LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

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Beyond Paraxial Approximation

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1 Introduction

For lenses used for the mode correction telescope designed by the Syracuse group, the estimation of the TEM00 mode loss needs to be estimated with an accuracy of 0.1%. For the size of focal lengths and beam sizes considered, simple formulas based on the paraxial approximation do not always work with this accuracy. Reflection of a field by a mirror and transmission through a lens with an arbitrary focal length is calculated using the Kirchhoff integral, without using the paraxial approximation.

When the input field is a TEM00 Gaussian field, the integral formula of the reflected or transmissted field at an arbitrary location can be expressed by a product of a simple TEM00 Gaussian field and a correction function, where the correction function can be written in an explicit integral form. The correction function becomes a unity in the limit of the paraxial approximation.

A matlab code is provided which calculates the field propagation based on this formula.

2 Reflection

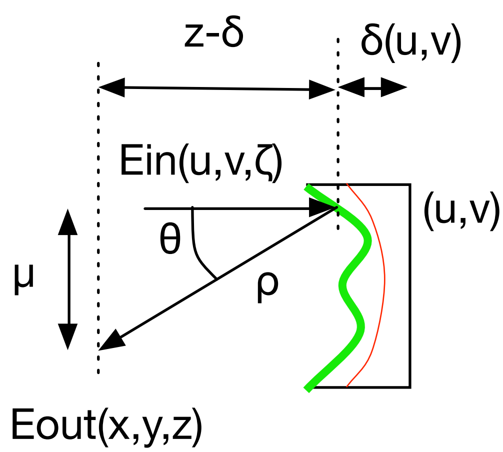

Figure 1 Reflection

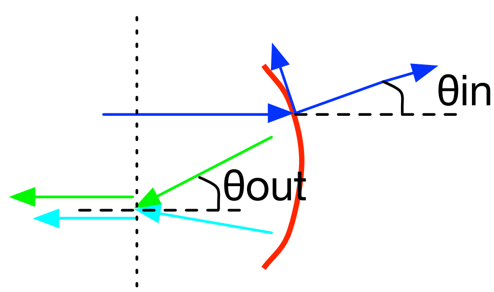

Figure 2 Effective component

Using the Kirchhoff integral, Eout, the field reflected by a mirror, can be expressed as follows using the incoming field, Ein, the surface height distribution δ, (see Fig.1) and two angles θ_{in} and θ_{out} (see Fig.2).

$$
\begin{aligned}
&E_{out}(x,y,z)=\frac{ik}{2\pi}\iint E_{in}(u,v,\zeta(u,v))\frac{\exp(-ik\rho)}{\rho}(1-\frac{i}{k\rho})\cos(\theta_{in})\cos(\theta_{out})dudv\\
&\mu^2=(u-x)^2+(v-y)^2\\
&\rho^2=\mu^2+(z-\delta)^2
\end{aligned} \tag{1}
$$

The factor $\cos(\theta_{in})$ picks up the component perpendicular to the optic surface, and $\cos(\theta_{out})$ picks up the component perpendicular to the output plane.

When the incoming field is almost a Gaussian shape,

$$
E_{in}(u,v,\zeta)=\sqrt{\frac{2}{\pi}}\frac{1}{w_{in}}\exp(-ik\zeta-(u^2+v^2)(\frac{ik}{2R_{in}}+\frac{1}{w_{in}{}^2}))F_{in}(u,v,\zeta) \tag{2}
$$

with F_{in} = 1 for a pure TEM00 field. When the surface height δ is expressed using the typical power term ~ r^{-2} and the rest d as

$$
\delta=d+\frac{u^2+v^2}{2R_{mir}} \tag{3}
$$

the original expression (1) can be written as follows.

$$
\begin{aligned}
&E_{out}(x,y,z)=\frac{ik}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{w_{in}}\exp(-ikz)\\
&\iint\exp(i2kd-(u^2+v^2)(\frac{ik}{2R_{out}}+\frac{1}{w_{in}{}^2}))\frac{\exp(-ik(\rho-(z-\delta))}{\rho}\frac{z-\delta}{\rho}(1-\frac{i}{k\rho})\cos(\theta_{in})dudv
\end{aligned} \tag{4}
$$

where R_{out} is expressed using the incoming field curvature and the mirror curvature as

$$
\frac{1}{R_{out}}=\frac{1}{R_{in}}-\frac{2}{R_{mir}} \tag{5}
$$

When the following variable change from (u,v) to (α,β), Eq.(6), is applied, eq.(1) can be expressed by a product of terms, one coming from the paraxial approximation and the correction due to the transverse effect.

$$
\begin{aligned}
&\alpha=\sqrt{A}(u-Bx),\ \beta=\sqrt{A}(v-By)\\
&A=\frac{1}{w_{in}{}^2}+i\frac{k}{2}(\frac{1}{R_{out}}+\frac{1}{z})\\
&B=i\frac{k}{2z}/A
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
&E_{out}(x,y,z)=TEM00(x,y,z)\times C\\
&C=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[ik(2d-(\rho-(z-\delta)-\frac{\mu^2}{2z}))-(\alpha^2+\beta^2)]\frac{z(z-\delta)}{\rho^2}(1-\frac{i}{k\rho})\cos(\theta_{in})d\alpha d\beta
\end{aligned} \tag{7}
$$

TEM00 in eq(7) is the Gaussian function with the curvature and width expected in the paraxial approximation, and C is the correction due to the transverse effect which is neglected in the paraxial approximation. For a parabolic mirror, d=0, and by neglecting the transverse effect in the limit of large z, the integrant of the correction factor C becomes $\exp(-\alpha^2-\beta^2)$, hence the correction factor C becomes 1, i.e., the reflected field is a pure Gaussian with the beam size and curvature expected by a paraxial approximation.

3 Transmission

3.1 Output field calculation

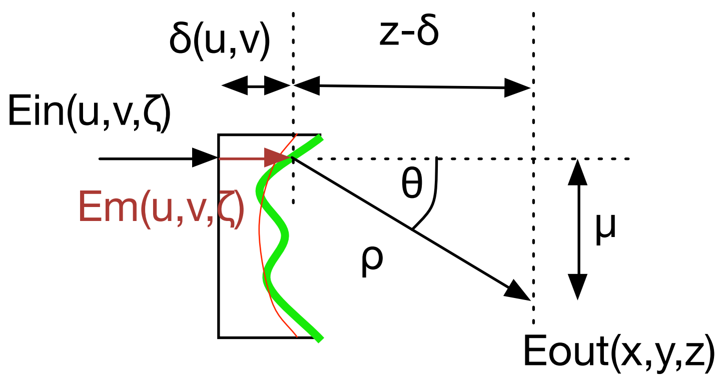

Figure 3 Transmission

The field after going through a lens, Fig.2, can be expressed using the following formula.

$$E_{out}(x,y,z)=\frac{ik}{2\pi}\iint E_m(u,v,\zeta(u,v))\frac{\exp(-ik\rho)}{\rho}(1-\frac{i}{k\rho})\cos(\theta_{in})\cos(\theta_{out})dudv$$
$$E_m(u,v)=F_{in}(u,v)\sqrt{\frac{2}{\pi}}\frac{1}{w_{in}}\exp(-ikn\delta)\exp(-(u^2+v^2)(\frac{ik}{2R_{in}}+\frac{1}{w_{in}{}^2}) \tag{8}$$

In this equation, Em is the field on the surface exiting the lens, R_{in} and w_{in} are the field curvature and width on the exiting surface, and n is the refractive index of the lens substrate.

By using the expression (3), the output field can be written as follows.

$$E_{out}(x,y,z)=\frac{ik}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{w_{in}}\exp(-ikz)$$
$$\iint\exp(-ik(n-1)d-(u^2+v^2)(\frac{ik}{2R_{out}}+\frac{1}{w_{in}{}^2}))\frac{\exp(-ik(\rho-(z-\delta))}{\rho}\frac{z-\delta}{\rho}(1-\frac{i}{k\rho})\cos(\theta_{in})dudv \tag{9}$$

where R_{out} is expressed using the incoming field curvature and the mirror curvature as

$$\frac{1}{R_{out}}=\frac{n-1}{R_{in}}+\frac{1}{R_{mir}} \tag{10}$$

This shows that the field curvature is 1/n of the curvature of the incoming field when Rmir=Rin.

Eq.(9) is essentially identical to eq.(4), and the same expression of the correction factor can be used to calculate the transmitted field. When calculating the correction, there are several differences between the reflection and transmission.

One is the effect of the deviation from the pure parabolic shape, i.e., i2kd in the exponent in Eq.(4) becomes –i(n-1)kd in Eq.(9). The variation d includes the error of the as-built curvature of the optic from the design value, and this difference makes the use of lenses be more tolerant about the requirements of the curvature error of optics.

Secondary, the curvatures of the field leaving the optic, R_{out}, are different for the two cases, Eq.(4) for reflection and Eq. (10) for transmission.

Another changes needed are that the field curvature (R_{in}) and width (W_{in}) used for the transmitted case are the values of the field after passing the finite thickness of the lens. This is discussed in the following section.

3.2 Finite thickness effect

The curvature and width of a Gaussian beam change as follows when the field goes through a substrate with thickness of d and refractive index of n.

$$\begin{aligned} \frac{1}{R(d)} &= \frac{1}{R}\left(1+\frac{d/n}{R}\left(\frac{R^2}{k^2w^4}-1\right)\right) \\ \frac{1}{w(d)^2} &= \frac{1}{w^2}\left(1-2\frac{d/n}{R}\right) \end{aligned} \tag{11}$$

On the right hand side, R and w are curvature and width before the propagation. As is shown in eq.(11), the change of the curvature and the width are proportional to d/R.

3.3 00 mode loss

The mixing of the 00 mode with different bases, (R,w) and (R', w'), can be expresses as follows.

$$\begin{aligned} \left|< 00\, with(R,w) \,|\, 00\, with(R',w')>\right|^2 &= \frac{\left(\frac{2}{1+w^2/w'^2}\right)\left(\frac{2}{1+w'^2/w^2}\right)}{1+\left(\frac{k}{2}\frac{w^2w'^2}{w^2+w'^2}\left(\frac{1}{R'}-\frac{1}{R}\right)\right)^2} \\ &\approx 1-\left(\frac{kw^2}{4R}\right)^2\varepsilon_R{}^2-\varepsilon_w{}^2 \\ \frac{1}{R'} = (1-\varepsilon_R)\frac{1}{R},\ & w' = (1-\varepsilon_w)w \end{aligned} \tag{12}$$

As eq.(12) shows, the loss of the 00 mode is in the second order of the change of the field parameters. But the coefficient of the curvature change can be large. E.g., with R=5cm and w=0.5mm, the coefficient of the curvature error is 50, and the curvature mismatch of 1% can induce 00 mode loss of 0.5%.

4 Correction factor

4.1 Expression of the correction factor

As has been discussed in Sec.2 and Sec.3, the outgoing field after reflection or transmission can be expressed as a production of a simple Gaussian form and a correction factor. Results are summarized in Eq.(13) and (14).

$$E_{out}(x,y,z) = TEM00(x,y,z) \times C$$
$$C = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[ik\phi - (\alpha^2+\beta^2)]\frac{z(z-\delta)}{\rho^2}(1-\frac{i}{k\rho})\cos(\theta_{in})d\alpha d\beta \tag{13}$$

$$\alpha = \sqrt{A}(u-Bx), \quad \beta = \sqrt{A}(v-By)$$
$$A = \frac{1}{{w_{in}}^2} + i\frac{k}{2}(\frac{1}{R_{out}}+\frac{1}{z}), \; B = i\frac{k}{2z}/A$$
$$\rho^2 = (x-u)^2+(y-v)^2+(z-\delta)^2, \, \mu^2 = u^2+v^2, \, \delta = d + \frac{\mu^2}{2R_{mir}}$$
$$reflection \quad : \phi = 2d - (\rho-(z-\delta)-\frac{\mu^2}{2z}), \qquad \frac{1}{R_{out}} = \frac{1}{R_{in}} - \frac{2}{R_{mir}}$$
$$transmission: \phi = -(n-1)d - (\rho-(z-\delta)-\frac{\mu^2}{2z}), \; \frac{1}{R_{out}} = \frac{n-1}{R_{in}} + \frac{1}{R_{mir}} \tag{14}$$
$$spherical\,surface: \cos(\theta_{in}) = \sqrt{\frac{1-2\frac{u^2+v^2}{{R_{mir}}^2}}{1-\frac{u^2+v^2}{{R_{mir}}^2}}}$$
$$parabolic surface: \cos(\theta_{in}) = \sqrt{1-\frac{u^2+v^2}{{R_{mir}}^2}}$$

4.2 Parabolic vs spherical shape

The main characteristic of the parabolic mirror is that the parallel field reflected by a parabolic mirror goes to the focal point. This is demonstrated by injecting a field with RoC of 1m and beam size of 1cm is injected to a parabolic and spherical mirrors with RoC of 2.3cm.

The reflected field is fit by a Gaussian shape using the small central region and the width of the beam is calculated from the slope of the power. Fig.4 shows the width at various locations. The focal point is 23mm/2 = 11.5mm. The black line is the width based on the paraxial approximation.

The red line is the width reflected by a parabolic mirror. The beam size goes to zero at the focal distance, which is consistent with the characteristic of the parabolic mirror. The beam reflected by a spherical mirror is complex. At around the focal distance, the power distribution in radial direction becomes wide spread than the prediction by the paraxial approximation. In a narrow range of distance, the power increases from the beam center in a very narrow radial region, and goes down outer wards. Two missing points on the blue line are those points.

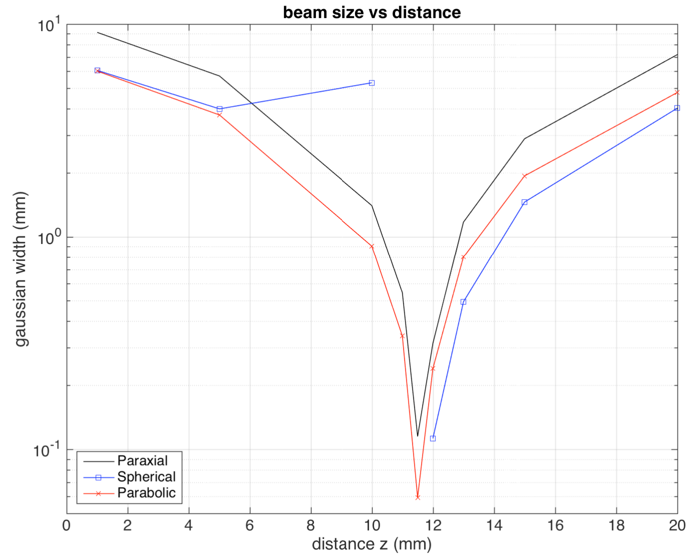

Figure 4 Reflected beam size vs distance

4.3 Distance dependence of Power and 00 mode

The total power of the field can be calculated as

$$P(z) = \iint |E_{out}(x,y,z)|^2\, dx\, dy = \frac{2}{\pi w(z)^2} \iint \exp(-2\frac{x^2+y^2}{w(z)^2}) |C(x,y,z)|^2\, dxdy \quad (15)$$

This is the power of the field going along the z axis after the interaction. The field at the point of interest, (x,y,z), is the sum of fields coming from all points on the optic surface, and the field is not necessarily parallel to the z axis. The factor $\cos(\theta_{out})$ picks up the component parallel to the z axis. Because of that, the energy P(z) may be less than 1. This is around the same effect as the reduction by $\cos(\theta_{in})$, and it can be taken as the systematic uncertainty of the paraxial approximation.

The amplitude of the TEM00 mode at a propagation distance z can be calculated by the following formula, i.e., the Gaussian weighted integral of C.

$$A00 = \iint TEM00(x,y)^* E_{out}(x,y,z) = \frac{2}{\pi w(z)^2} \iint \exp(-2\frac{x^2+y^2}{w(z)^2}) C(x,y,z) dxdy \quad (16)$$

In the paraxial approximation, the mode fractions are independent on the propagation distance. When this mode fraction is calculated for the aLIGO ITM mirror, the 00 mode fraction is essentially unity, independent on the propagation distance, both for the parabolic and spherical case.

In the rest of this section, details of the correction factor is studied using a simple reflection of a field with RoC of 2.3cm and width of 1mm by a mirror with RoC of 2.3cm.

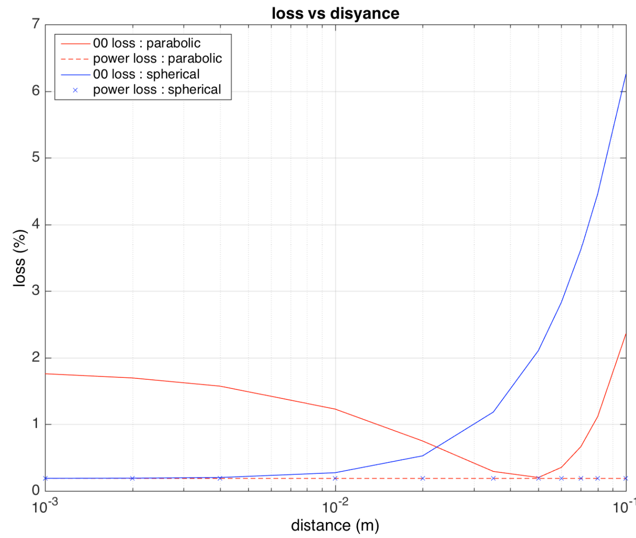

Figure 5 00 mode and power loss vs distance

Fig.5 shows the 00 mode loss, $1-|A00|^2$ (eq.14), as a function of the propagation distance reflected by a parabolic and spherical mirrors.

The power loss, 1 – P(z) (eq.15), is essentially identical for both mirror shapes and independent of the distance. The power loss comes from the loss to the non-normal components of the incoming and out going fields, or by $\cos(\theta_{in})$ and $\cos(\theta_{out})$ in eq.(1) and (8) (see Fig.2). The contributions of these two effects are almost the same. In the paraxial approximation, fields are propagating along the chosen beam axis, and this kind of loss is not taken into account. This loss can be approximated by the following formula, which is the power weighted $\cos(\theta_{in})^2$.

$$(\sqrt{\frac{2}{\pi}}\frac{1}{w})^2 \iint \exp(-2\frac{u^2+v^2}{w^2})(1-\frac{u^2+v^2}{{R_{mir}}^2}) \approx 1-\frac{w^2}{2{R_{mir}}^2} \qquad (17)$$

There are two losses, $\cos(\theta_{in})$ and $\cos(\theta_{out})$, and the total loss due to the non parallel component can be approximated by w^2 / R_{mir}^2. This is an approximation, which can be used to estimation the error of the paraxial approximation.

Fig.6 shows the radial distribution of the correction function at three different locations, 1mm, 2cm and 5cm from the mirror. The radial distance is normalized by the beam size at each location, 0.956mm, 0.131mm and 1.174mm respectively. The position dependence of the 00 mode loss sown in Fig.5 comes from the change of the correction function as the propagation distance changes. Two figures in the right hand side in Fig.6 is the magnified figures of the left hand side, in the radius within one beam size, w(z).

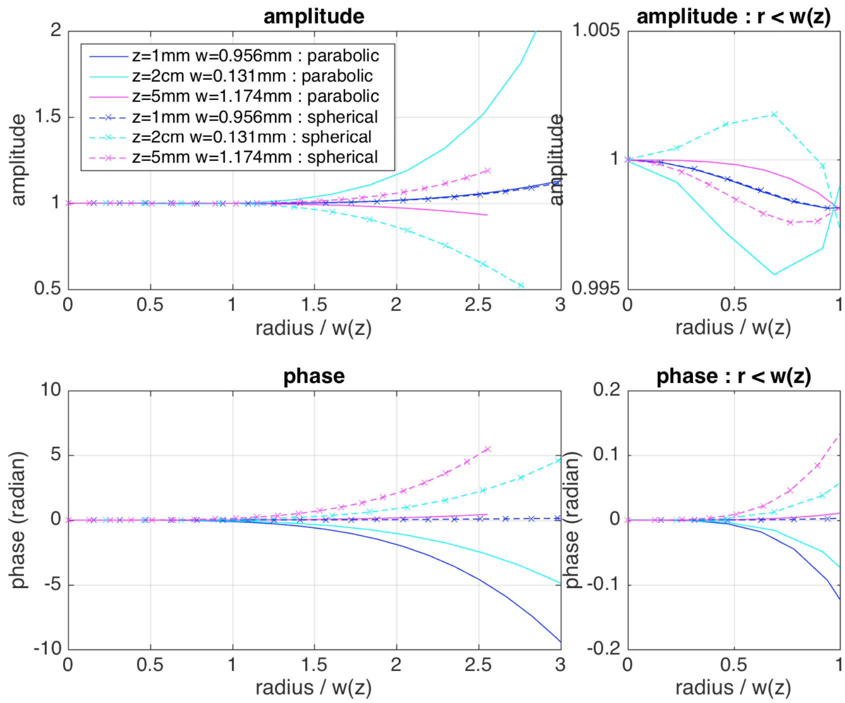

Figure 6 correction function at different locations

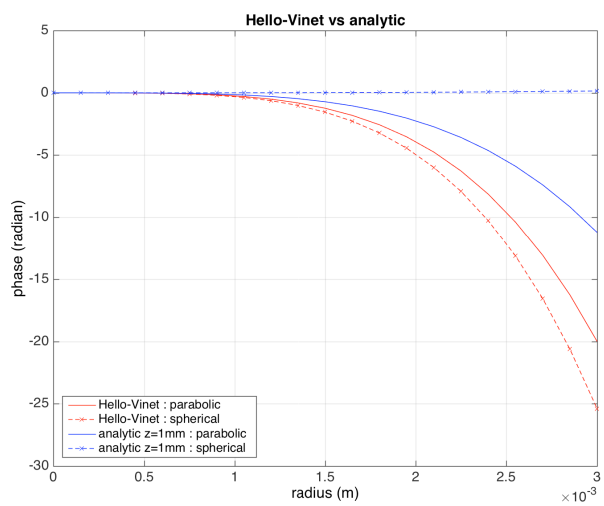

Figure 7 Hello-Vinet calculation and correction function

Fig.7 compares the phase change calculation done by Hello-Vinet and the same quantity using the correction function. The Hello-Vinet formula does not take into account the variation of the beam power, or the beam size is assumed to be much wider than other length of interest. The result will be valid near the reflection point, and phase by the correction is calculated at z = 1mm.

The correction function is compared numerically with a brute force calculation using Eq.(1) as is, which is much slower and harder to apply, and they agreed very well.

5 Numerical results for the telescope lenses

For one design of the telescope, various losses are calculated.

```
n = 1.45; th = 0.000; k=2*pi/1.064e-6;

tele(1).Ropt = -0.0498;
tele(1).Rin = 6.746;
tele(1).win = 638.7e-6;
tele(1).dist = 0.05;

tele(2).Ropt = 0.0258;
tele(2).Rin = -0.061;
tele(2).win = 350.3e-6;
tele(2).dist = 0.2809;

tele(3).Ropt = 0.0522;
tele(3).Rin = 0.623;
tele(3).win = 557e-6;
tele(3).dist = 0.05;

tele(4).Ropt = -0.0610;
tele(4).Rin = 0.146;
tele(4).win = 846e-6;
tele(4).dist = 0.9991;
```

The result running Paraxialscript.m is as follows.

```
n = 1.45, lens th = 0 =================
----
lens 1 : Labr (eq 4.4) = 2.50398e-07, win = 0.0006387, wout = 0.000355854
spherical : 00 mode loss = 4.68039e-05, dPwr = 4.5838e-05
parabolic : 00 mode loss = 4.60691e-05, dPwr = 4.58334e-05

----
lens 2 : Labr (eq 4.4) = 1.06031e-07, win = 0.0003503, wout = 0.000528571
spherical : 00 mode loss = 8.98746e-05, dPwr = 8.97645e-05
parabolic : 00 mode loss = 8.96872e-05, dPwr = 8.9685e-05

----
lens 3 : Labr (eq 4.4) = 6.31612e-08, win = 0.000557, wout = 0.000842338
spherical : 00 mode loss = 2.68254e-05, dPwr = 2.65453e-05
parabolic : 00 mode loss = 2.66193e-05, dPwr = 2.65413e-05

----
lens 4 : Labr (eq 4.4) = 7.0245e-07, win = 0.000846, wout = 0.000565623
spherical : 00 mode loss = 9.34488e-05, dPwr = 9.27361e-05
parabolic : 00 mode loss = 9.30759e-05, dPwr = 9.30734e-05
```

The mode loss using the analytic calculation is given in Eq.4.4 in T1500372.

$$L_{aberration} = \frac{5}{256}(n-1)^2 k^2 \frac{w^8}{R_{mir}^6} \tag{18}$$

dPwr is the loss of the out going power along the z axis, which can be approximated by Eq.(17). For all cases, the loss due to the field going out of the z axis is larger than the mode loss calculated using a simple lens formula.