Appendix A

Pages from class notes of 'Advanced Structural Dynamics' class at MIT by Professor Eduardo Kausel. Course number 2.060. September 2006.

The vibration absorber

Let K, M denote the stiffness and mass of a structure, modeled as a single dof system, and let k, m be the stiffness and mass of a vibration absorber or tuned-mass damper (TMD). The TMD is used to ameliorate the intensity of motion in the structure when the latter is subjected to dynamic forces F. We define

$\Omega = \sqrt{\dfrac{K}{M}}$	Frequency of structure alone (without the TMD)
$\omega_o = \sqrt{\dfrac{k}{m}}$	Frequency of TMD alone (without the structure)
$\omega_r = \sqrt{\dfrac{K}{m+M}}$	Frequency of structure with infinitely rigid oscillator
ω_1, ω_2	Frequencies of coupled system
$\mu = \dfrac{m}{M}$	Mass ratio

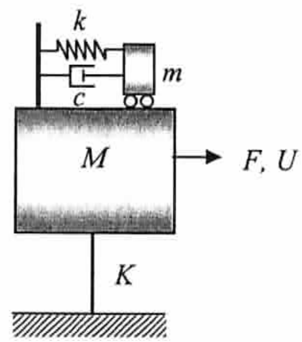

Fig. A1

The stiffness and mass matrices of the coupled system as well as the displacement and load vectors are

$$\mathbf{K} = \begin{Bmatrix} k & -k \\ -k & k+K \end{Bmatrix} \qquad \mathbf{M} = \begin{Bmatrix} m & \\ & M \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} u \\ U \end{Bmatrix} \qquad \mathbf{p} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Solving the eigenvalue problem for this 2-DOF system, we obtain the coupled frequencies

$$\left(\frac{\omega_j}{\Omega}\right)^2 = \frac{1}{2}\left\{(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2 + 1 \mp \sqrt{\left[(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2 - 1\right]^2 + 4\mu\left(\frac{\omega_o}{\Omega}\right)^2}\right\} \qquad j=1,2$$

Next, we evaluate the dynamic response in the structure elicited by a harmonic force with amplitude F acting on the structural mass M. Neglecting damping in the structure (but not in the oscillator), the dynamic equilibrium equation is then

$$\begin{Bmatrix} k + \mathrm{i}\omega c - \omega^2 m & -(k+\mathrm{i}\omega c) \\ -(k+\mathrm{i}\omega c) & K + k + \mathrm{i}\omega c - \omega^2 M \end{Bmatrix} \begin{Bmatrix} u \\ U \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Solving for the response in the structure U, we obtain after brief algebra

$$U = \frac{F\left[k - \omega^2 m + \mathrm{i}\omega c\right]}{\left[K - \omega^2(m+M)\right]\left[k - \dfrac{\left(K - \omega^2 M\right)\omega^2 m}{K - \omega^2(m+M)} + \mathrm{i}\omega c\right]} \qquad (1) \quad \text{Eq. A1}$$

We shall show now that there exist two frequencies for which the response is independent of the damping constant c. Hence, all amplification functions have these points in common, no matter what the damping. These two points occur when the complex terms in the numerator and denominator cancel identically. This is satisfied if, and only if, the two real parts are equal, that is, if

$$\pm\left[k-\omega^2 m\right]=k-\frac{\left(K-\omega^2 M\right)\omega^2 m}{K-\omega^2(m+M)} \qquad (2)$$

Eq. A2

The plus/minus sign on the left-hand side is to allow for frequencies greater than that of the tuned mass damper (a condition that would make the left-hand side term negative). If we consider first the positive sign, we find that it is satisfied only if $\omega=0$. This solution is not interesting, because it represents a static problem. On the other hand, if we consider the negative sign, we obtain the biquadratic equation

$$m(m+2M)\omega^4-2\left[k(m+M)+Km\right]\omega^2+2Kk=0$$

whose solution is

$$\left(\frac{\omega_{P,Q}}{\Omega}\right)^2=\frac{1}{2+\mu}\left\{(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2+1\mp\sqrt{\left[\left(\frac{\omega_o}{\Omega}\right)^2-1\right]^2+\mu(2+\mu)\left(\frac{\omega_o}{\Omega}\right)^2}\right\} \qquad (3)$$

Eq. A3

The two frequencies given by this equation represent two points P, Q through which all transfer functions must pass, no matter what their damping should be. In particular, these two points must

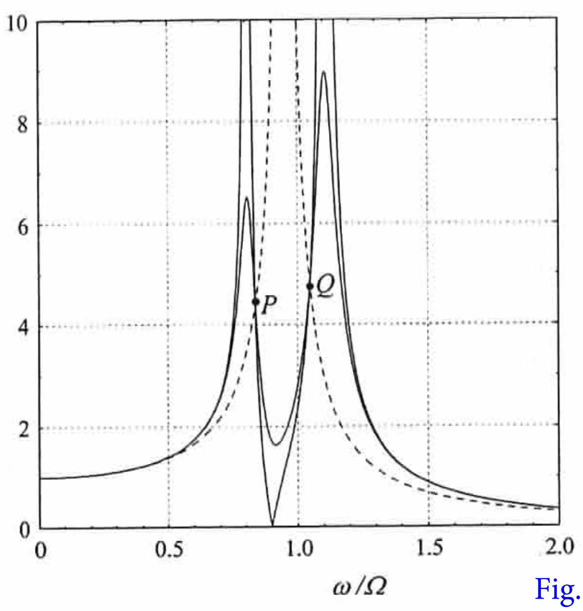

Fig. A2

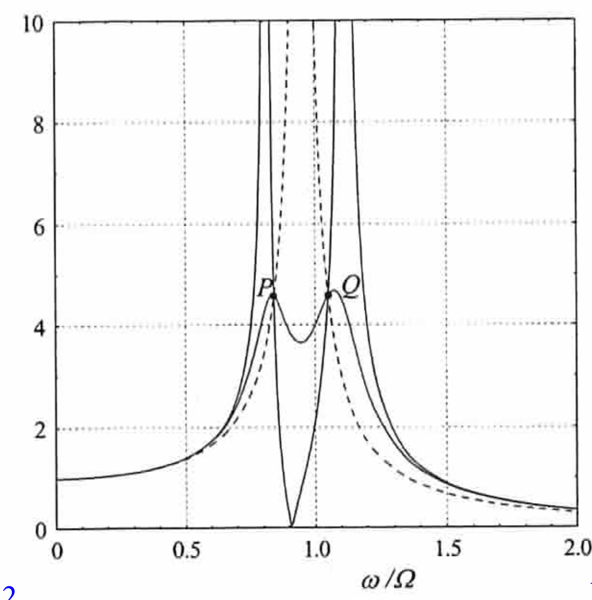

Fig. A3

also be traversed by the two transfer functions that correspond to zero damping and to infinite damping, i.e. $c=0$ and $c=\infty$. In the latter case, the system behaves the same as if the TMD was perfectly rigid. This in turn represents an undamped SDOF system with stiffness K, total mass m+M, and frequency ω_r as defined earlier. This frequency, and that of the oscillator alone (ω_o), are bracketed by the two natural frequencies of the coupled system, as can be shown by considering Rayleigh's quotient with an arbitrary vector $\mathbf{v}^T=\{a \quad b\}$. From the enclosure theorem, we have

$$\omega_1^2 \le R = \frac{\mathbf{v}^T \mathbf{K}\,\mathbf{v}}{\mathbf{v}^T \mathbf{M}\,\mathbf{v}} = \frac{(a-b)^2 k + b^2 K}{a^2 m + b^2 M} \le \omega_2^2 \qquad \text{Eq. A4}$$

in which a and b can be chosen arbitrarily; if we consider in turn the two choices a=b as well as b=0, we obtain the two inequalities

$$\omega_1^2 \le \frac{K}{m+M} \le \omega_2^2 \quad \text{and} \quad \omega_1^2 \le \frac{k}{m} \le \omega_2^2 \qquad \text{Eq. A5}$$

that is, $\omega_1 \le \omega_r \le \omega_2$ and $\omega_1 \le \omega_o \le \omega_2$. It follows that the two points P, Q lie somewhere in between the two natural frequencies at the intersection of the amplification function of the undamped coupled system and the undamped SDOF system with augmented mass m+M. The response amplitudes at these two frequencies are obtained by substituting the left-hand side of eq. 2 (negative sign case) into the denominator of eq. 1, and setting the dashpot constant to zero. The result is

$$U_{P,Q} = -\frac{F}{K - \omega_{P,Q}^2 (m+M)} \qquad \text{Eq. A6}$$

In general, the response amplitudes at the two frequencies for P, Q will not be equal. Optimal tuning of the mass damper can be achieved by enforcing these two amplitudes to be the same:

$$K - \omega_P^2 (m+M) = \pm\left[K - \omega_Q^2 (m+M)\right] \qquad \text{Eq. A7}$$

The case where both amplitudes are equal and have the same sign cannot be satisfied, since it implies equal frequencies for P and Q. Alternatively, if we consider equal amplitudes and opposite phase, we obtain

$$\omega_P^2 + \omega_Q^2 = \frac{2K}{m+M} = \frac{2\Omega^2}{1+\mu} \qquad \text{Eq. A8}$$

Equating this to the sum of the two roots in eq. 3, we obtain

$$\frac{\omega_P^2 + \omega_Q^2}{2\Omega^2} = \frac{1}{1+\mu} = \frac{(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2 + 1}{2+\mu} \qquad \text{Eq. A9}$$

From here, we obtain the optimal tuning condition

$$\boxed{\frac{\omega_o}{\Omega} = \frac{1}{1+\mu} = \sqrt{\frac{k\,M}{K\,m}}} \qquad \text{Eq. A10}$$

which relates the optimal frequency of the oscillator to the design mass ratio. This ratio ensures that the two points P, Q have the same height. The coupled frequencies observed with optimal tuning are

$$\boxed{\frac{\omega_j}{\Omega} = \sqrt{\frac{1+\frac{1}{2}\mu \mp \sqrt{\mu + \frac{1}{4}\mu^2}}{1+\mu}}} \qquad \text{Eq. A11}$$

The optimal damping constant that should be assigned to an optimally tuned mass damper is the one that would cause the transfer function at the two points P, Q to have a horizontal slope. However, the analysis for this condition is rather cumbersome, and exact expressions are not available. A reasonably close approximation is given by the expression[9]

$$\boxed{\frac{c}{2m\Omega} \equiv \frac{\xi\,\omega_o}{\Omega} = \sqrt{\frac{3\mu}{8(1+\mu)^3}}} \qquad \text{Eq. A12}$$

The transfer function for an optimally tuned mass damper is shown on the figure on the right. The maximum amplification for this case is $A_{\max} = \sqrt{1+2/\mu}$.

Lanchester damping

A Lanchester tuned mass damper is one in which the stiffness of the damper is zero (or nearly zero). The optimal parameters for this case are

Eq. A13 $\qquad A_{\max} = 1 + 2/\mu$

Eq. A14 $\qquad \xi = \dfrac{c}{2m\Omega} = \sqrt{\dfrac{1}{2(2+\mu)(1+\mu)}}$

Eq. A15 $\qquad \dfrac{\omega_Q}{\Omega} = \sqrt{1 - \dfrac{\mu}{2(1+\mu)}} \approx \sqrt{\dfrac{1}{1+\frac{1}{2}\mu}}$

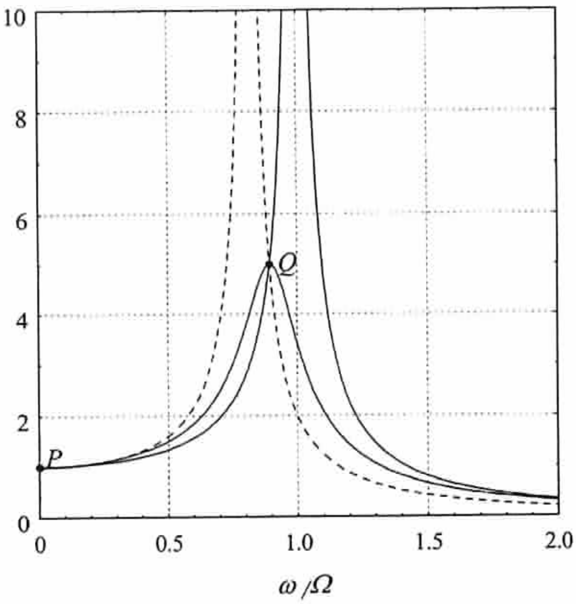

Fig. A4

Torsional Tuned Mass Damper

The torsional mass damper is a pendulum damper that automatically adjusts its resonant frequency to the rotational speed of wheels and shafts. It is used to ameliorate vibrations in these systems and dissipates energy through friction instead of viscous dashpots.

Consider a wheel to which two simple penduli of length L are attached. The penduli pivot about diametrically opposite points that are distant a from the axis, and are maintained in place by springs. As the wheel turns with rotational speed ω, the pivoting points experience a centripetal

[9] Den Hartog, J.P. : Mechanical Vibration (4th edition), McGraw-Hill, New York, 1956

acceleration that elicits a fictitious centrifugal gravity field $g' = \omega^2 (a+L) \gg g$, which in turn imparts on the penduli a resonant frequency

$$\omega_o = \sqrt{\frac{g'}{L}} = \omega \sqrt{1+\frac{a}{L}} \qquad \text{Eq. A16}$$

If $a \ll L$, then $\omega_o \approx \omega$, implying an oscillator that is tuned to the rotational speed.

As an example of application, consider a machine shaft whose flexural vibration mode is being excited by unavoidable eccentricities, and assume that the frequency of this mode is twice the operational speed of the shaft. To suppress this vibration, we must design a tuned mass damper that is tuned to that frequency, that is, having a natural frequency $\omega_o = 2\omega$. This can be accomplished by setting $4\omega^2 = \omega^2 (1 + a/L)$, which yields a=3L.

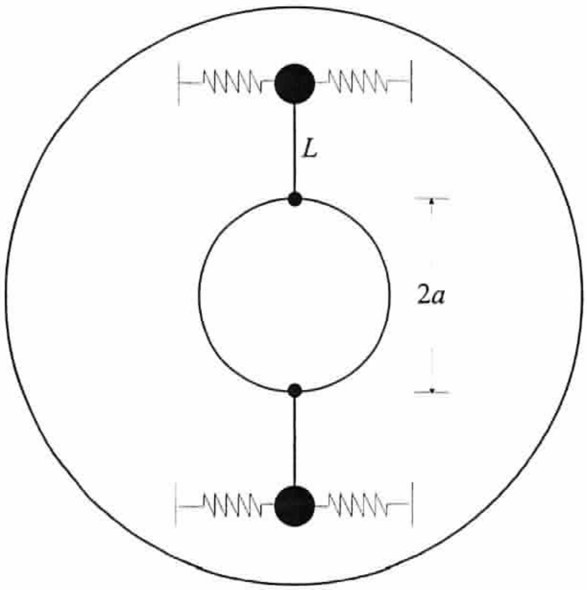

Fig. A5