Testing The Strong Field Dynamics of General Relativity Using Compact Binary Systems
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Einstein's General Theory of Relativity (GR) has been well tested in the weak field regime over the past century. However, such tests have not been carried out in the highly dynamical and inherently non-linear strong field regime. Recent advancements in ground based gravitational wave detectors, (e.g., Advanced LIGO, VIRGO), will allow us probe this regime of general relativity by investigating gravitational waves produced by astrophysical systems with strong gravitational fields such as compact binary coalescences. While current search techniques utilize standard GR waveforms to identify weak GW signals in the presence of noisy data, alternative theories of gravity predict signals that differ significantly from GR. We investigate our ability to find non-GR effects in detected waveforms of an astrophysical source in an alternative theory of gravity by introducing an arbitrary parameter, $\alpha_{\rm nGR}$, to modify standard GR waveform features, such as ringdown frequency, merger frequency, and amplitude. We then perform statistical methods such as matched filtering and bayesian inference to quantify how well future detectors will be able to distinguish between the gravitational waveforms in the event that GR is not the complete theory of gravity.

I. INTRODUCTION

General Relativity is a theory of gravity originally proposed by Albert Einstein in 1915 to generalize special relativity and Newton's law of universal gravitation. Led by the fundamental principle of equivalence, Einstein used GR to describe the motion of accelerating, massive particles by describing the associated field strength as the extent to which these particles warp the four-dimensional geometry of space and time, or *spacetime*. The principle of equivalence is now often viewed as the broader overall idea that spacetime is curved. Coupled with this fundamental postulate, [1] proposed a set of ideas which became known as the Einstein equivalence principle (EEP). Simply stated, the EEP is comprised of three broad ideas: the Weak Equivalence Principle (WEP) is valid, the outcome of any local non-gravitational experiment is independent of the velocity of of the reference frame in which it is performed (local Lorentz invariance), and that the outcome of an experiment is independent of when and where it takes place in the Universe (local position invariance) [2]. The EEP has allowed for a wide range of experimental tests that aim to test the foundation of GR and the notion of curved spacetime describing the nature of gravity.

Experiments aimed at testing the foundations of GR include the perihelion shift of Mercury, the orbital decay of the Hulse-Taylor binary pulsar B1913+16, and laboratory based tests of the WEP. GR predicted the rate of perihelion shift of Mercury, $\dot{\tilde{\omega}} \sim 42.''98$ (arc seconds per century), a problem previously unsolved since announced by Le Verrier in 1859. The orbital decay of the Hulse-Taylor binary pulsar, $\dot{P}_b$, as predicted by GR yields a value of $\dot{P}_b \sim -2.402531 \pm 0.000014 \times 10^{-12}$, which when compared to the corrected observed value, $\dot{P}_b^{\rm corr}/\dot{P}_b^{\rm GR} \sim 0.997 \pm 0.002$ [3]. Tests of the WEP include measuring the fractional difference in acceleration between two bodies. This difference is referred to as the "Eötvös ratio" and is defined by $\eta \sim 2|a_1 - a_2|/|a_1 + a_2|$, where a_1 and a_2 refer to the acceleration of the respective bodies considered for the test. One specific example performed at the University of Washington was able to reach a value of $\eta \sim 2 \times 10^{-13}$ [4–6], with continued efforts to further constrain this parameter ongoing [7].

The strength of the gravitational field is often characterized by the "compactness" parameter, $\epsilon \sim GM/Rc^2$, where G is the gravitational constant, M is the characteristic mass, R characteristic radius, and c the speed of light in a vacuum. Despite the many successful tests of GR described here, all previous tests have been carried out in the dynamically slow and *weak* field regime. For example, within our Solar system, the field strength takes a value of $\epsilon \sim 10^{-6}$. Although many alternative theories of gravity predict solutions in agreement with GR within the weak field, they also give rise to deviations from GR in the highly dynamical and non-linear *strong* field regime.

Alternative theories of gravity that predict deviations from GR include: scalar-tensor, massive graviton, modified quadratic gravity, $f(R)$, variable G, noncommutative geometry and gravitational parity violation

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[8–14]. The parameterized post-Einsteinian (ppE) framework [15] allows one to quantify the extent to which an alternative theory may produce changes to the physical nature of systems in which GR is taken to be the complete theory of gravitation. For example, modified quadratic gravity predicts a change to the strong-field interaction of compact binary coalescences (CBCs) by introducing corrections to the Einstein-Hilbert action which depends on higher powers of the curvature. Such alternative theories can be tested directly by investigating these extremely relativistic and dynamical systems wherein the gravitational field strength $\epsilon \sim 0.1$ to near unity. The final moments of CBC's are extremely luminous in gravitational radiation and encoded within these gravitational wave signals is the necessary information to further constrain alternative theories of gravitation [2, 16–19].

Gravitational waves (GWs) are propagating oscillations in the gravitational field caused by the acceleration of massive bodies [20]. The amplitude of a gravitational wave is characterized by the wave amplitude, or strain, $h \sim \Delta L/L$, where ΔL denotes the total change in length L between two objects such as a pair of mirrors forming an optical cavity of an interferometer in the presence of gravitational radiation [19]. Carrying energy, angular momentum, and inducing orbital decay in tight compact binary systems, gravitational waves propagate unimpeded through the Universe at the speed of light. GWs have two independent polarizations: h_+ and $h_\times$, where the distinction between the two is the way in which a circular ring of test particles in the (x, y) is affected by the presence of a transverse wave propagating in the z direction. It has been shown that GW signals can be systematically analyzed to extract intrinsic information about their astrophysical source [21–26]. Compact binary coalescences (CBCs), inspiralling binary star systems consisting of neutron stars (NSs) or black holes (BHs), are promising sources for the direct detection of gravitational wave signals for next generation gravitational wave detectors [27–29].

These highly compact objects have field strengths that range from $\epsilon \sim 0.1$ to near unity and during the final stages of coalescence reach relativistic orbital velocities of $0.1c \lesssim v \lesssim 0.6c$, thus providing a direct probe of the highly non-linear and dynamical strong field. Beyond tests of GR, NS-NS binary systems can be used to constrain the nuclear equation of state (EOS) of ultra relativistic, degenerate neutron star matter deep within the core or to probe tidal deformations of the companion stars approaching the innermost stable circular orbit (ISCO) [30–33]. Contrary to tests on the behavior of matter in NS-NS binaries, BH-BH binaries provide the cleanest, direct test of GR as black holes are purely mass, or curved spacetime, allowing one to neglect the complex behavior of matter [34]. Recent advancement in ground based gravitational wave detectors such as Advanced LIGO (aLIGO), VIRGO, and KAGRA will be capable of detecting signals from these compact binary systems while also expanding the detector coverage area leading to an increase of expected detections [35–38].

The Laser Interferometer Gravitational-Wave Observatory (LIGO) project is involved in the development and operation of increasingly sensitive gravitational wave detectors. The Initial LIGO detectors were built in the late 1990's and operated at and beyond design sensitivity from 2005 to 2007 [39]. Advanced LIGO is the next generation of detectors wherein the infrastructure created with iLIGO will be upgraded and expanded to significantly increase range and detector sensitivity [35].

For example, these upgrades are expected to give aLIGO a maximum sensitivity to strain, for a frequency band of $f \sim$ 100-200 Hz, of $h_{\rm rms} \sim 4 \times 10^{-23}$, along with an increased horizon distance of up to ~ 450 Mpc. This increase in detector sensitivity also increases the number of possible events to 10-100 events per year, resulting a higher likelihood of detection. Compact binary systems are a particularly promising candidate for direct detection of gravitational waves as well as direct tests of general relativity in the strong field regime [40].

Current techniques for determining if an observed signal originated from an astrophysical source as opposed to a non gaussian glitch or instrumental error include checking for coincident triggers within a small timeframe, statistically minimizing the known noise from the detector, and then using statistical methods to compare the observed signal with a template bank of approximate waveforms [41–44]. The techniques used to model the gravitational radiation emitted from different astrophysical sources, have been shown to provide an accurate description in the static slow moving, weak field regime where $v \ll c$ [45, 46]. However, alternative theories of gravity lead to solutions of GR in the weak-field but could diverge strongly in events beyond that such as the merging of two compact objects [47–49]. In the event that GR is not the complete theory of gravitation, a detection from such a highly relativistic source that emits a GW that deviates significantly from GR could bypass detection for a template bank utilizing only standard GR waveforms and also introduce unexpected degeneracies with inferred intrinsic parameters of the system. Therefore, it is proposed that the methods by which incident signals are analyzed thoroughly account for physically motivated deviations from GR that have been inferred by alternative theories of gravity.

In this paper we investigate the effect of non-GR deviations in simulated gravitational waveforms used to determine detection of a GW signal. For this investigation, we consider binary systems composed of binary black holes in the context the next generation gravitational wave detector, aLIGO. We perform numerical calculations to model these gravitational waveforms from a variety of binary systems. We then introduce a parameterized phenomenological function function to modify the gravitational waveform which produces a significant deviation from GR. Then, we perform a quantitative assessment of the properties of these modified waveforms and their implications on possible detection signals. In Sect. II we discuss our methods, in Sect. IV we present our numerical calculations, in Sect. VII we discuss our results and their implications to next generation GW detectors, and in Sect. VIII we present our conclusions.

II. METHODS

We begin our investigation by considering briefly a first order approximation of gravitational radiation in compact binary systems. We then calculate gravitational waveforms for non-spinning, equal mass ratio BBH systems for a range of total mass of $M \in [10, 250]\ M_{\odot}$ using the `IMRPhenomC` waveform approximation, an implementation within the `lalsimulation` algorithm library following the phenomenological model presented in [50]. Once a set of standard unmodified waveforms have been constructed, we explore the effects of altering the parameters associated with our phenomenological model by introducing a multiplicative parameter, $\alpha_{\rm nGR}$. To quantify the effects of this modification to the standard waveform we perform statistical techniques such as matched filtering and bayesian inference.

A. Compact Binary Systems

Compact binary systems containing neutron stars or black holes are thought to be a promising source for direct detection by next generation gravitational wave detectors [51–53]. The evolution of binary systems include in order: inspiral, merger, and ringdown. The inspiral is the orbit of two bodies about a common center of mass wherein the orbital radius decreases with time due to energy loss from sources including gravitational radiation. The merger is characterized by the orbital radius below that of $a_{\rm isco}$ wherein the two bodies begin to interact whether through tidal deformation in neutron stars or direct collision. The final stage is the ringdown wherein the newly formed massive BH or NS will oscillate at a damping ringdown frequency, emitting gravitational radiation as it settles to its new state. To illustrate these systems we consider the inspiral phase of a binary black hole system in a nearly circular orbit with companion masses of $m_1 = m_2 = 5M_{\odot}$ and an initial orbital period of $T = 0.1$ (s). Using Kepler's Law of Orbits, we write the relation between the orbital period and orbital separation for a circular orbit with $e \approx 0$,

$$\frac{4\pi^2}{GM}T^2 = a^3 \ , \tag{1}$$

where $M = m_1 + m_2 \equiv 10\ {\rm M}_{\odot}$, for this example. The initial orbital velocity of the binary objects can be deduced using these values

$$v_{\rm orb} = \frac{2\pi a}{T} \ , \tag{2}$$

where the initial orbital velocities of the system considered here are an appreciable fraction of the speed of light,

$$\beta = v_{\rm orb}/c \approx 0.15 \ . \tag{3}$$

The frequency of the gravitational waves emitted for such a system can be expressed as

$$f_{\rm GW} \sim 2 f_{\rm orb} \ , \tag{4}$$

The approximate amplitude of the gravitational wave strain can be calculated using the stationary phase, quadrupole approximation

$$h_+(t) \sim \frac{4G^2\mu M}{Dc^4 a(t)}\cos(\Phi(\mathrm{t})), \tag{5}$$

where the phase can be expressed as

$$\Phi(t) \sim \int 2\pi f_{\rm GW}(t)\ dt, \tag{6}$$

where $f_{\rm GW}$ is the frequency of the emitted GW, D is the distance from the system to the observer, μ is the reduced mass $(m_1 m_2/M)$, and a is the orbital separation of two objects in a tight circular orbit. For the system here we only consider the "plus" polarization of the GW emitted. As these compact systems evolve, they rapidly spiral inwards becoming more efficient in radiating energy via gravitational waves as the orbital radius shrinks and the orbital velocities increase. We can determine the time frame in which the binary will coalesce by calculating the innermost stable circular orbit (ISCO), the innermost stable orbital radius as determined by the Schwarzschild solution,

$$a_{\rm isco} \sim \ 6\frac{GM}{c^2}. \tag{7}$$

The timescale in which a binary system will reach this orbit, from an initial orbital period, T, is often referred to as the "'chirp' time,

$$t_{\rm chirp} \sim \frac{5GM^2}{256\mu c^3 \beta^8}. \tag{8}$$

We find for the system considered here, $t_{\rm chirp} \approx 18.92$ (s), with orbital velocities at $\beta \approx 0.41$ and the approximate GW strain of $|h_+| \approx 2.1 \times 10^{-20}$ (strain) at a frequency of $f_{\rm GW} \approx 439$ (Hz). The compactness of this system at ISCO yields a value of $\epsilon \sim GM/Rc^2 \sim 0.2$. The final seconds of the inspiral occur well within the non-linear, highly dynamical strong field regime providing direct tests of alternative theories of gravity that predict observable deviations from GR. The Fourier transform of the gravitational wave strain can also be useful for determining the amplitude of the GW signal in the frequency domain. We numerically compute the Fast Fourier Transform (FFT) as the following,

$$\mathcal{F}[h_+(t)] = \tilde{h}_+(f) \equiv \int_{-\infty}^{+\infty} h_+(t) e^{i2\pi f t} dt \ . \tag{9}$$

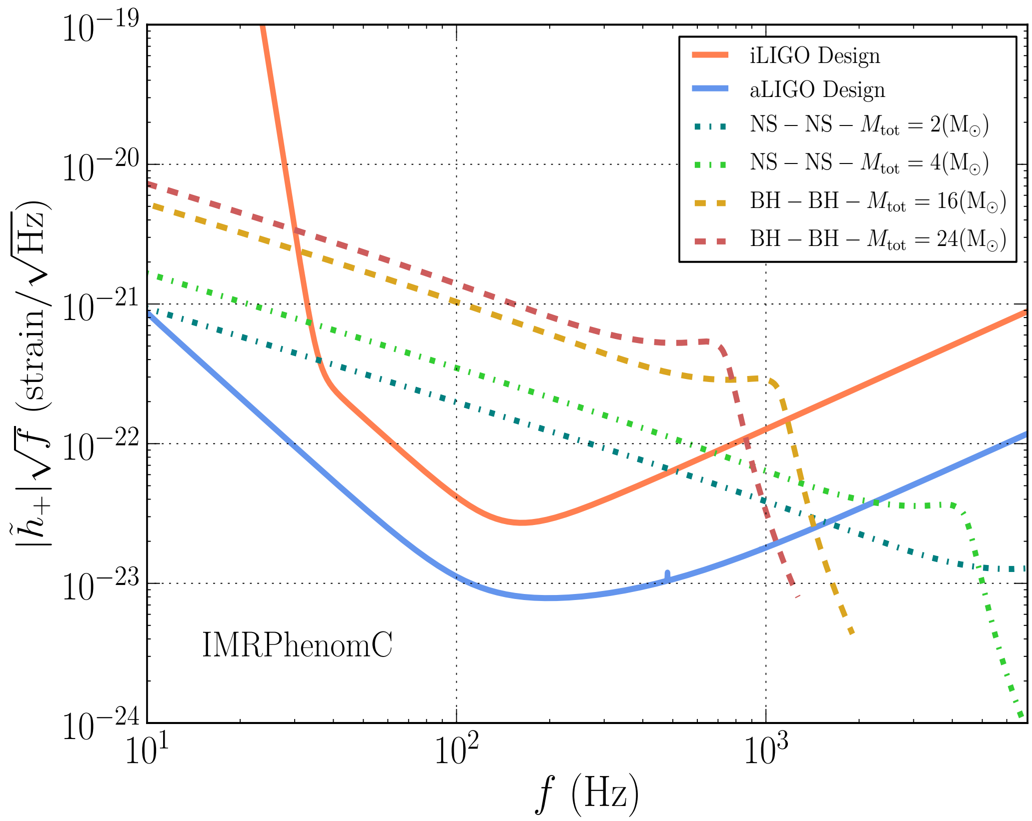

FIG. 1. Diagram showing the design amplitude spectral density of Initial LIGO (iLIGO), Advanced LIGO (aLIGO), and the equivalent gravitational wave strain for sources located at $D = 16$ Mpc with mass ratio, $q = 1$. The dark green (dash-dot) line shows a binary neutron star system (BNS) with total mass of $M_{\rm tot} \sim 2M_\odot$, while the light green (dash-dot) line shows a BNS with total mass, $M_{\rm tot} \sim 4M_\odot$. The gold (dashed) line shows a binary black hole system (BBH) with total mass of $M_{\rm tot} \sim 16M_\odot$ and the red (dashed) line shows a BBH with $M_{\rm tot} \sim 24M_\odot$.

To determine the frequencies at which the gravitational wave detector will be likely to observe a detection, the incoherent sum of the various sources of the noise within the detector (e,g. seismic noise, thermal noise, and shot noise) are used in determining a power spectral density. The power spectral density (PSD) is the power of the noise at a given frequency, while the amplitude spectral density (ASD) can be considered as the amplitude below which the interferometer is insensitive to a detection. Therefore, it is often useful to compare the ASD of a given detector to a variety of gravitational wave signals to determine the most likely astrophysical sources. The expected amplitude spectral density for Initial LIGO (iLIGO) [39], Advanced LIGO (aLIGO) [35], and approximate GW signals from possible astrophysical sources are shown in Figure (1). The sources shown were constructed using the `IMRPhenomC` waveform approximant as described in [50] with source distances $D = 16$ (Mpc), the approximate distance to the Virgo Cluster. Figure (1) suggest that for binary neutron star systems nearing coalescence, the inspiral phase will emit gravitational radiation within the frequency band of aLIGO but the merger phase will likely occur at frequencies outside of this range. However, binary black hole (BBH) sources are predicted to experience inspiral, merger, and ringdown well within aLIGO sensitivity, such as the compact binary system considered above.

B. Modeling Gravitational Waves Emitted by Compact Binary Systems

Waveform approximations for various astrophysical sources have been constructed by combining Nth order post newtonian (PN) approximation waveforms with those calculated by the data available from numerical relativity (NR) [54–56]. `IMRPhenomC` is a phenomenological waveform model constructed using this method to describe non-precessing BBH systems [50]. These model waveforms have been shown to yield an overlap with the hybrid waveforms of greater than 97% for all BBH systems observable by Advanced LIGO and are in agreement with results obtained in [46]. For this study, we perform calculations of non-spinning, equal mass ratio binary systems for a range of total mass using the `IMRPhenomC` template waveform as implemented in `lalsimulation`, a set of numerical routines used to compute the gravitational wave signal for a particular waveform template, contained in the LALSuite algorithm library [57].

C. Matched Filtering

Proper analysis of a gravitational wave signal requires the ability to determine as much about the astrophysical source as possible. One such technique used for estimating the parameters of a given GW signal is known as matched filtering. Matched filtering is a technique in which a GW signal can be analyzed to determine how well it correlates or *matches* a template waveform for a particular set of intrinsic input parameters. A match threshold may be set such that a time series, $s(t)$, must match with a particular waveform by more than, say, 97%. To calculate the match, $\mathcal{M}$ of a GW signal and a particular waveform we first determine the the overlap, $\mathcal{O}$, using the following noise weighted inner product

$$\langle A|B\rangle = 4\Re\int_{-\infty}^{+\infty}\frac{\widetilde{A}(f)\widetilde{B}^{*}(f)}{S_n(f)}df \ , \tag{10}$$

where $S_{\rm n}$ is the power spectral density of the detector, $\tilde{A}$ is the FFT of a signal A, and $\tilde{B}^{*}$ is the complex conjugate of the FFT of a waveform template B. The optimal signal to noise ratio (SNR) of the filter is

$$\rho_{\rm opt} = \sqrt{\langle A|A\rangle}, \tag{11}$$

and the normalized match can then be calculated by maximizing the overlap for ϕ_0 and t_0

$$\mathcal{M} = \max_{\{\phi_0,{\rm t}_0\}}\frac{\mathcal{O}}{\sqrt{\langle {\rm A}|{\rm A}\rangle\,\langle {\rm B}|{\rm B}\rangle}} \ , \tag{12}$$

For a given signal A, such that $A \approx B$, the match will yield a value of unity, matching the template waveform to 100%. Matched filtering allows for a quantitative assessment of the differences between signals and template waveforms as well as a means by which non GR deviations to the waveform approximations can affect the match.

D. Modeling Gravitational Waves Predicted by Alternative Theories of Gravity

Among the many alternative theories of gravity, only a few have been studied to the extent which observable deviations from GR can be inferred [2]. To investigate the effect of these alternatives one may consider a variety of methods. The two main methods considered are often referred to as the *top down* approach and the *bottom up* approach. We will discuss briefly the two different approaches and their different consequences.

The *bottom up* approach involves considering a specific alternative theory and its physical influence on the system and the extent to which it will modify the GR approximation of the system. For example, Jordan-Fierz-Brans-Dicke (BD) scalar tensor theory predicts dipole radiation in addition to the quadrupole radiation predicted by GR [58–61]. This radiation is expressed by

$$\dot{E}_{\rm BD} \sim -\frac{2G^3\eta^2M^4\mathcal{S}^2}{3c^5a^4\omega_{\rm BD}} \ , \tag{13}$$

where η is the symmetric mass ratio (m_1m_2/M^2), M the total mass, a the orbital separation, $\omega_{\rm BD}$ is the dimensionless Dicke coupling constant and $\mathcal{S}$ is the difference in sensitivity of the two objects with sensitivity, $s_i = (\partial \ln(m_i)/\partial \ln(G))_N$ at fixed baryon number, N. The additional radiation term predicted by BD scalar tensor theory suggests that systems will be more efficient in losing the energy contained in a binary system resulting in a different cutoff frequency, $f_{\rm cut}$, than that which is predicted by GR. Therefore, one can investigate the possible consequences of a particular alternative theories and their influence on systems modeled using GR waveforms.

Alternatively, in the *top down* approach one may adopt a general framework considering deviations from GR not particular to any one alternative theory but rather adopting phenomenological parameters constrained by weak field measurements in the context of different theories. A theoretical framework for introducing such modifications to standard GR waveforms has been proposed in [15, 62]. This framework, called the "Parameterized Post-Einsteinian" (ppE) framework introduces a minimal set of parameters from which "non GR" waveforms may be constructed for the inspiral, merger, and ringdown stages of binary systems. The ppE waveform model in the stationary phase approximation is constructed by

$$\tilde{h}_{\rm nGR}(f) \sim \tilde{h}_{\rm GR}(f)[1+\alpha u^a]e^{i\beta u^b} \ , \tag{14}$$

with

$$u \sim \pi\mathcal{M}f_{\rm GW} \ . \tag{15}$$

Eqn. (15) is the reduced GW frequency and $\mathcal{M} = M\eta^{3/5}$, commonly referred to as the chirp mass. For values of $\alpha = \beta = a = b = 0$, the ppE model results in the waveform model predicted by GR. In this representation, the exponents a and b are fixed exponents for a specific modified theory of gravity, while α and β correspond to the magnitude of modification to the amplitude and phase, respectively. For example, one recovers the leading ppE corrections to the Brans-Dicke theory for $(a, \alpha, b, \beta) = (a, 0, -7/3, \beta)$ [60, 63, 64] where tracking of the Cassini spacecraft has provided constraints on the parameters, α and β [65].

For this investigation, we consider a combination of the two approaches described above by introducing an arbitrary multiplicative parameter to a set standard GR waveforms. This parameter will allow for generic modifications to the GW signals that could arise from physically motivated deviations from GR as predicted by alternative

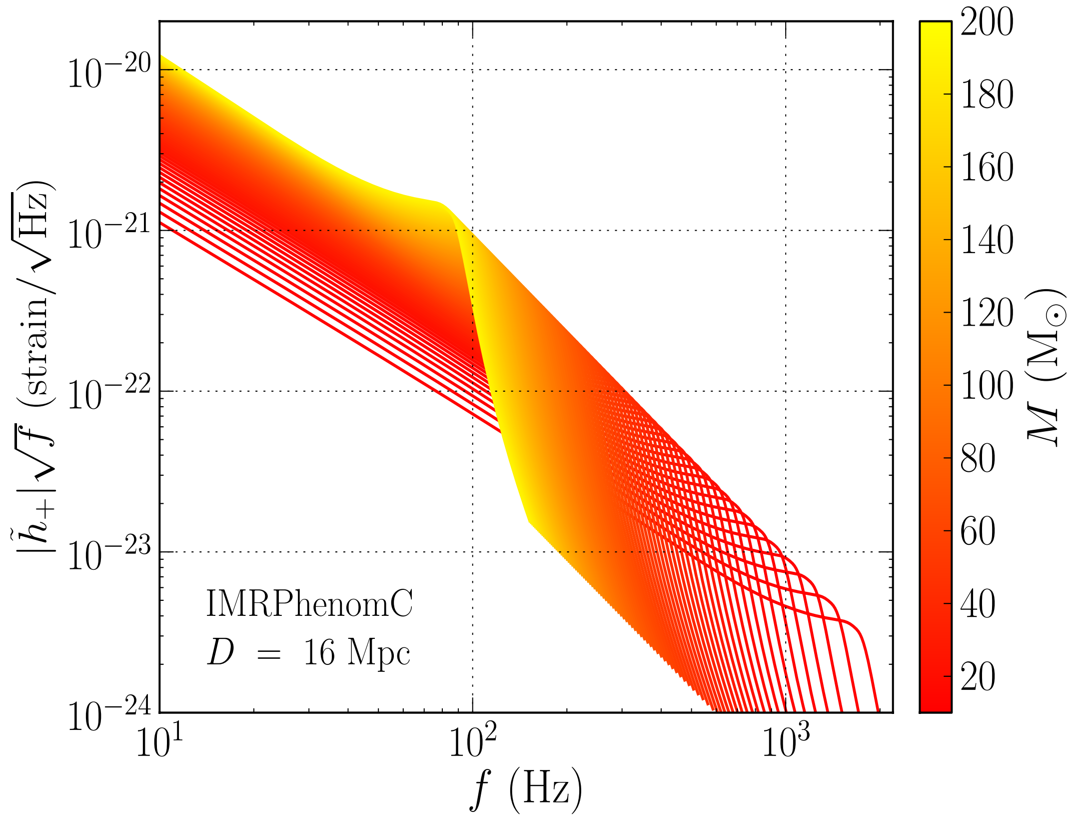

FIG. 2. Diagram showing the inspiral, merger, and ringdown gravitational wave strain as a function of frequency for a set of different systems. The waveform template used for these calculations was `IMRPhenomC`, a phenomenological waveform template constructed to describe non-precessing binary black hole systems. The systems considered here were non spinning, $\chi = 0$, and were of the mass range $M \in [10, 200]\ M_\odot$, a range chosen to encompass low to intermediate mass BH binaries. The most massive binaries, those with $M \geq 160\ M_\odot$, merge before $f_{\rm GW} \sim 100$ (Hz) while the less massive systems take longer to merge, with merger frequency increasing as total mass decreases.

theories. After modification of standard waveforms, we wish to quantify the effect these modifications will have on the signal analysis for gravitational wave searches with Advanced LIGO.

III. STANDARD GR WAVEFORMS

To begin our investigation, we first consider a set of standard waveforms as predicted by GR. The waveform template considered is the `IMRPhenomC` model [46]. We use this template to compute the inspiral, merger, and ringdown phase of binary BH systems in the mass of $M \in [10, 200]\ M_\odot$. To compute this GW signal requires a choice of input parameters corresponding to intrinsic parameters of the astrophysical source. These parameters include the orbital phase of the binary system, the frequency interval (sampling rate), starting and stopping frequencies, the BH reduced mass spin parameter, χ, and lastly, the distance from the source. We wish to study deviations from GR that could potentially manifest themselves as very small changes to the overall waveform. Therefore, it is important that one not introduce unnecessary uncertainty to the system by an impractical choice of input parameters [66–68]. To focus this study on tests of modifications to GR, we restrict our investigation to non-spinning, BBH systems at the optimal orbital phase and at a relatively close distance. The frequency interval, or sampling rate, of the calculation is arbitrary except in that we require it be sufficiently resolved to properly compute the waveform. The component masses considered are equivalent to half of the total mass for an equal mass ratio of $q = m_1/m_2 = 1$. The computation interval, the start and ending frequencies, were chosen such that the entire evolution of the system was calculated for frequencies detectable by aLIGO, $f_{\rm min} \geq 10$ (Hz). Choosing the input parameters in the following manner allowed us to provide a baseline set of waveforms from which we can begin our investigation.

Figure (2) shows the inspiral, merger, and ringdown gravitational wave strain in the frequency domain for a grid of models from total mass, M from $M \leq 10 \leq 200 M_\odot$. We can see that the most massive BBH systems ($M \geq M = 160\ M_\odot$), the merger occurs at a GW frequency of $f_{\rm GW} \sim 100$ (Hz). This frequency corresponds to the approximate frequency at which aLIGO is designed

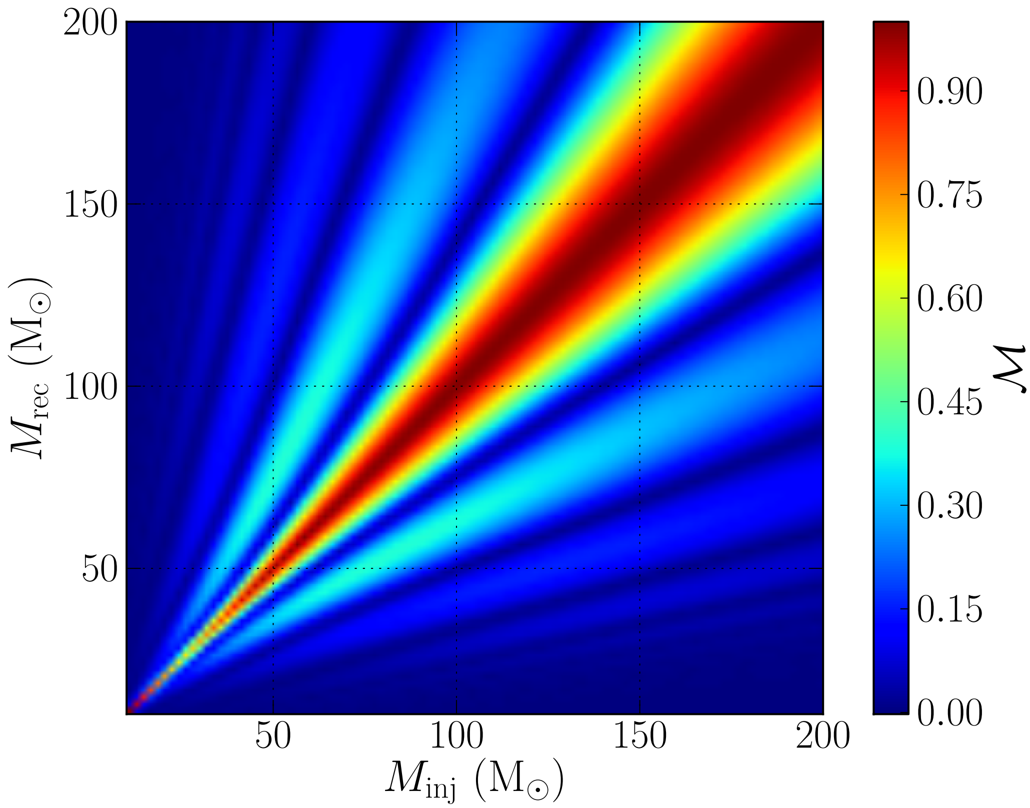

FIG. 3. A contour diagram showing the match of the injected waveform at a particular total mass, M_{inj} with another waveform of total mass, M_{rec} a value of unity is expected for the match of a waveform with itself as these waveforms do not contain any modifications. For systems with $M \geq 25\ M_\odot$, the precision in the match calculation decreases, permitting multiple waveforms to trigger matches above our set match threshold of $\mathcal{M}_{\mathrm{min}} \geq 0.97$.

to be most sensitive (See Figure 1).

Let us now consider how well aLIGO will be able to detect such GW signals by calculating the match as defined in Eqn. (12). We will be able to assess our ability to determine the astrophysical source from which the model originated, a crucial step in determining additional information about the system from the information stored with the gravitational wave strain. In Figure (3) we show the calculated match for a set of waveforms. The injected total mass M_{inj}, represents the input GW signal for our calculation (corresponding to $\tilde{A}$ in Eq. (12)), while the recovered mass, M_{rec} corresponds to the waveform template at a particular total mass. Because the comparison made here is the ideal case in which the waveform is compared with itself, we expect a value of unity for $M_{\mathrm{inj}} = M_{\mathrm{rec}}$. However, in Figure (3) we also see that for small systems of total mass ($M \lesssim 25\ M_\odot$), the match value falls steeply to a value below our minimum match criteria, $\mathcal{M}_{\mathrm{min}} \geq 0.97$, while systems above this mass have a more gradual decrease in match. This increased uncertainty for systems of higher mass may cause multiple match triggers above our minimum criterion, thus introducing possible degenericies when determining information about the astrophysical source.

IV. MODIFYING GR WAVEFORMS

Here we introduce our method to implement deviations from general relativity that may have non-negligible effects on the gravitational waveform used in detection of GW signals. We consider several waveforms and discuss the associated parameters and our motivation in altering these waveforms. The SNR lost by introducing these deviations are examined to determine our ability to detect gravitational waves by using waveforms predicted by general relativity. Before implementing modifications to the standard GR waveforms, we briefly review the steps required for construction of the physically motivated gravitational waveforms.

A. Constructing Standard GR Gravitational Waveforms

The first stage of compact binary coalescence is the inspiral. This stage can be adequately modeled under the assumption of a weak gravitational field using the Post Newtonian (PN) approximation to general relativity. The GW phase of the early adiabatic inspiral of a BBH coalescence is based on a stationary phase approximation and has been well modeled by the following analytical formula

$$\psi_{\rm SPA}(f) = 2\pi f t_0 - \phi_0 - \frac{\pi}{4} + \frac{3}{128\eta}(\pi f)^{-5/3}\sum_{k=0}^{7}\alpha_k(\pi f)^{k/3}\ , \tag{16}$$

where f is the GW frequency, ϕ_0 is the orbital phase of the binary, and α_k corresponds to the k'th coefficient of the TaylorF2 description of the Fourier phase [69–72].

As mentioned in Section II A, the assumption that the gravitational field is sufficiently weak begins to break down as the binary approaches the pre-merger phase, $a \to a_{\rm isco}$. Modeling of the premerger phase, $\psi_{\rm PM}$ was done by [50] using

$$\psi_{\rm PM}(f) = \eta^{-1}(\alpha_1 f^{-5/3} + \alpha_2 f^{-1} + \alpha_3 f^{-1/3} + \alpha_4 + \alpha_5 f^{2/3} + \alpha_6 f) \tag{17}$$

where α_k are phenomenological coefficients fitted to agree with hybrid waveforms for $0.1 f_{\rm RD} \lesssim f \lesssim f_{\rm RD}$. The ringown frequency, $f_{\rm RD}$ is determined from the spins and masses of the black holes considered and can be computed by the following form

$$f_{\rm RD} = \frac{c^3}{2\pi GM}[k_1 + k_2(1+a)^{k_3}] \tag{18}$$

where $k_{\rm i} = \{1.521, -1.1568, 0.1292\}$ as given in Table VIII of [73].

Lastly, the linear ansatz proposed by [50] for the ringdown phase is

$$\psi_{\rm RD} = \beta_1 + \beta_2 f \tag{19}$$

with coefficient determined by the pre-merger phase. The final phenomenological phase, $\Phi(f)$, is from these three sources with appropriate transition frequencies.

The PN model of the GW amplitude obtained from the stationary phase approximation can be expressed as

$$A_{\rm PN}(f) = \mathcal{C} f^{-7/6}(1 + \sum_{i=2}^{3}\alpha_i \nu^i) \quad \text{for } f < f_{\rm isco}, \tag{20}$$

where $\nu = (\pi M f)^{1/3}$ and $\mathcal{C}$ is a numerical constant that depends on the sky location, orientation and masses [46]. While this amplitude can be considered adequate for weak gravitational fields, as the binary approaches merger, a more descriptive approach is required. A pre-merger amplitude is proposed by the reexpansion of eqn. (20) leading to

$$A_{\rm PM} = A_{\rm PN} + \gamma_1 f^{5/3}\ , \tag{21}$$

where γ_1 is fit to obtain the pre-merger amplitude in NR simulations of binary black hole mergers [74, 75].

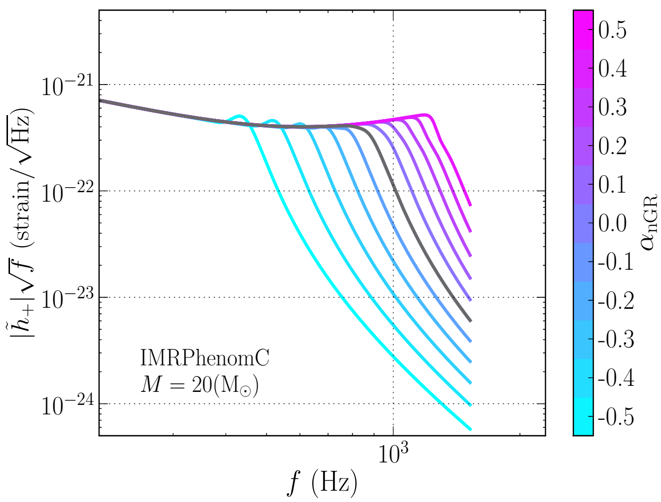

FIG. 4. A figure showing the gravitational wave strain as a function of frequency for a system with total mass, $M = 20M_\odot$. The gray line represents the GW for $\alpha_{\rm nGR} = 0$, corresponding to the model as predicted by GR.

The rigndown amplitude is approximated using a Lorentzian

$$A_{\rm RD} = \delta_1 \mathcal{L}(f, f_{\rm RD}, \delta_2 Q) f^{-7/6}\ , \tag{22}$$

with phenomenological coefficients fit to the hybrid data. The final amplitude function, $A_{\rm phen}$ can be expressed a combination to fully capture the entire evolution of the gravitational waveform. For a detailed review of the calculation of the fitting parameters for this phenomenological model, see Table II of [50].

B. Altering Standard GR Waveforms

To explore potential deviations from GR that may arise in highly dynamical and non-linear systems, we consider physical quantities used in the construction of the phenomenological models for template waveforms.

We first consider the ringdown frequency in the `IMRPhenomC` waveform template. Defined in eqn. (18), the ringdown frequency $f_{\rm RD}$ is modified in the following way

$$f_{\rm RD}^{\rm nGR} = f_{\rm RD}(1 + \alpha_{\rm nGR}), \tag{23}$$

where $\alpha_{\rm nGR}$ is an arbitrary parameter introduced to modify the ringdown frequency predicted by GR. Figure (4) shows the effect of this modification for a range of values $-0.5 < \alpha_{\rm nGR} < 0.5$. The GW strain for $\alpha_{\rm nGR} = 0$, corresponds to the unmodified waveform, gray line in figure (4), for a system with total mass $M = 20M_\odot$.

Modifying our gravitational waveform by means of eqn. (23) shows a significant shift in our waveform, to quantify this effect we wish to determine also the amount of SNR lost by a signal corresponding to a value of $\alpha_{\rm nGR} \neq 0$.

V. FUTURE WORK

Continued work involves quantifying the extent to which our current methods of parameter estimation will be able to analyze non-GR waveforms. We will continue our investigation for different aspects of the `IMRPhenomC`, waveform e.g., amplitude and pre merger phase. Then, we will prepare a final patch for which our new parameter, $\alpha_{\rm nGR}$ will be able to efficiently passed into a LALSuite GW simulation. This will expedite future work and allow for the focus to be placed on analyzing these modified waveforms. Below I will describe in detail the work done in the past ~ 4 weeks.

VI. PROGRESS UPDATE

Week 3 was spent determining what waveforms can be modified to introduce non-GR deviations. However, before I can introduce a modified waveform I was to write a program to compute the overlap of two waveforms A and B. After the script was written I altered the waveforms of IMRPhenomA and IMRPhenomC by adding an arbitrary modification to the strain of the form $h_+^{\rm nGR} = h_+^{\rm GR}[1+\alpha^2]$ where $\alpha = f/86.4$. This initial exploration helped show what to expect from deviations implemented into lalsuite. Additionally, I computed the inner product of the form $\langle u|v\rangle = \mathbf{u}\cdot\mathbf{v}/|u||v|$. However, after discussing with my mentors I found that my implementation did not take into account the power spectral density for the detector. I will revisit my code and determine the proper noise weighted inner product in the context of aLIGO should be and how to implement it.

Week 4 This week was spent participating in the Caltech Gravitational Wave and Astrophysics Summer School. Additional work on calculating the overlap was done during down time. I was also able to attend a lecture by Dr. Chen on Testing Alternative Theories of GR.

Week 5 A significant amount of time this week was spent addressing comments in my progress report #1 to bring my report closer to the final draft. For example, following Tjonnie's suggestion, I implemented a new structure in my introduction that allowed my ideas to flow more smoothly. In addition to this, I worked on computing the match, and with help from my mentor was able to finalize a script that calculates the match of two waveforms for a given noise curve. This script was used to produce figure (2) to illustrate how rapidly the match falls off for different waveforms and why matched filtering is so effective.

Week 6 Most of this week was spent in LIGO Livingston. However, I was able to add an overview of the methods and techniques used in my work, e.g. compact binaries in sec.(II A), matched filtering techniques in sec. (II C), and discussion on techniques/motivation for tests of GR and alternative theories in sec. (II D). I also spent a significant amount of time discussing the standard non modified waveforms to give the reader a detailed exmaple of what the GW signals will look like. This provided the platform for me to go beyond these standard waveforms by implementing modifications into lalsuite.

We have had to confront the following challenges: computing the match, enabling the user to pass a non-GR parameter to lalsuite, choosing which waveforms and or parameters to modifiy and over what range of variables. Computing the match was difficult for me as there are many deifinitions of this value in the literature however, my mentor was able to assist me in learning how to calculate it and enabling me to move to the next step of my prject. The waveform we are considering is arbitrary, but also mildly focused on BBH as they are potentially the most promising sources for direct detection of gravitational waves.

Updated steps by which our goals will be met are detailed below:

Objectives	
Week	Focused Efforts
7-8	Post processing, Bayesian Analysis *Parameter estimation using lalinference* *analyze results, ability to test GR*
9-10	Create final patch, conclude findings *Generalize method for future use (patch)* *Draw conclusion, prepare final presentation*

VII. DISCUSSION

VIII. CONCLUSIONS

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