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Linear systems

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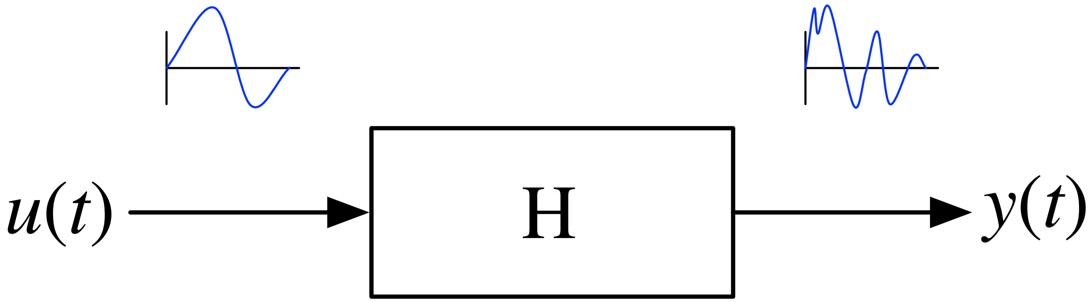
What is a "system"?

- **A "system"**

A physical instance that has internal DOFs (states)

Here we are interested in the response of the system

i.e. excitation -> response



Because...

- we can develop a better understanding of the system (i.e. modeling)
- it has a lot of applications

Particularly, **we are interested in Linear Time Invariant (LTI) systems**

Why do we need to learn about LTI systems?

- **Raw detector output ≠ GW waveforms**
  We need to consider:
  - interferometer / sensor / actuator responses
  - signal conditioning filters
  - effect of feedback controls

- **Understanding the dynamics of various systems**
  - mechanics, electronics, thermodynamics, optics, ...

- **Designing feedback/feedforward control system**

- **Signal processing: calibration / signal filtering**

Example of systems

- **“Filters” and “Transducers”**

If the input and output have:

a same unit, the system is categorized as a “filter”

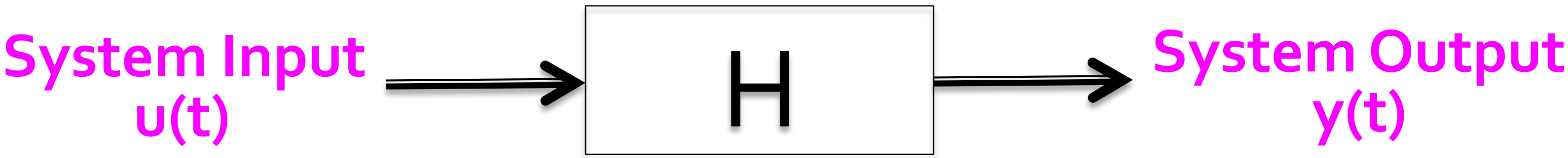
- electrical filter (V->V, A->A),
- mechanical filter (m->m), optical filter (E->E),
- digital filter (number -> number)

different units, the system is categorized as a “transducer”

- force-to-displacement actuator (m/s^2->m),
- electro-magnetic actuator (V->m/s^2),
- displacement sensors (m->V)
- electrostatic transducer (C (charge) -> N/m^2 (sound))
- transimpedance amplifier (A->V), current driver (V->A)
- gravitational wave detector (“gravito-optic modulator”)

LTI: Linearity and Time Invariance

- **LTI systems fullfil the following two conditions**



- **Linearity (super position)**

$$y_1(t) = H\left\{u_1(t)\right\}$$
$$y_2(t) = H\left\{u_2(t)\right\}$$
$$\Longrightarrow \alpha y_1(t) + \beta y_2(t) = H\left\{\alpha u_1(t) + \beta u_2(t)\right\}$$

- **Time invariance**

$$y(t) = H\left\{u(t)\right\}$$
$$\Longrightarrow y(t-\tau) = H\left\{u(t-\tau)\right\}$$

Why LTI?

- Why do we limit the discussion within LTI systems?
  - They are simple,
    but still gives a lot of characteristics about them
  - Nonlinear systems can be reduced to a linear system at a local region of the state space (cf. a pendulum)
  - If change of the system states is slow,
    most of the LTI arguments are still applicable

Impulse response

- **Impulse response:** A way to characterize the system H:

$$u(t) = \delta(t)$$
$$\implies y(t) = H\left\{u(t)\right\} \equiv h(t)$$

- **Constructing an arbitrary response:**

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\mathrm{d}\tau$$
$$\implies y(t) = H\left\{u(t)\right\}$$
$$= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

(Convolution)

Frequency response

- **Arbitrary response:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\mathrm{d}\tau$$

- **What is the system response to a sinusoidal excitation?**

$$u(t) = e^{i\omega t}$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} e^{i\omega\tau} h(t-\tau)\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} e^{i\omega(t-\sigma)} h(\sigma)\mathrm{d}\sigma \quad (\tau \to t-\sigma) \\ &= \left[\int_{-\infty}^{\infty} e^{-i\omega\sigma} h(\sigma)\mathrm{d}\sigma\right] e^{i\omega t} \\ &= \boxed{H(\omega)e^{i\omega t}} \end{aligned}$$

Fourier Transform of h(t)

Frequency response

$$u(t) = e^{i\omega t} \Longrightarrow y(t) = H(\omega)e^{i\omega t}$$

- Important consequences for LTI systems
  - Sinusoidal excitation induces sinusoidal response at the same frequency
  - The frequency response H(w) is, in fact, the fourier transform of the impulse response h(t)

Transfer functions

- **Impulse response:**

$$h(t)$$

- **Frequency response:** **(aka transfer function in freq domain)**

$$H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} h(t) \mathrm{d}t$$

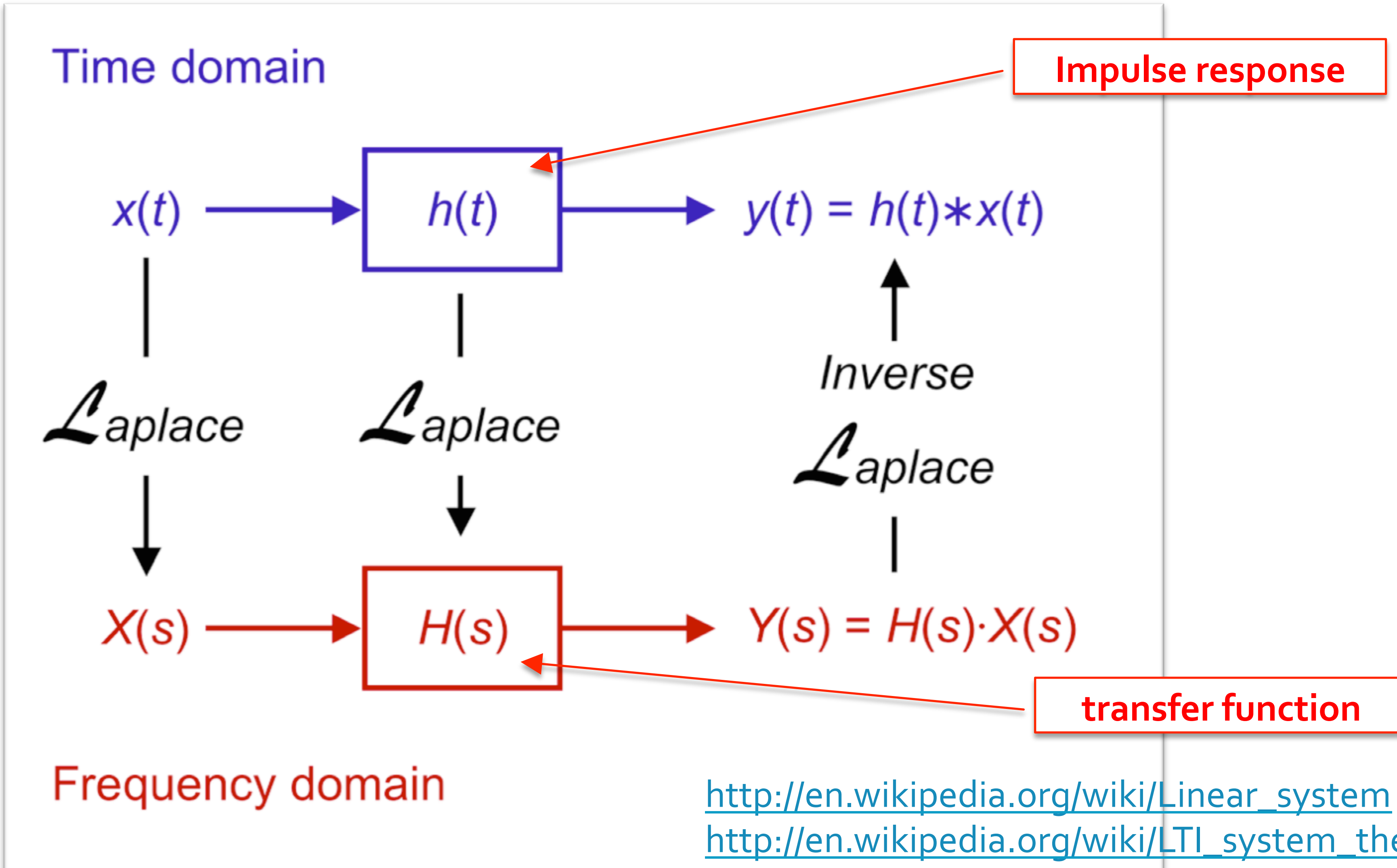
- **Transfer function in Laplace domain**

$$H(s) = \int_{-\infty}^{\infty} e^{-st} h(t) \mathrm{d}t$$

s is a natural extension of “frequency” in a complex plane
for most of the applications, we can just use

$$s = i\omega (= i2\pi f)$$

Time domain vs Laplace (or Fourier) domain



http://en.wikipedia.org/wiki/Linear_system
http://en.wikipedia.org/wiki/LTI_system_theory

Deriving a transfer function from a t-domain Diff Eq.

- In many cases, an LTI system can be described by a linear ODE
- It is easy to convert from an ODE to a transfer function

$$\frac{d}{dt} \Longrightarrow s$$

Laplace Transform

$$\Longrightarrow i\omega = i2\pi f$$

Fourier Transform

- e.g. Forced oscillation of a damped oscillator

$$m\ddot{x}(t) = -kx(t) - \gamma\dot{x}(t) + F(t)$$

$$ms^2X(s) = -kX(s) - \gamma sX(s) + F(s)$$

$$H(s) \equiv \frac{X(s)}{F(s)} = \frac{1}{ms^2 + \gamma s + k}$$

Transfer function of a mechanical system

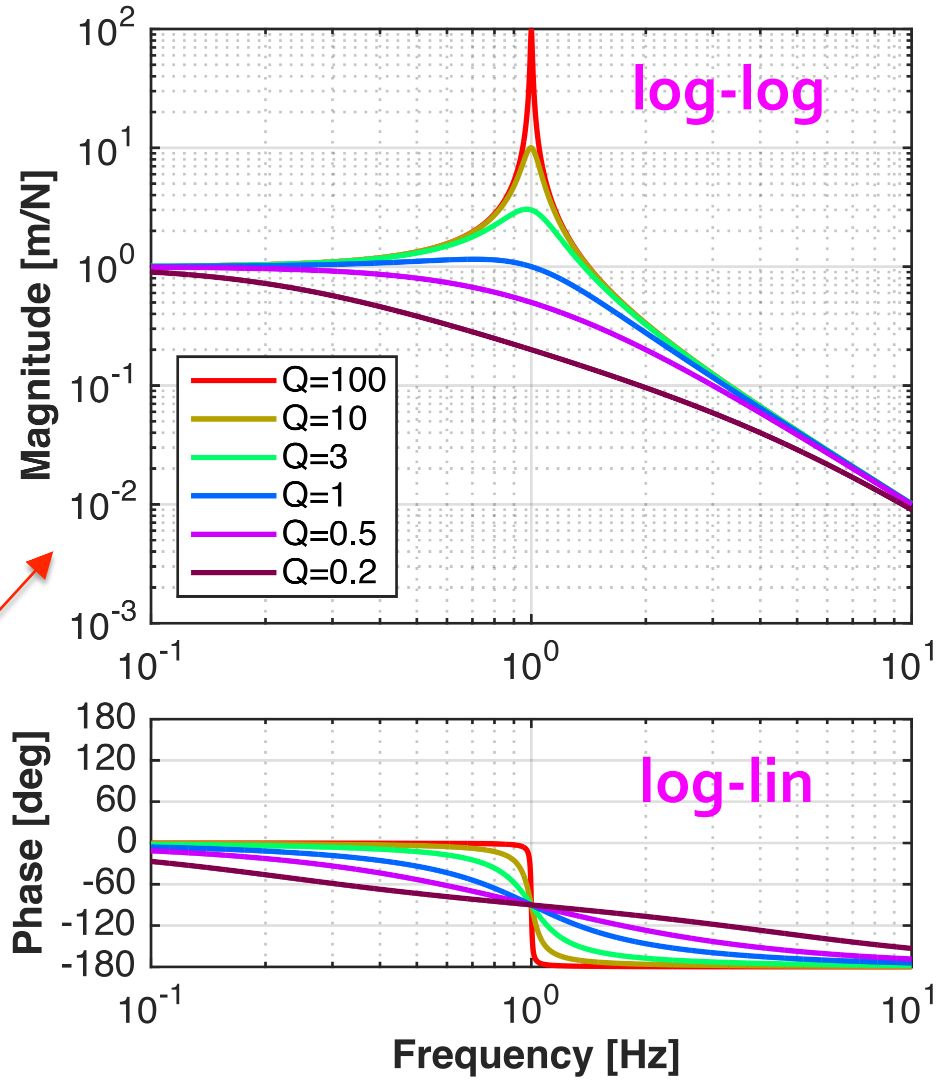
- **e.g. Forced oscillation of a damped oscillator**

$$H(s) = \frac{1}{ms^2 + \gamma s + k}$$

$$H(s) = \frac{1}{m} \frac{1}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

$$H(\omega) = \frac{1}{m} \frac{1}{-\omega^2 + i\frac{\omega_0}{Q}\omega + \omega_0^2}$$

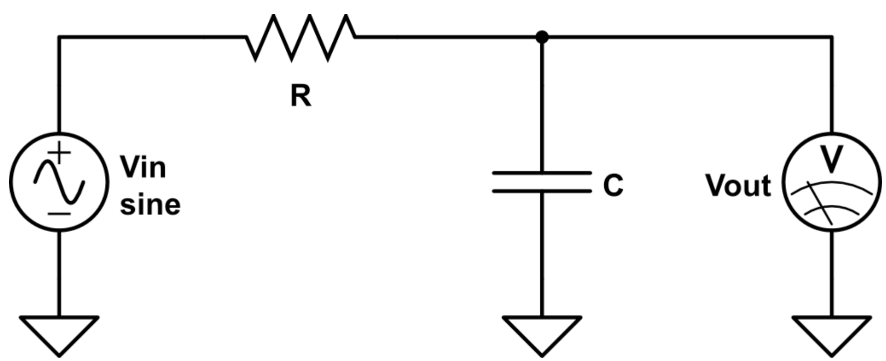
$$\omega_0 = \sqrt{k/m}, \ \ \gamma = m\omega_0/Q$$

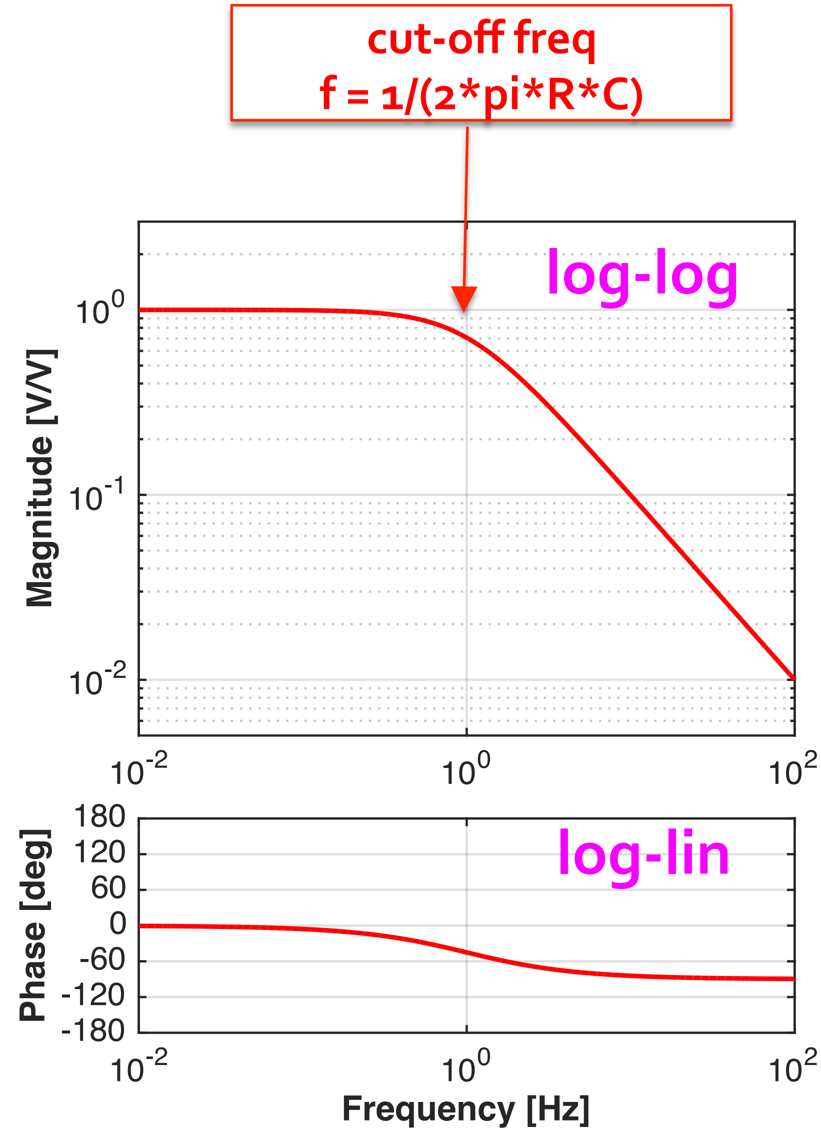


Bode diagram

Transfer function of an electrical system

- **e.g. RC filter**





$$V_{\rm out} = q/C$$

$$\dot{q} = (V_{\rm in} - V_{\rm out})/R$$

$$\Longrightarrow i\omega C V_{\rm out}(\omega) = (V_{\rm in}(\omega) - V_{\rm out}(\omega))/R$$

$$\Longrightarrow \frac{V_{\rm out}(\omega)}{V_{\rm in}} = \frac{1}{1 + i\omega RC}$$

Zeros, Poles, and Gain decomposition

- The transfer function of a system with an ODEcan be expressed as:

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n}$$

- The roots of the numerator are called as "zeros" and the roots of the denominator are called as "poles"

$$H(s) = \frac{b_m \prod_{i=1}^{m} (s - s_{zi})}{a_n \prod_{j=1}^{n} (s - s_{pj})}$$

- Zeros (s_{zi}) and poles (s_{pi}) are

  real numbers (single zeros/poles)

  or pairs of complex conjugates (complex zeros/poles)

  (fundamental theorem of algebra)

Linear systems and their stability

- **Poles rule the stability of the system!**

  H(s) can be rewritten as

$$H(s) = \sum_{j=1}^{n} \frac{K_j}{(s - s_{pj})}$$

(partial fraction decomposition)

- **Each term imposes exponential time impulse response**

$$\text{T.F.: } H_j(s) = \frac{1}{s + s_{pj}} \Longleftrightarrow \text{I.R.: } h_j(t) = e^{s_{pj} t}$$

- **Therefore, if there is ANY pole with $\mathrm{Re}(s_{pj}) > 0$ the response of the system diverges**

Linear systems and their stability

- **Poles rule the stability of the system!**
- **Location of the pole (pair) and the impulse response**

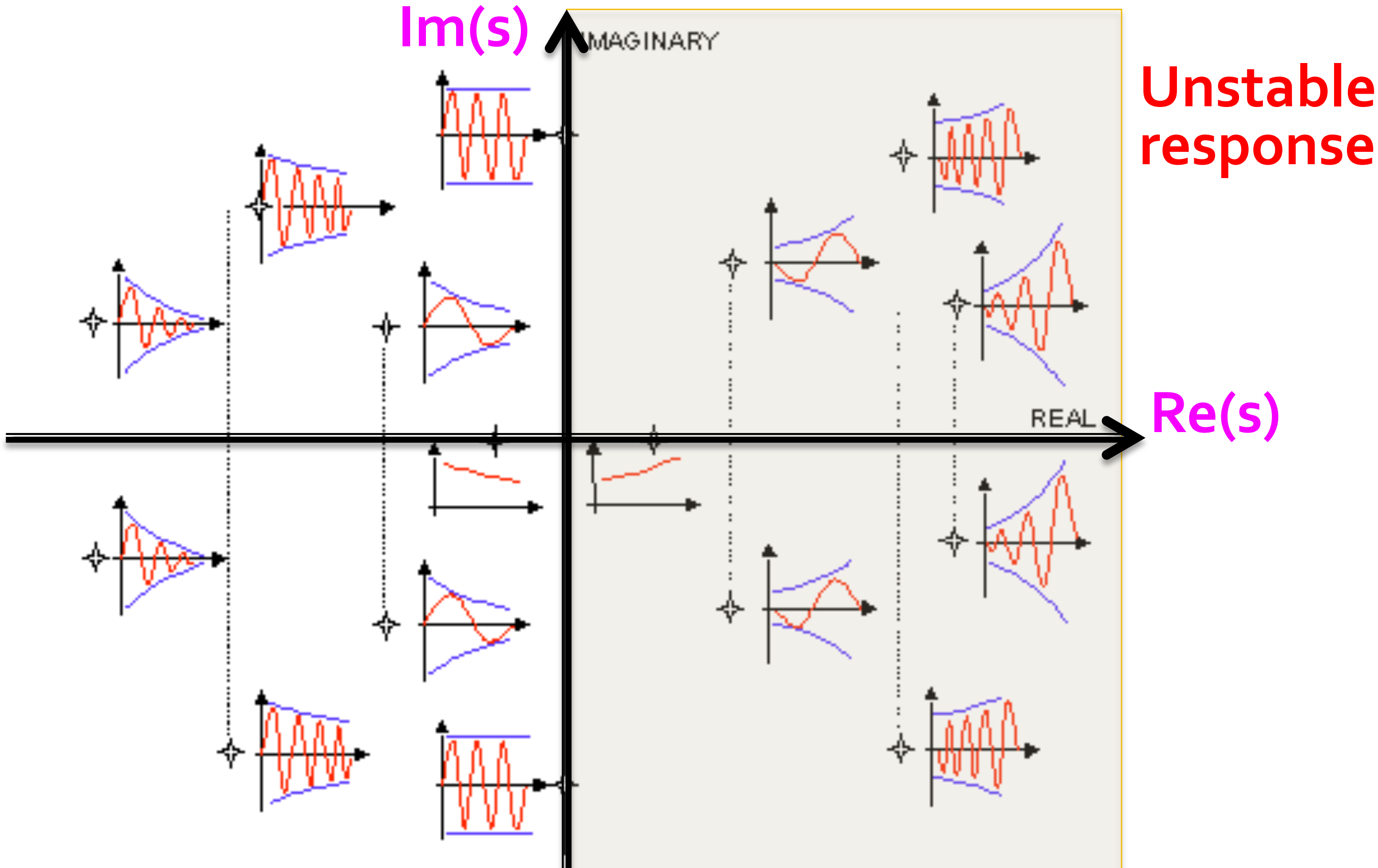
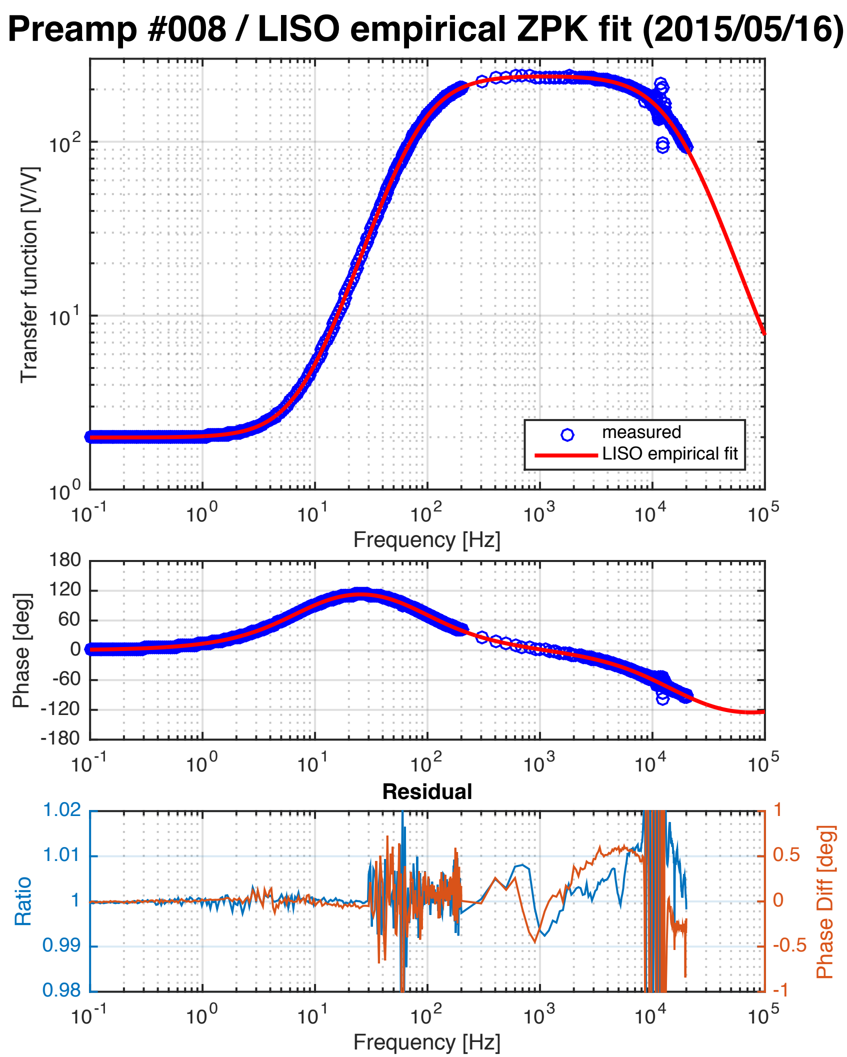


Figure 12: Root locus for different arrangements of the eigen values

http://nupet.daelt.ct.utfpr.edu.br/_ontomos/paginas/AMESim4.2.0/demo/Misc/la/SecondOrder/SecondOrder.htm

System identification

- **Modeling of the system: usually done in the freq. domain**




```
#LISO SOURCE
zero 7.7035143792  ### fitted (name = zero0)
zero 7.7035143792  ### fitted (name = zero1)
zero 134.5024181647k  ### fitted (name = zero2)

pole 84.5723492984  ### fitted (name = pole0)
pole 84.5723492984  ### fitted (name = pole1)
pole 22.3835023628k  ### fitted (name = pole2)
pole 11.9020537576k  ### fitted (name = pole3)

factor 1.9914251238  ### fitted

param zero0:f 1 10
sparam zero1:f
param zero2:f 1 1M

param pole0:f .1 1M
sparam pole1:f
param pole2:f .1 1M
param pole3:f .1 1M

param factor 1p 1M

fit Preamp008.bod absdeg rel

gnuterm pdf

rewrite samebetter

freq log 0.01 100k 1000 ### from data file


============================


#Parameter Estimation

#Best parameter estimates:
#zero0:f =        7.7035143791999995955 +- 33.87m (0.44%)
#--> zero1:f = 7.7035143792 +- 33.87m (0.44%)
#zero2:f =        134502.41816470012418 +- 46.44k (34.5%)
#pole0:f =        84.572349298399998929 +- 355.9m (0.421%)
#--> pole1:f = 84.5723492984 +- 355.9m (0.421%)
#pole2:f =        22383.502362800008996 +- 3.336k (14.9%)
#pole3:f =        11902.053757599991513 +- 809.9 (6.8%)
#factor =        1.9914251238000000299 +- 10.8m (0.542%)
```

System Identification Tools: e.g. LISO, Vectfit

Zero, Pole, Gain representation

- **Building blocks ("zpk" representation)**
  - **Single pole**

$$H(s) = \frac{s_p}{s + s_p} \qquad (s_p \in \mathbb{R}, s_p > 0)$$

  - **Single zero**

$$H(s) = \frac{s + s_z}{s_z} \qquad (s_z \in \mathbb{R}, s_z > 0)$$

  - **A pair of complex poles**

$$H(s) = \frac{s_p s_p^*}{(s + s_p)(s + s_p^*)} \quad (s_p \in \mathbb{C}, \ \Re(s_p) > 0)$$

  - **A pair of complex zeros**

$$H(s) = \frac{(s + s_z)(s + s_z^*)}{s_z s_z^*} \quad (s_z \in \mathbb{C}, \ \Re(s_z) > 0)$$

  - **Gain**

$$H(s) = K \qquad (K \in \mathbb{R})$$

Linear systems

- Summary
  - LTI systems
  - Description of the LTI system:

    Impulse response <==> transfer function
  - Zero, Pole, Gain representation of transfer functions
  - Pole locations determine the stability of the system

    System identification

Zero, Pole, K representation

- **Relationship between pole/zero locations and wo&Q**

$$H(s) = \frac{s_p s_p^*}{(s+s_p)(s+s_p^*)}$$
$$= \frac{|s_p|^2}{s^2 + 2\Re(s_p)s + |s_p|^2}$$

- **To be compared with**

$$H(\omega) = \frac{\omega_0^2}{-\omega^2 + i\omega_0\omega/Q + \omega_0^2}$$

$$\Longrightarrow \quad \omega_0 = |s_p|,\ Q = \frac{|s_p|}{2\Re(s_p)}$$

Zero, Pole, K representation