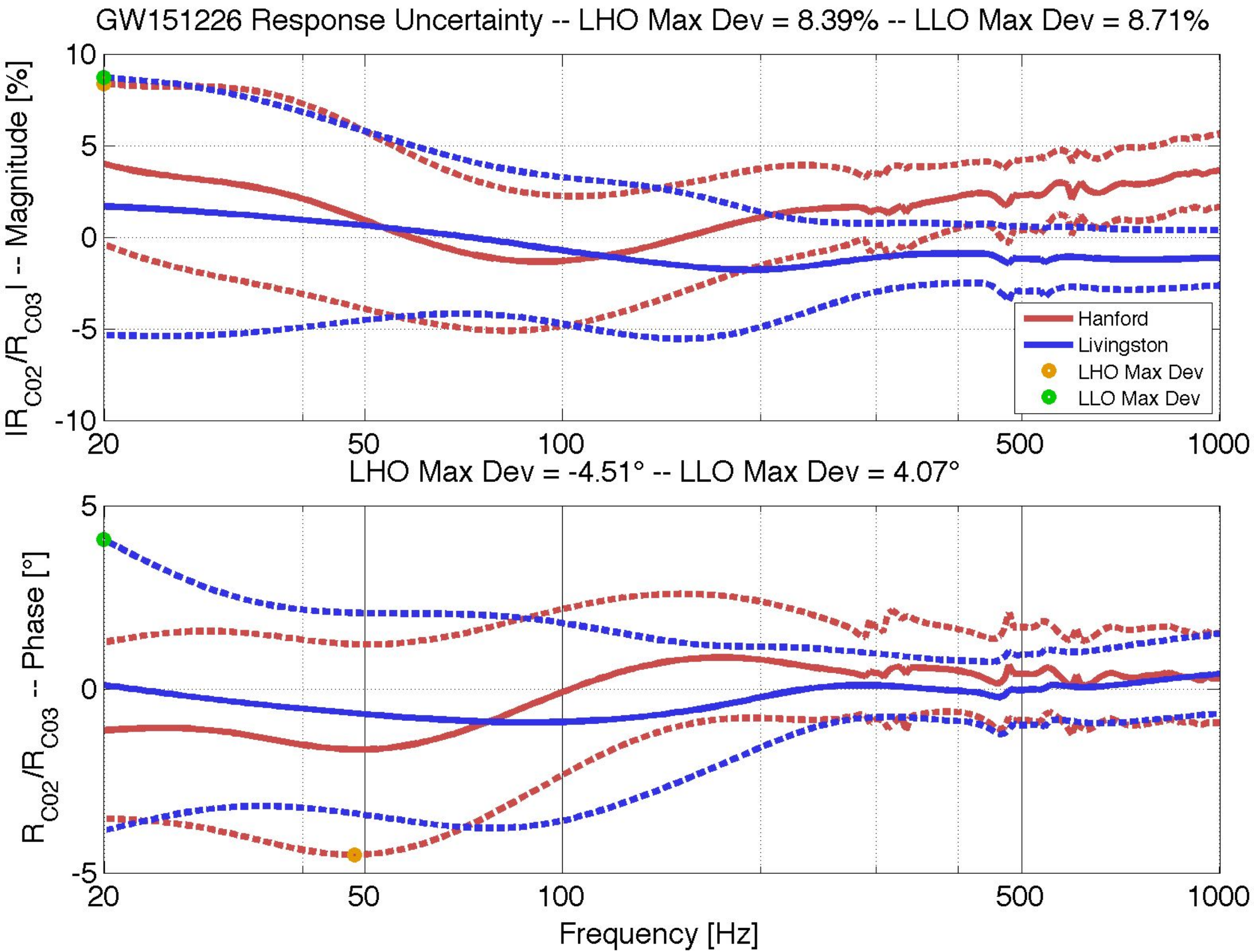
L1 Cavity Pole:
385.0 Hz



Freq Range:
20 - 1000 Hz

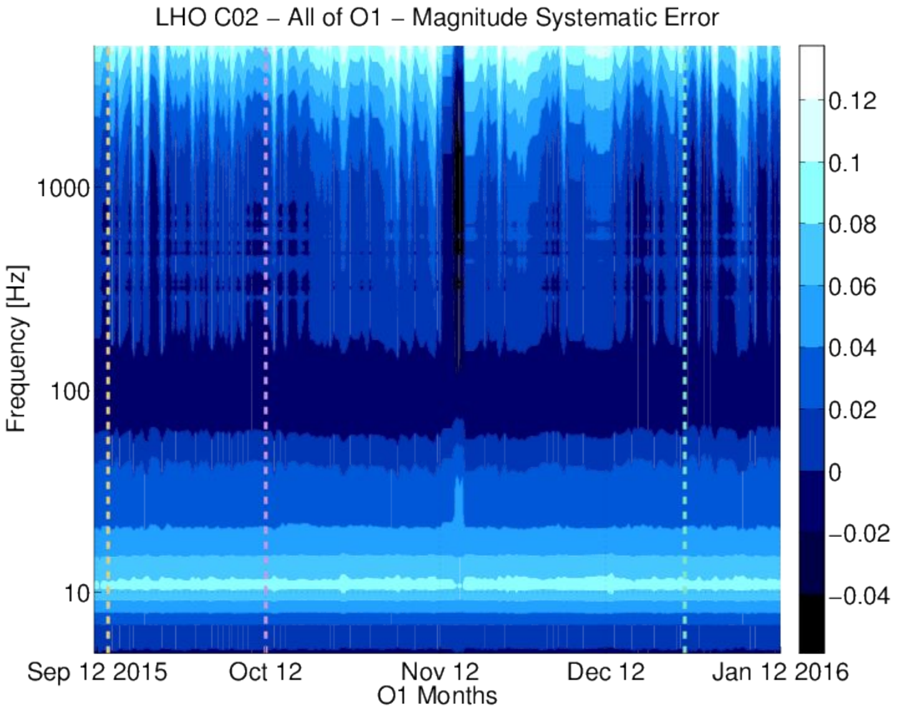


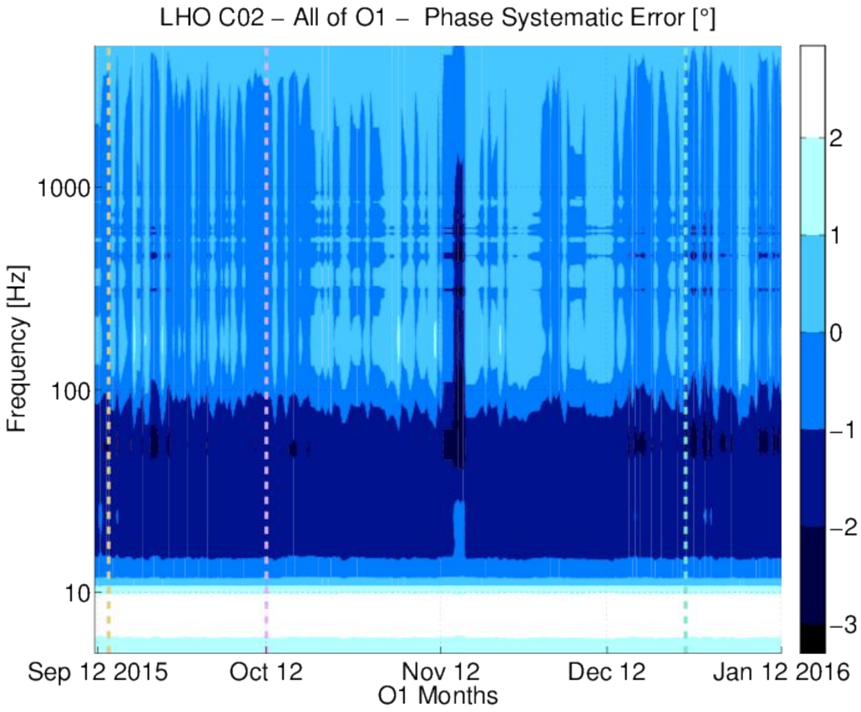
Freq Range: 20 - 1000 Hz



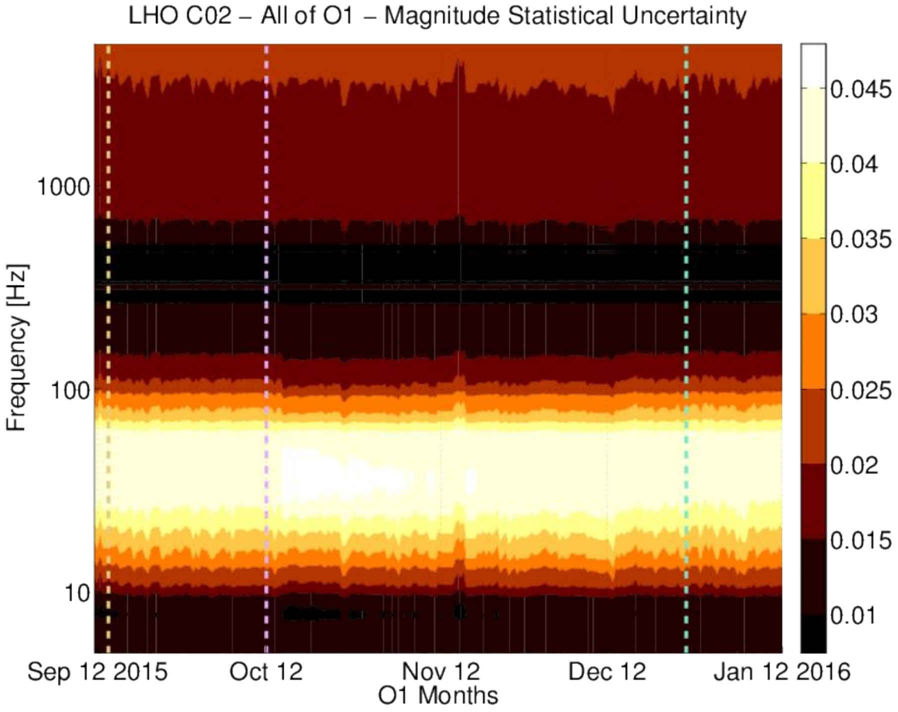
Freq Range:
20 - 1000 Hz

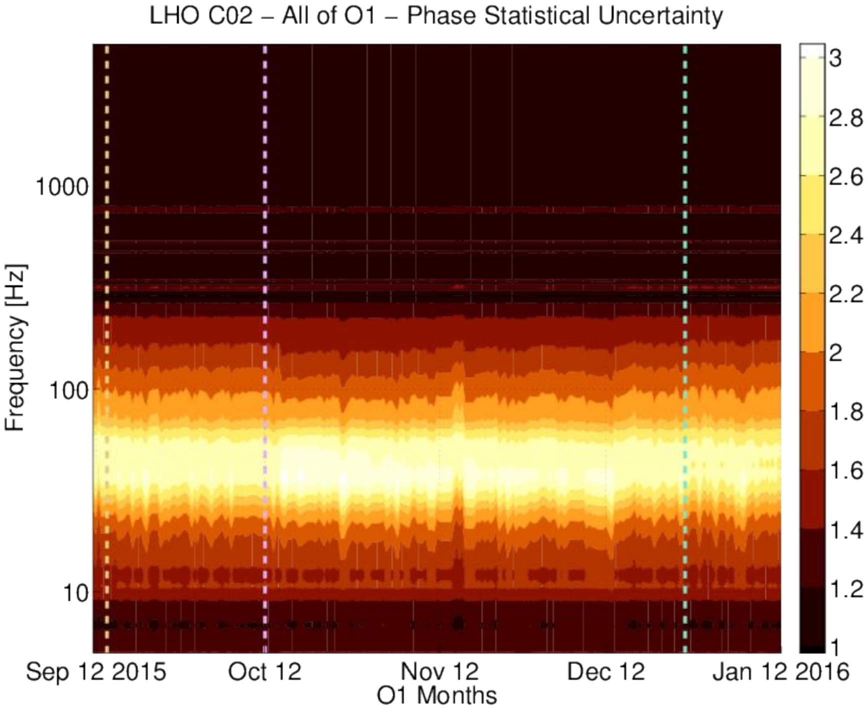
All of O1 **LHO** Systematic Error



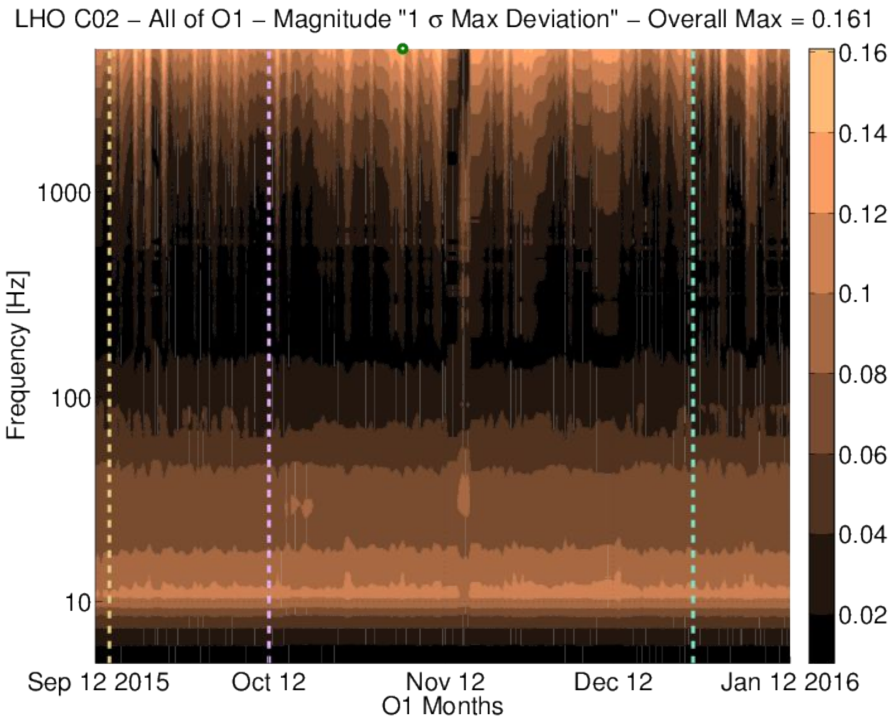


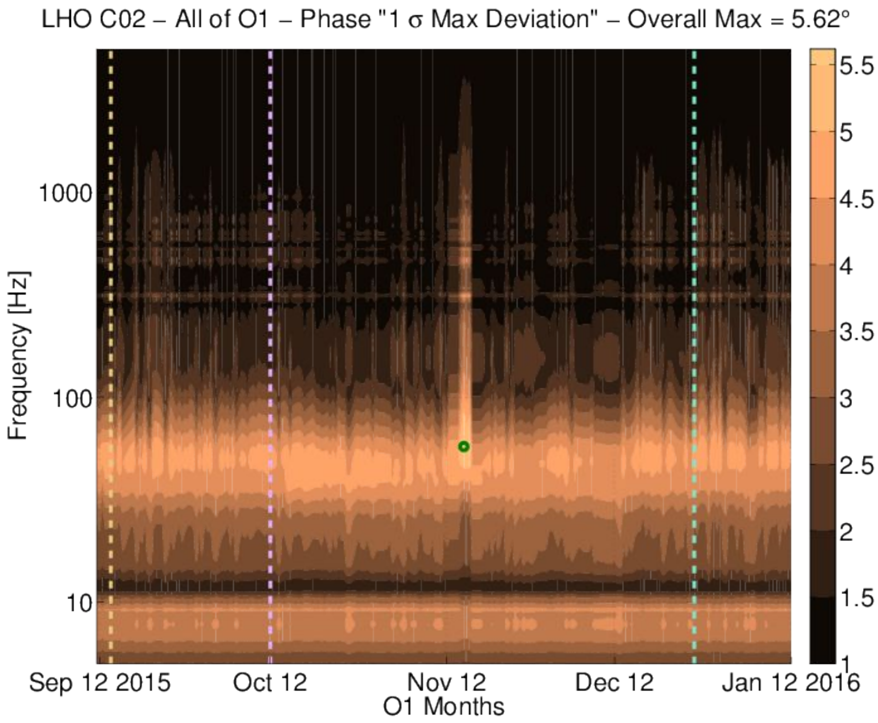
All of O1 **LHO** Statistical Uncertainty



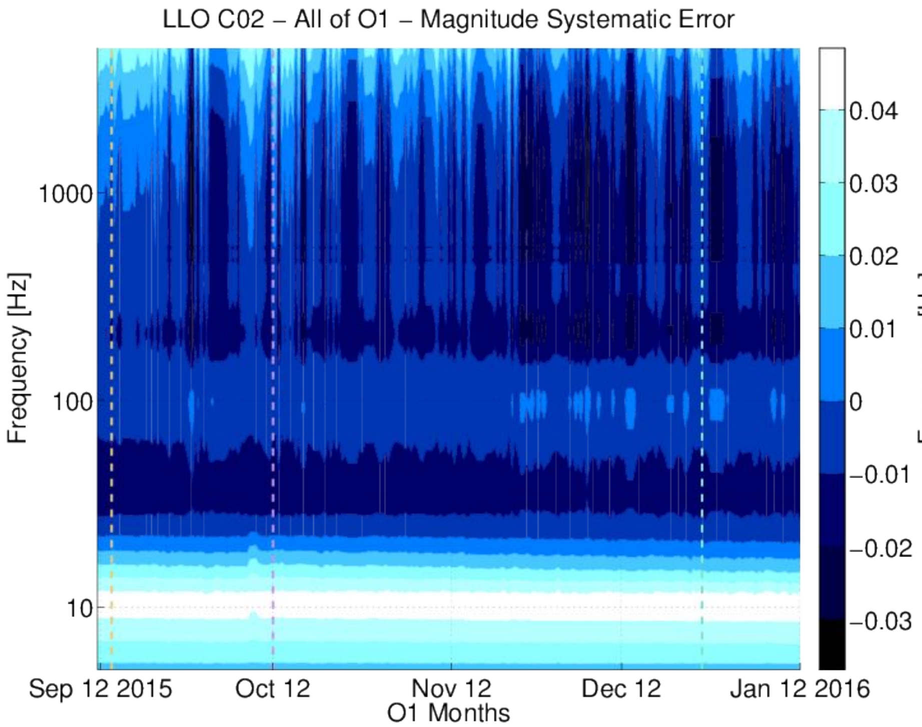


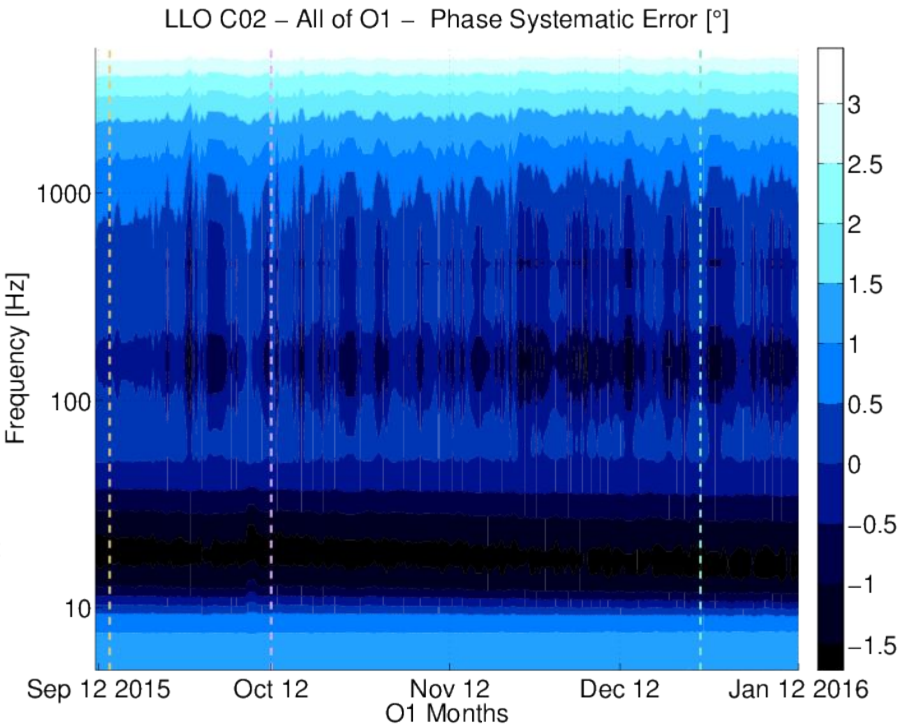
All of O1 **LHO** “Maximum Deviation”



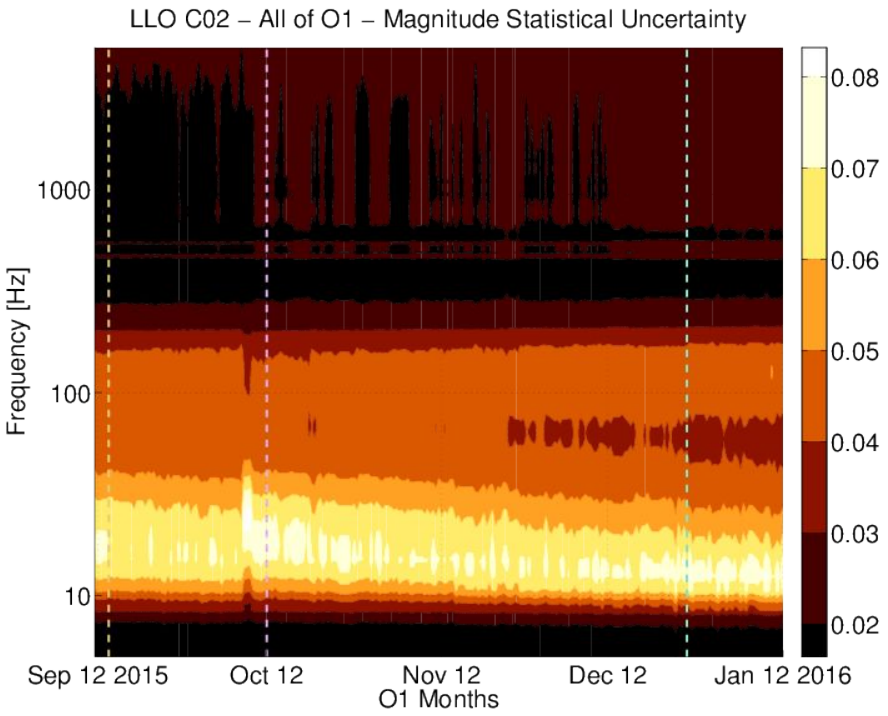


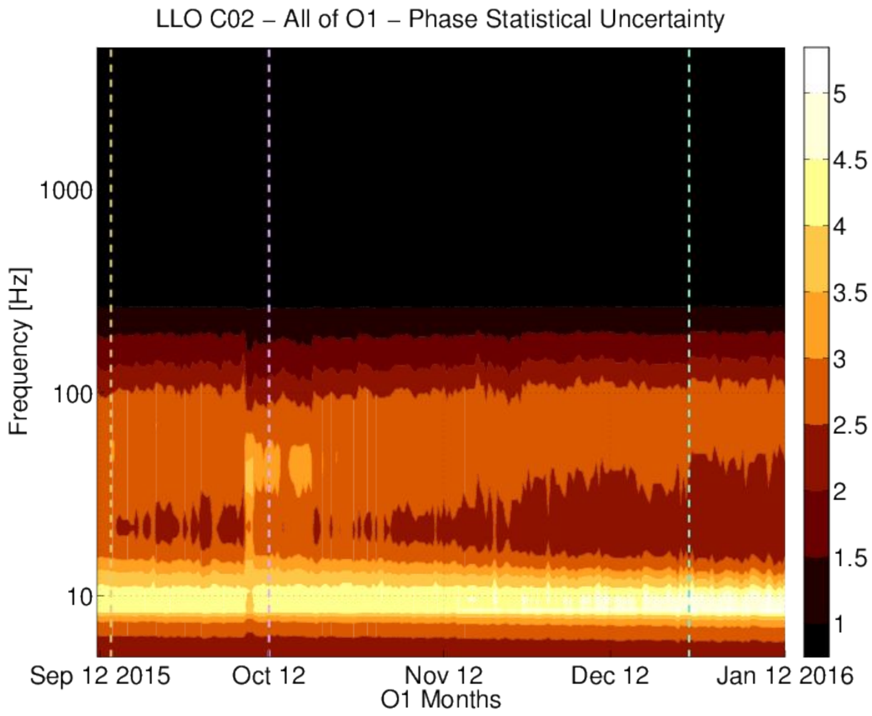
All of O1 **LLO** Systematic Error



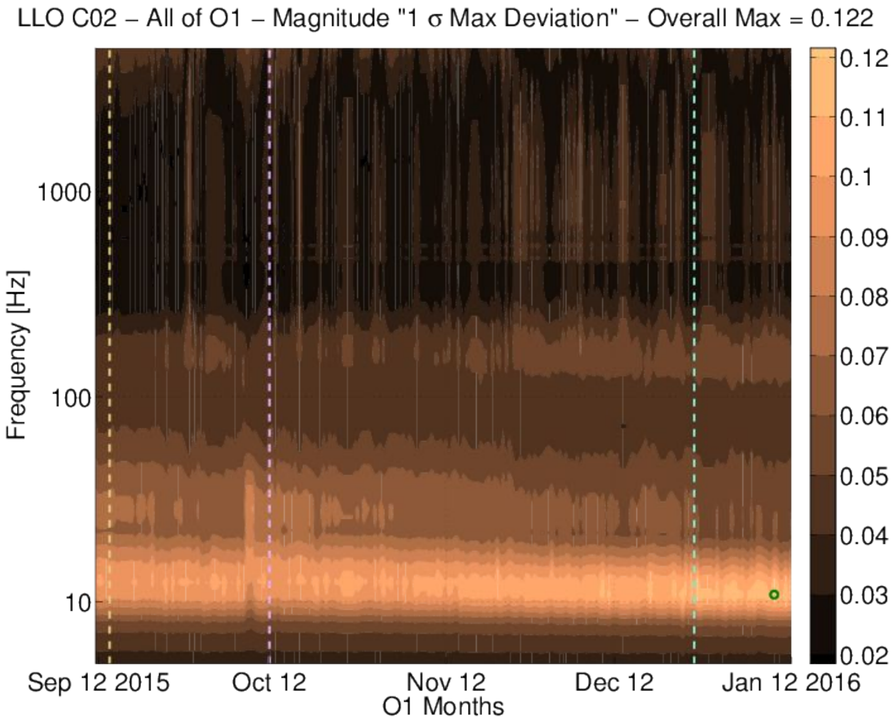


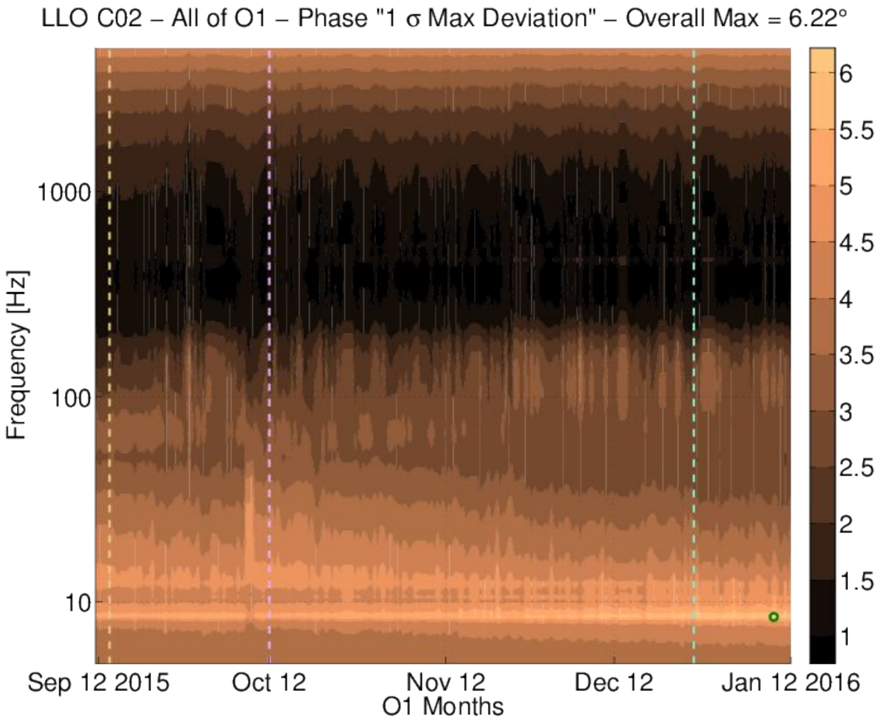
All of O1 LLO Statistical Uncertainty





All of O1 **LLO** "Maximum Deviation"





Conclusions

- For the C02 calibrated data, the interferometer time dependence is drastically reduced, leaving only the motion of the cavity pole to adversely affect our response function.
  - As a result, **our error and uncertainty budget are mostly stationary throughout all of O1**.
- Throughout all of O1, the highest "maximum deviations" between 5 and 5000 Hz were:

All of O1 LHO		**All of O1 LLO**	
Frequency Range = 20 - 1000 Hz			
Magnitude	Phase	Magnitude	Phase
8.5%	5.62 degrees	9.0%	4.72 degrees
Frequency Range = 5 - 5000 Hz			
Magnitude	Phase	Magnitude	Phase
16.1%	5.62 degrees	12.2%	6.22 degrees

Conclusions

Each Event's "Maximium Deviation" for 20 - 1000 Hz.

Frequency Range = 20 - 1000 Hz				
	LHO		LLO	
Event	Magnitude	Phase	Magnitude	Phase
GW150914	8.37%	-4.68 degrees	9.11%	4.37 degrees
LVT151012	8.25%	-4.57 degrees	9.64%	4.56 degrees
GW151226	8.39%	-4.51 degrees	8.71%	4.07 degrees

Where does the code live?

Calibration SVN:

Actuation and Sensing Fitting Functions:

```
/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/${IFO}/Scripts/Uncertainty/
{sensing,actuation}_measurement_uncertainty_calcs_${IFO}_v2.m
```

Covariance Matrix Production Script:

```
/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Scripts/Uncertainty/strainUncertaintyCovariance_PE_O1.m
```

Response Uncertainty Calculation Script:

```
/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Scripts/Uncertainty/ComplexUncertaintyPropagation.m
```

Response Uncertianty Plotter:

```
/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Scripts/Uncertainty/complexUncertaintyPlotter.m
```

Where does the code live?

Plot Results

Slide 35:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/21-Aug-2016_Both_IFOs_GW150914_C02_1126259462_Response_Function_Uncertainty_incl_Covariance.png

Slide 36:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/21-Aug-2016_Both_IFOs_LVT151012_C02_1128678900_Response_Function_Uncertainty_incl_Covariance.png

Slide 37:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/21-Aug-2016_Both_IFOs_GW151226_C02_1135136350_Response_Function_Uncertainty_incl_Covariance.png

Slide 38:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/23-Aug-2016_Both_IFOs_GW150914_C02_1126259462_Response_Function_Uncertainty_incl_Covariance.png

Slide 39:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/23-Aug-2016_Both_IFOs_LVT151012_C02_1128678900_Response_Function_Uncertainty_incl_Covariance.png

Slide 40:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/Common/Results/Uncertainty/23-Aug-2016_Both_IFOs_GW151226_C02_1135136350_Response_Function_Uncertainty_incl_Covariance.png

Where does the code live?

Plot Results

Slide 41:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/H1/Results/Uncertainty/28-Jul-2016_H1_C02_All_of_O1_Spectrograms_Systematic_Error_{Magnitude,Phase}.png

Slide 42:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/H1/Results/Uncertainty/28-Jul-2016_H1_C02_All_of_O1_Spectrograms_Statistical_Uncertainty_{Magnitude,Phase}.png

Slide 43:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/H1/Results/Uncertainty/28-Jul-2016_H1_C02_All_of_O1_Spectrograms_Maximum_Deviation_{Magnitude,Phase}.png

Slide 44:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/L1/Results/Uncertainty/21-Aug-2016_L1_C02_All_of_O1_Spectrograms_Systematic_Error_{Magnitude,Phase}.png

Slide 45:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/L1/Results/Uncertainty/21-Aug-2016_L1_C02_All_of_O1_Spectrograms_Statistical_Uncertainty_{Magnitude,Phase}.png

Slide 46:

/ligo/svncommon/CalSVN/aligocalibration/trunk/Runs/O1/L1/Results/Uncertainty/21-Aug-2016_L1_C02_All_of_O1_Spectrograms_Maximum_Deviation_{Magnitude,Phase}.png

Future Work (loosely in order of importance)

- Detuning seen at LHO and not LLO.
  - Compare SRCs of two IFOs to determine cause.
- Bayesian Calibration Uncertainty
  - MCMC over DARM model parameters fitting to plant measurements.
- Measurement technique is long and arduous. It is susceptible to glitches and detector non-stationarity.
  - Future: Schroeder Phase measurement technique pings plant a many frequencies at the same time
- LALInference Calibration C code
  - Code is written, but not incorporated into LALInference.
  - Does not yet include covariance calculations. Might wait until full Bayesian model is implemented.
- Check extrapolation of DARM Model uncertainty past measurement frequency vectors
  - Gaussian Process allows unmodeled uncertainty propagation, completely measurement driven.
    - This sort of thing would have automatically characterized detuning + its uncertainty during O1.