Scaling up coating Brownian noise measurements

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1 Intro

The purpose of this document is to calculate levels of coating Brownian thermal noise given laboratory direct measurements at smaller scales. We will do this by applying some scaling rules derived from coating Brownian noise theory. By the end of this document, we should have calculated the level of coating Brownian thermal noise expected in the aLIGO interferometers if they were coated with the same coatings as measured in the TNI [1, 2, 3] and in the Caltech rigid cavity measurement (TNC) [4, 5], but with a film thickness scaled to be comparable with the HR films on the ETMs and ITMs in aLIGO. We can then use these values to compare with the noise levels seen the the aLIGO interferometers to determine if they have reached coating thermal noise limitations.

2 Lab Measurements

2.1 TNI

The bulk of the information on the TIN can be found in reference [1]. However, I had to dig up the mirror dimensions from the DCC [6]. The mirrors are 4" in diameter and 4" in height. The beam waist, $w0$, was about 160 mum for all measurements. Four coatings were measured in the TNI:

1. Quarter-wave stack: silica/tantala
2. Thermal noise optimized stack: silica/tantala
3. Quarter-wave stack: silica/Ti:tantala
4. Thermal noise optimized dichromic stack: silica/Ti:tantala.

These are listed as TNI(1-4) in table 1. All are on silica substrates

The values for S_x(100Hz) were calculated by taking the values for ϕ_{SiO_2}, $\phi_{Ta_2O_5}$, and $\phi_{TiO_2:Ta_2O_5}$ given in tables IV and V of [1], and plugging them back into equations (1) and (2) of that same paper. This should give a good approximation of the actual value of S_x they would have measured.

Optic	w_0 [μm]	a [cm]	d [μm]	H [cm]	S_x(100Hz) [m^2 Hz^{-1}]
TNI(1)	160	5.08	4.55	10.16	1.06e-35
TNI(2)	160	5.08	5.41	10.16	8.82e-36
TNI(3)	160	5.08	4.21	10.16	7.79e-36
TNI(4)	160	5.08	3.81	10.16	6.84e-36
TNC	1.82	1.27	4.53	0.635	1.1e-35
ITM	53000	17	2.8	20	–
ETM	62000	17	5.9	20	–

Table 1: Values useful for converting coating Brownian noise. TNI(1-4) refers to the four different coatings measured in the Thermal Noise Interferometer at Caltech [1, 2, 3]. TNC refers to the reference cavity measurements made at Caltech by Tara Chalermsongsak [4, 5]. Values for ITM and ETM are for the LIGO Input Test Mass and End Test Mass, respectively [7, 8]

2.2 TNC

These measurements are nicely covered in [4], with the exception of the substrate thickness, H, which had to be found in [5]. The silica substrates were 1" in diameter and 1/4" in thickness. The film measured here was an old quarter-wave stack produced by REO, made of silica/tantala. To get the value of S_x(100Hz), I plugged their measured value of $\phi_c = 4.43 \times 10^{-4}$ into their equation (8). Again, this should give an approximation of the coating Brownian noise they would have measured at 100 Hz.

3 Scaling Rules

I think we need to apply two scaling rules, first for the beam-spot and substrate size, and then for the film thickness. For the beam-spot and substrate dependence, we rely on the work by Somiya and Yamamoto [9]. In that paper, they calculate the Brownian noise of a coating on a finite-sized substrate, which we can write as:

$$S_{\mathrm{x}}^{\mathrm{FIN}}(\Omega) = \frac{8k_{\mathrm{B}}T}{\Omega}\Phi_{\mathrm{c}}U^{\mathrm{FIN}} \tag{1}$$

(from their equation (3)). Here, $S_{\mathrm{x}}^{\mathrm{FIN}}(\Omega)$ is the coating Brownian noise for a finite size mirror at frequency Ω, k_{B} is Boltzmann's constant, T is the temperature of the optic, Φ_{c} is the mechanical loss of the coating material, and U^{FIN} is the energy stored in the film from an imaginary force used to probe the fluctuations (in the shape of the beam intensity). We can then take the ratio of this value to that of a coating with the same properties on an infinite substrate $S_{\mathrm{x}}^{\mathrm{INF}}(\Omega)$:

$$R = \frac{S_{\mathrm{x}}^{\mathrm{FIN}}(\Omega)}{S_{\mathrm{x}}^{\mathrm{INF}}(\Omega)}, \tag{2}$$

where $S_{\mathrm{x}}^{\mathrm{INF}}(\Omega)$ is given by their equation (29):

$$S_{\mathrm{x}}^{\mathrm{INF}}(\Omega) = \frac{4k_{\mathrm{B}}T}{\Omega}\frac{d}{\pi w_0^2}\Phi_{\mathrm{c}}\frac{Y_{\mathrm{c}}^2(1+\nu_{\mathrm{s}})^2(1-2\nu_{\mathrm{s}})^2 + Y_{\mathrm{s}}^2(1+\nu_{\mathrm{c}})^2(1-2\nu_{\mathrm{c}})}{Y_{\mathrm{s}}^2 Y_{\mathrm{c}}(1-\nu_{\mathrm{c}}^2)}. \tag{3}$$

Here, Y is the Young's modulus and ν is the Poisson ratio, subscripts c and s indicate properties of the coating and substrate, respectively, and d indicates the film thickness. Combining equations 1, 2, and 3, we come to the relation:

$$R = 2U^{\mathrm{FIN}}\frac{\pi w_0^2}{d}\left[\frac{Y_{\mathrm{c}}^2(1+\nu_{\mathrm{s}})^2(1-2\nu_{\mathrm{s}})^2 + Y_{\mathrm{s}}^2(1+\nu_{\mathrm{c}})^2(1-2\nu_{\mathrm{c}})}{Y_{\mathrm{s}}^2 Y_{\mathrm{c}}(1-\nu_{\mathrm{c}}^2)}\right]^{-1}. \tag{4}$$

The attached Mathematica notebook is written to calculate this ratio using the equations in [9], but for some reason, it isn't giving reasonable values

Once we are able to calculate R, we should make them for each of the laboratory measurements, as well as for the ITM and ETM mirrors, but assuming the same thickness as the laboratory measurements. For example, we can calculate $R_{\mathrm{TNI(1)}}$, the ratio of the TNI(1) measurement to it's imaginary infinite substrate case, and $R_{\mathrm{ITM}}(d = d_{\mathrm{TNI(1)}})$, which will be the same ratio for the noise we expect from the ITM mirror if it had a coating the same thickness as the TNI(1) measurement. Then we have the following:

$$R_{\mathrm{TNI(1)}} = \frac{S_{\mathrm{TNI(1)}}^{\mathrm{measured}}}{S^{\mathrm{INF}}(d = d_{\mathrm{TNI(1)}})} \tag{5}$$

$$R_{\mathrm{ITM}}(d = d_{\mathrm{TNI(1)}}) = \frac{S_{\mathrm{ITM}}^{\mathrm{predicted}}(d = d_{\mathrm{TNI(1)}})}{S^{\mathrm{INF}}(d = d_{\mathrm{TNI(1)}})} \tag{6}$$

$$S_{\mathrm{ITM}}^{\mathrm{predicted}}(d = d_{\mathrm{TNI(1)}}) = S_{\mathrm{TNI(1)}}^{\mathrm{measured}}\frac{R_{\mathrm{ITM}}(d = d_{\mathrm{TNI(1)}})}{R_{\mathrm{TNI(1)}}} \tag{7}$$

This line of reasoning only gets us halfway there, as we're really interested in $S_{\mathrm{ITM}}^{\mathrm{predicted}}(d = d_{\mathrm{ITM}})$. This is easy to scale to, since the noise is directly proportional to the coating thickness, as in equation 3. Therefore, we can make the final scaling using the relation:

$$S_{\mathrm{ITM}}^{\mathrm{predicted}} = \frac{d_{\mathrm{ITM}}}{d_{\mathrm{TNI(1)}}}S_{\mathrm{ITM}}^{\mathrm{predicted}}(d = d_{\mathrm{TNI(1)}}), \tag{8}$$

which is exactly the value we are trying to calculate.

Looking back, we can actually skip the calculation of S^{INF}, as it cancels in all of the equations, and the ratio of Rs in equation 7 can be replaced with the ratios of U^{FIN}s. We can also combine equation 8 with equation 7 to get a final equation:

$$S_{\mathrm{ITM}}^{\mathrm{predicted}} = \frac{d_{\mathrm{ITM}}}{d_{\mathrm{TNI(1)}}}\frac{U_{\mathrm{ITM}}(d = d_{\mathrm{TNI(1)}})}{U_{\mathrm{TNI(1)}}}S_{\mathrm{TNI(1)}}^{\mathrm{measured}}. \tag{9}$$

Optic	U^{FIN} [J]
TNI	2.37e-8
TNC	2.088e-6
ITM	5.096e-11
ETM	4.056e-11

Table 2: Values useful for converting coating Brownian noise, as calculated using Liu and Thorne [10]. All TNI measurements have the same value because they all used the same substrate size and beam size. TNC value is basically equal to U^{INF} due to the extreme spot size to mirror radius ratio.

Optic	S^{ITM}_{pred} [$\mathrm{m^2\ Hz^{-1}}$]	S^{ETM}_{pred} [$\mathrm{m^2\ Hz^{-1}}$]	S_{IFO} [$\mathrm{Hz^{-1}}$]
TNI(1)	1.40e-38	2.35e-38	4.69e-45
TNI(2)	1.04e-38	1.75e-38	3.49e-45
TNI(3)	1.14e-38	1.86e-38	3.75e-45
TNI(4)	1.08e-38	1.81e-38	3.61e-45
TNC	1.66e-40	2.78e-40	5.55e-47

Table 3: Calculated values of $S_x(100\mathrm{Hz})$ for End Test Masses (ETM) and Input Test Masses (ITM), as predicted by scaling the measured noises for each experiment using Liu and Thorne and equation 9.

4 Cheating, with Liu and Thorne

Since I was having difficulty getting the Somiya [9] calculations to work out (still trying though), Eric Gustafson suggested that I try using Liu and Thorne [10] instead. Liu and Thorne calculate the correction to the thermal noise for a finite-sized substrate. They do not consider coatings at all. However, as both Nakagawa [11] and Levin [12] have shown (for infinite substrates, anyway), the loss of a total coated substrate is:

$$S^{\mathrm{Total}} = S^{\mathrm{Substrate}} \left[1 + \frac{2}{\sqrt{\pi}} \frac{(1-2\sigma)}{(1-\sigma)} \frac{\phi_{\mathrm{coating}}}{\phi_{\mathrm{substrate}}} \left(\frac{d}{w} \right) \right]. \tag{10}$$

So it's not terribly unreasonable to assume that the only correction we need to make is one to $S^{\mathrm{Substrate}}$, since it's going to get carried through anyway.

The further beauty of using Liu and Thorne's calculations is that I can get them to work in Mathematica (barely), and that the energy ratios don't have a film thickness dependence. Running them through the calculations I get the Energy values (U^{FIN}) discussed in equations 1 and 9. These are listed in table 2. If I plug them into equation 9, I get the results in table 3. These might be close representatives to what we would see if we were to place these coatings on aLIGO mirrors and somehow make them the same thickness as those on the ETM and ITM mirrors.

Finally, to turn displacement noise, S_x, for a single mirror into strain noise in the interferometer, we just add the noise for each mirror and divide by the arm length:

$$S_{\mathrm{IFO}} = 2 * (S_x^{\mathrm{ITM}} + S_x^{\mathrm{ETM}})/L^2, \tag{11}$$

where S_{IFO} is the total power spectral density from mirror Brownian thermal noise in the interferometer, S_x^{ITM} and S_x^{ETM} are the displacement noise we've calculated for the ITM and ETM, respectively, and L is the 4 km arm length of aLIGO. This gives us the final column of table 3. For comparison, the value of $\sqrt{S_{\mathrm{IFO}}}$ from GWINC is 4.25e-24 $\mathrm{Hz}^{-1/2}$, or S_{IFO} = 1.81e-47 Hz^{-1}.

4.1 Discussion on Liu and Thorne

As much as I'd like to stop here and add a pretty graph or something, I'm not really happy with using Liu and Thorne for coatings. First of all, Liu and Thorne is not for coatings, and both Nakagawa and Levin are considering semi-infinite test masses. Only Somiya actually calculates the true effect of finite-sized coating thermal noise. So I will keep trying to get that to work. Additionally, I did try to use the Liu and Thorne correction to calculate the results Somiya get in their paper, and I get nothing like their results. However, my calculations can replicate what Liu and Thorne get in their paper, so I at least know it's not my code's fault that this fails.

Additionally, it is hard for me to believe that the coatings measured at the TNI; especially TNI(3) and (4), which are Ti:tantala mirrors, would have higher noise than the TNC coating, which was just some old tantala/silica mirror lying around. In fact, the TNC coating should be almost the same as TNI(1). I have not yet figured out why they are so different.

References

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