LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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LIGO SCIENTIFIC COLLABORATION

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Tuned Mass Damper Simulations for Quad Bounce and Roll Modes

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Contents

1 Introduction

Due to the regular problems posed to the interferometer by the ringing up of the quadruple suspensions' highest frequency bounce and roll modes, this note explores the possibility of damping them with tuned mass dampers (TMDs). This note simulates the damping and thermal noise performance of TMDs applied to the springs tips in the upper-intermediate-mass (UIM). See Fig. 1.

Section 2 gives a list of variables used in Section 3. The design of the TMDs in Section 3 follows class notes appended to this document in Appendix A. These notes are from a Mechanical Engineering class at MIT in the fall of 2006. They are an excerpt from an unfinished text book the professor was writing at the time called 'Advanced Structural Dynamics'. I haven't been able to find a published version of the book.

The results of the simulations are given in Section 4 and discussed in Section 5. You can read these sections without the details of Sections 2 and 3.

The simulations apply models for each damper (4 total) to the full quad model since each damper will be influenced by both bounce and roll. This also permits us to explore asymmetric designs, such as putting just a bounce damper on one spring and a roll damper on the other.

The MATLAB code for these simulations is on the SVN at
`.../SusSVN/sus/trunk/QUAD/Common/MatlabTools/QuadModel_Production/`
`TMD_BounceAndRollModes_fullquad.m`

The quad model parameter file referenced in this document is in the same directory. The filename is `quadopt_fiber.m`.

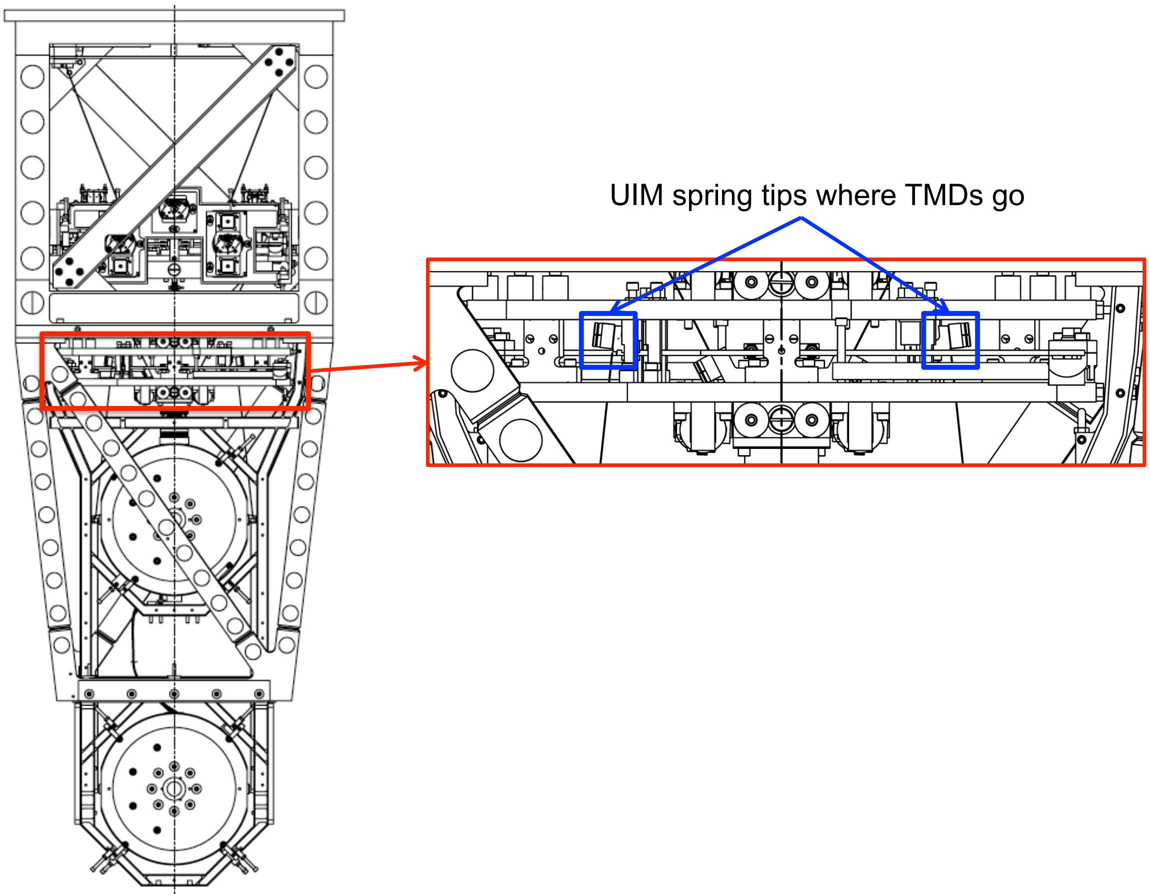

Figure 1: Left: A SolidWorks sketch of the quadruple pendulum. Right: a blow up of the UIM, highlighting the blade spring tips where the dampers are being considered to go. Reference D0901346.

2 Notation

I will follow the notation in Appendix A and the quad model parameter file as closely as possible. The list below summarizes the variables used in Section 3.

- Ω is the undamped bounce or roll frequency in rad/s.
- ω_0 is the frequency of the damper alone, in rad/s.
- μ_b is the mass ratio of the bounce damper.
- μ_r is the mass ratio of the roll damper.
- m_b is the bounce damper mass on each spring tip.
- m_r is the roll damper mass on each spring tip.
- k is the damper stiffness.
- c is the damper viscous damping coefficient.

- d is the damper structural damping factor.
- M_2 is the mass of the quad's penultimate mass (PUM), following the model's notation.
- I_{2x} is the roll inertia of the PUM, following the model's notation.
- n_3 is half the horizontal distance between the wire prisms on the PUM.
- M_r is the effective roll mass of the quadruple suspension.
- x is the variable for the position of the damper mass.
- x_g is the variable for the position of the damper's 'ground' (the spring tip).

3 Tuned Mass Damper Design

This section presents the equations for damper mass, resonant frequency, stiffness, and damping. It then presents the state space matrices for the damper model. Finally, it discusses how the thermal noise is calculated.

3.1 Damper Mass

The mass of each bounce damper is given by

$$m_b = 0.5\mu_b M_2 \tag{1}$$

The 0.5 is because we have two of them, one on each spring. M_2 is probably not the theoretically correct mass value to use. Likely the modal mass of the bounce mode is more correct. However, since the PUM and test mass are the same weight, the modal mass is also the same.

The mass of each roll damper is

$$m_r = 0.5\mu_r M_r \tag{2}$$

$$M_r = I_{2x}/n_3^2 \tag{3}$$

As for the bounce damper, it is probably more correct to reference the modal mass rather than the inertia of the PUM. However, since the PUM and test mass are the same, this works out OK.

Note that the effective roll mass, M_r, is obtained by dividing the PUM inertia I_{2x} by n_3^2, the square of half the wire spacing at the PUM, not the the UIM spring tips. The wire is slightly angled, so the wire spacing is not the same in both places (though it is close). I believe it is better to use the PUM because for small roll rotations the vertical displacement of the spring tip is the same as that of the PUM wire prism. Thus, we can think of the dampers as being located at the tips of the PUM prisms.

3.2 Damper Frequency and Stiffness

The resonant frequency of an ideal damper (on its own) is not exactly the frequency of the mode to be damped. As given by Eq. (A10), the frequencies are scaled by the mass ratio

$$\omega_0 = \frac{\Omega}{1+\mu} \tag{4}$$

Likely, the mass ratios in this case will be so small that we would do just as well targeting the actual bounce and roll frequencies.

Given the damper frequency and mass, the stiffness is straight forward.

$$k = m\omega_0^2 \tag{5}$$

3.3 Damper Lossiness

There are two types of lossiness we must consider. Appendix A assumes viscous or velocity damping (dashpots). This is a good model for eddy current damping. However, a better model for the damping inherent in many materials is structural damping, where the loss is expressed as an imaginary term in the damper stiffness. Fortunately, we can use Appendix A for structural as well as viscous by making the loss terms equal at the resonant frequency.

For viscous damping the equation of motion of the damper by itself is

$$m\ddot{x} + c\dot{x} + kx = c\dot{x}_g + kx_g \tag{6}$$

where x is the displacement of the damper mass, and x_g is the displacement of the damper's 'ground'. c is the viscous damping coefficient.

In the frequency domain, this becomes

$$-m\omega^2 x + ic\omega x + kx = ic\omega x_g + kx_g \tag{7}$$

where ω is an arbitrary frequency in rad/s.

For structural damping, in the frequency domain, this is

$$-m\omega^2 x + k(1+id)x = k(1+id)x_g \tag{8}$$

Where, d is the loss factor on the stiffness. This loss is assumed to be frequency independent. Rearranging terms to compare with viscous damping

$$-m\omega^2 x + ikdx + ks = k(1+id)x_g \tag{9}$$

If the loss terms on the left of Eqs. (7) and (9) are equal at resonance $\omega = \omega_0$, then

$$d = \frac{\omega_0}{k}c \tag{10}$$

Eq. (A12) gives an approximation of the ideal viscous damping coefficient, c, where ideal means the most damping for a given mass ratio.

$$\boxed{c = 2m\Omega\sqrt{\frac{3\mu}{8(1+\mu)^3}} \quad [\mathrm{N/(m/s)}]} \tag{11}$$

Therefore, the structural damping constant d is

$$\boxed{d = 2\frac{m}{k}\omega_0\Omega\sqrt{\frac{3\mu}{8(1+\mu)^3}} = 2\frac{\Omega}{\omega_0}\sqrt{\frac{3\mu}{8(1+\mu)^3}}} \tag{12}$$

Since equation A12 is an approximation, I found it necessary to multiply by a correction factor. For mass ratios of 10^{-5}, this turns out to be 1.16. For 10^{-4} it is 1.14. By 10^{-2}, unity works well (it is insensitive to the correction at that point).

Note, the damping values c and d can be normalized to a damping ratio ζ for easier comparison (when the damper is by itself), where $\zeta = 1$ is critical damping. ζ is also inversely related to the Q factor.

$$\zeta = \frac{1}{2m\Omega}c = \frac{1}{2}d = \frac{1}{2Q} \tag{13}$$

3.4 Damper State Space Model

State space models of the dampers are needed in order to connect them to the quad state space model. There are two states in each damper model, the vertical displacement x and velocity x_g of the damper mass.

This subsection builds the state space matrices for both the viscous and the structural damping cases.

Note, structural damping is not quite linear and so the state space will have non-complimentary pair poles. This results in an unstable system. This is fine for thermal noise calculations, but one must use the equivalent viscous case for all controls simulations.

The $\boldsymbol{A}$ matrix for each structural damper is

$$\boldsymbol{A}_{SD} = \begin{bmatrix} 0 & 1 \\ -k(1+id)/m & 0 \end{bmatrix} \tag{14}$$

The $\boldsymbol{A}$ matrix for each viscous damper is

$$\boldsymbol{A}_{VD} = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} \tag{15}$$

The remaining state space matrices depend on which spring the damper is located, since a roll of the quad will move the damper either up or down depending on where it is. I follow the sign convention that positive roll points down the beam tube axis.

For the left spring tip, as seen from behind the quad, the structural damper $\boldsymbol{B}$ matrix is

$$\boldsymbol{B}_{SD,L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k(1+id)/m & n_3 k(1+id)/m & 0 & 0 \end{bmatrix} \tag{16}$$

where the first two columns correspond to vertical and roll displacements respectively of the PUM, and the second to vertical and roll velocities. The bottom row corresponds to the acceleration imposed on the damper mass.

For the right spring tip, as seen from behind the quad, the structural damper $\boldsymbol{B}$ matrix is

$$\boldsymbol{B}_{SD,R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k(1+id)/m & -n_3 k(1+id)/m & 0 & 0 \end{bmatrix} \tag{17}$$

For the viscous damping case, these $\boldsymbol{B}$ matrices are

$$\boldsymbol{B}_{VD,L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k/m & n_3 k/m & c/m & n_3 c/m \end{bmatrix} \tag{18}$$

$$\boldsymbol{B}_{VD,R} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k/m & -n_3 k/m & c/m & -n_3 c/m \end{bmatrix} \tag{19}$$

For the left spring tip, as seen from behind the quad, the structural damper $\boldsymbol{C}$ matrix is

$$\boldsymbol{C}_{SD,L} = \begin{bmatrix} k(1+id) & 0 \\ n_3 k(1+id) & 0 \end{bmatrix} \tag{20}$$

where the first row corresponds to the vertical reaction force on the PUM and the second a roll reaction torque. The first column corresponds to the vertical displacement of the damper mass, the second to its velocity.

For the right spring tip, as seen from behind the quad, the structural damper $\boldsymbol{C}$ matrix is

$$\boldsymbol{C}_{SD,R} = \begin{bmatrix} k(1+id) & 0 \\ -n_3 k(1+id) & 0 \end{bmatrix} \tag{21}$$

For the viscous damping case, the $\boldsymbol{C}$ matrices are

$$\boldsymbol{C}_{VD,L} = \begin{bmatrix} k & c \\ n_3 k & n_3 c \end{bmatrix} \tag{22}$$

$$\boldsymbol{C}_{VD,R} = \begin{bmatrix} k & c \\ -n_3 k & -n_3 c \end{bmatrix} \tag{23}$$

For the left spring tip, as seen from behind the quad, the structural damper $\boldsymbol{D}$ matrix is

$$\boldsymbol{D}_{SD,L} = \begin{bmatrix} -k(1+id) & -n_3 k(1+id) & 0 & 0 \\ -n_3 k(1+id) & -n_3^2 k(1+id) & 0 & 0 \end{bmatrix} \tag{24}$$

where, like the $\boldsymbol{C}$ matrix, the first row corresponds to the vertical reaction force on the PUM and the second the roll reaction torque. Like the $\boldsymbol{B}$ matrix, the first two columns correspond to vertical and roll displacements respectively of the PUM, and the second to vertical and roll velocities.

For the right spring tip, as seen from behind the quad, the structural damper $\boldsymbol{D}$ matrix is

$$\boldsymbol{D}_{SD,R} = \begin{bmatrix} -k(1+id) & n_3 k(1+id) & 0 & 0 \\ n_3 k(1+id) & -n_3^2 k(1+id) & 0 & 0 \end{bmatrix} \tag{25}$$

For the viscous damping case, the $\boldsymbol{D}$ matrices are

$$\boldsymbol{D}_{VD,L} = \begin{bmatrix} -k & -n_3 k & -c & -n_3 c \\ -n_3 k & -n_3^2 k & -n_3 c & -n_3^2 c \end{bmatrix} \tag{26}$$

$$\boldsymbol{D}_{VD,R} = \begin{bmatrix} -k & n_3 k & -c & n_3 c \\ n_3 k & -n_3^2 k & n_3 c & -n_3^2 c \end{bmatrix} \tag{27}$$

It is easy to confuse the meaning of the $\boldsymbol{D}$ matrix, since it is not often used. Here it can be interpreted as the reaction forces you get on the quad when the spring tip the damper sits on displaces. The $\boldsymbol{C}$ matrix on the other hand is the reaction forces you get on the quad when the damper mass displaces. E.g. the simple spring equation says the spring force is $F = k(x - x_g)$. The $\boldsymbol{C}$ matrix represents the kx part. The $\boldsymbol{D}$ matrix represents $-kx_g$.

The models of the dampers are then plugged into the quad model using the `append` and `connect` commands. See the MATLAB code for the details.

3.5 Thermal Noise Modeling

The thermal noise calculations follow the fluctuation dissipation theorem given by Equation 7.12 at the bottom of page 110 in Peter Saulson's book *Fundamentals of Interferometric Gravitational Wave Detectors*. I rewrite here in Eq. (28).

$$\text{DARM thermal noise} = \frac{\text{coupling factor}}{\pi f}\sqrt{K_B T |\Re(Y(f))|} \tag{28}$$

where, f is the frequency in Hz, K_B is Boltzmann's constant, T is the temperature in Kelvin, $\Re$ is the real number operator, and $Y(f)$ is the complex valued transfer function from a force on the test mass to a velocity of the test mass.

For vertical motion, the transfer function Y is along the vertical degree of freedom. The coupling factor is assumed to be 0.001, which is the expected coupling from vertical motion to DARM. For roll motion, the transfer function Y is along the roll degree of freedom. The coupling factor is assumed to be 5e-5, which is just a guess chosen so that the roll thermal noise contributes about as much to DARM as vertical. I don't know what it actually is.

4 Results

This section considers dampers of different sizes and tuning errors. Subsection 4.1 considers bounce dampers with mass ratios of 10^{-5}. Subsection 4.2 considers bounce dampers with mass ratios of 2.5×10^{-4}.

As mentioned in the previous section, the roll coupling to DARM was guessed. I don't know what it is, so I simply chose a value that appears to contribute similar thermal noise to DARM as vertical. This value is 5×10^{-5}. I also assume the damper loss is frequency independent.

The vertical thermal noise curves for the undamped suspension are set to reproduce Figure 10 of 'Design and development of the advanced LIGO monolithic fused silica suspension' 2012 Class. Quantum Grav. 29 035003, A. Cumming et al. The resulting loss factor yields a Q of 4 million. The undamped loss factor for roll was guessed as the same value as vertical. The Qs were recently measured at LHO as 600,000 for bounce and 450,000 for roll.
Bounce Q: `https://alog.ligo-wa.caltech.edu/aLOG/index.php?callRep=18823`
Roll Q: `https://alog.ligo-wa.caltech.edu/aLOG/index.php?callRep=18848`
However, these are most likely dominated by losses higher up the chain. The Q of 4 million is assumed to represent the loss coming from just the fibers.

In all cases, the thermal noise curves shown are for an individual suspension. If all 4 test mass suspensions are given dampers you must multiply by 2.

4.1 Small Dampers - mass ratio of 10^{-5}

This subsection considers dampers with mass ratios of 10^{-5}. This yields bounce dampers of 0.2 grams on each spring and roll dampers of 0.09 grams on each spring. The structural damping loss factor of the damper stiffness is about $1/225 = 0.0045$.

Fig. 2 shows the thermal noise for the ideal case where the dampers are perfectly tuned. The Qs are close to the minimum possible value of about 460 for both bounce and roll. The solid lines are for the damped noise seen by DARM. The dashed line is the undamped noise. The green line shows the structural damping case, the magenta line viscous damping. The black line is the overall undamped suspension thermal noise predicted by Glasgow in T1500278. Note, the bounce mode in the Glasgow curve is slightly lower than what we built.

Fig. 3 shows the thermal noise for dampers that are detuned by 0.1%. This raises the Qs to about 1060 but does little to the thermal noise.

Fig. 4 shows the thermal noise for dampers that are detuned by 1%. This raises the Qs to about 18,000 but does little to the thermal noise. If anything, the thermal noise is somewhat worse because the peak is staring to spread out.

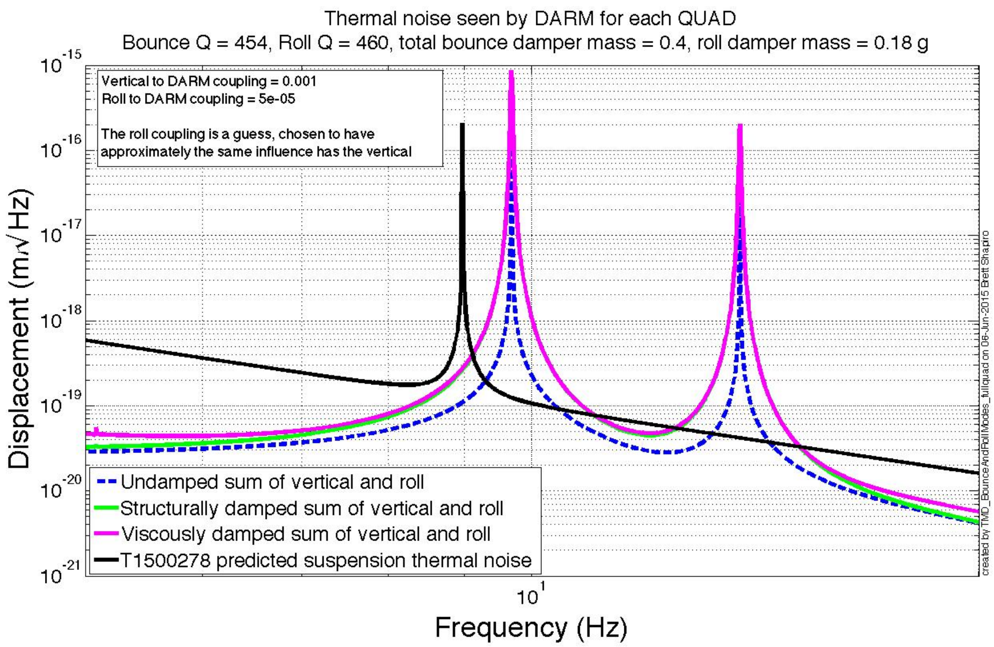

Figure 2: Thermal noise seen in DARM for each quad with 10^{-5} mass ratios. The dampers are perfectly tuned resulting in bounce and roll Qs of about 460. The loss factor of the damping material is $1/225 = 0.0045$.

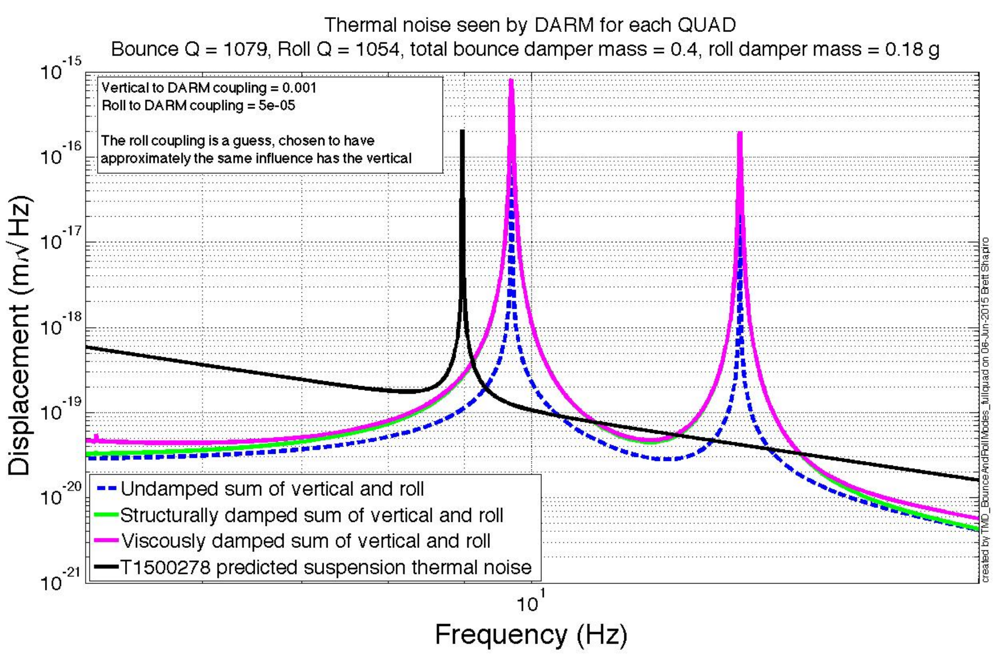

Figure 3: Thermal noise seen in DARM for each quad with 10^{-5} mass ratios. The dampers are detuned by +0.1% resulting in bounce and roll Qs of about 1060. The thermal noise is about the same as the perfectly tuned case.

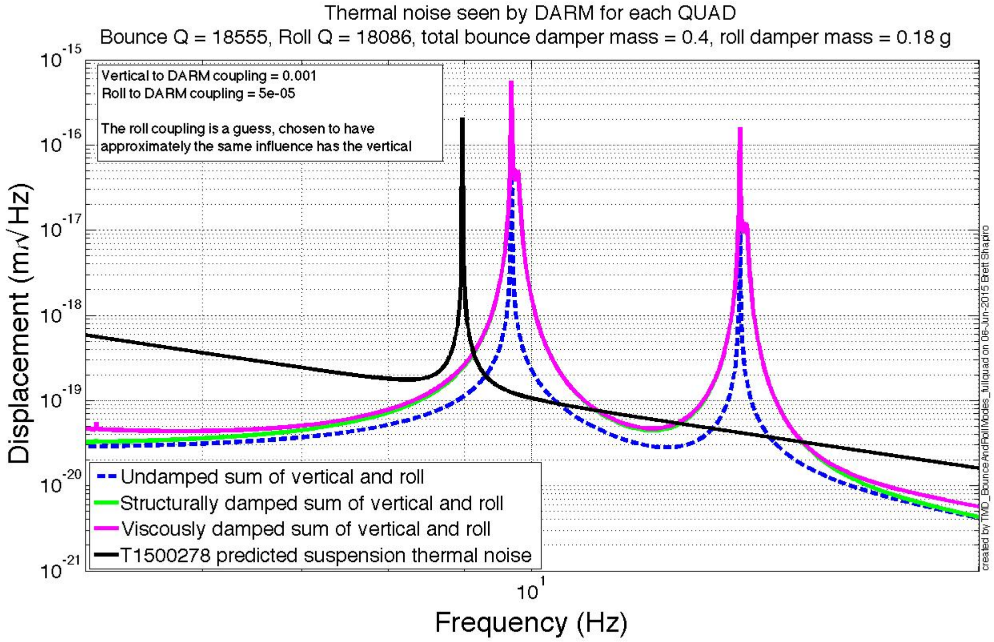

Figure 4: Thermal noise seen in DARM for each quad with 10^{-5} mass ratios. The dampers are detuned by +1% resulting in bounce and roll Qs of about 18,000. The thermal noise is about the same as the perfectly tuned case.

4.2 Large Dampers - mass ratio of 2.5×10^{-4}

This subsection considers dampers with mass ratios of 2.5×10^{-4}. This yields bounce dampers of about 5 grams on each spring and roll dampers of about 2.2 grams on each spring. The structural damping loss factor of the damper stiffness is about $1/45 = 0.022$.

Fig. 5 shows the thermal noise for the ideal case where the dampers are perfectly tuned. The Qs are close to the minimum possible value of about 100 for both bounce and roll. The increased mass yields lower Qs, but higher thermal noise than the smaller damper case.

Fig. 6 shows the thermal noise for dampers that are detuned by 0.1%. This hardly influences the Qs or the thermal noise. The higher mass has less sensitivity to error.

Fig. 7 shows the thermal noise for dampers that are detuned by 1%. This does influence the Qs, rasing them to about 350. This is still a pretty descent value. The thermal noise is not impacted much. Again, the higher mass has less sensitivity to error.

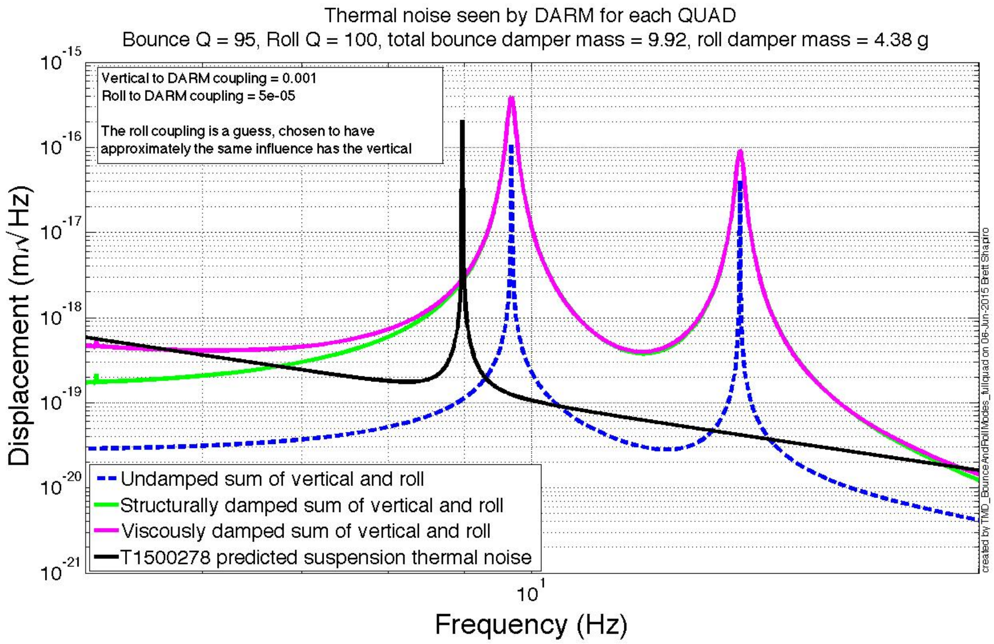

Figure 5: Thermal noise seen in DARM for each quad with 2.5×10^{-4} mass ratios. The dampers are perfectly tuned resulting in bounce and roll Qs of about 100. The loss factor of the damping material is $1/45 = 0.022$.

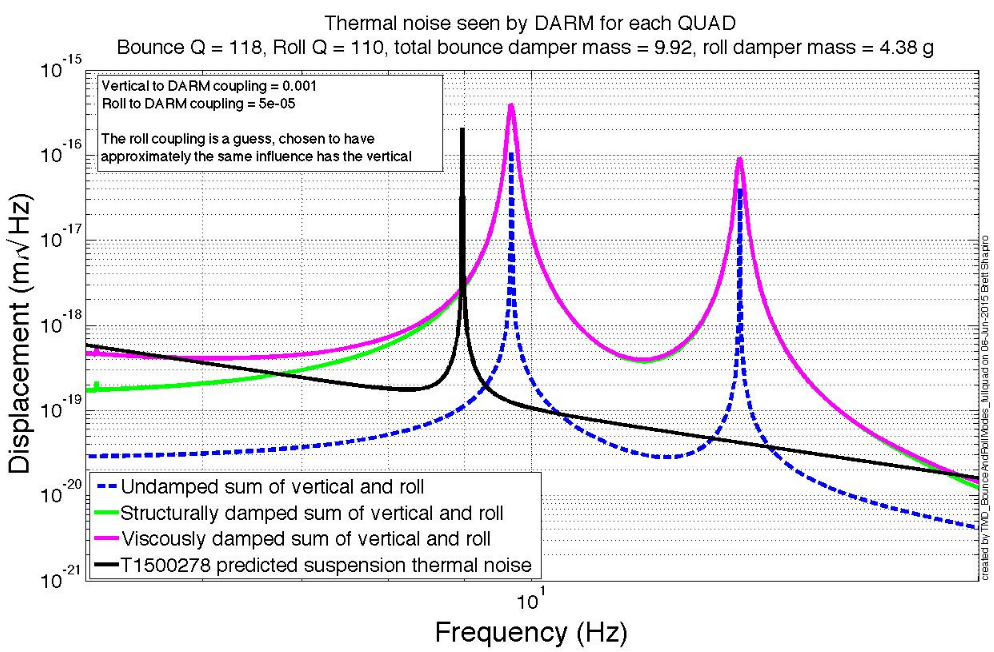

Figure 6: Thermal noise seen in DARM for each quad with 2.5×10^{-4} mass ratios. The dampers are detuned by $+0.1\%$ resulting in bounce and roll Qs of about 115. The thermal noise is about the same as the perfectly tuned case.

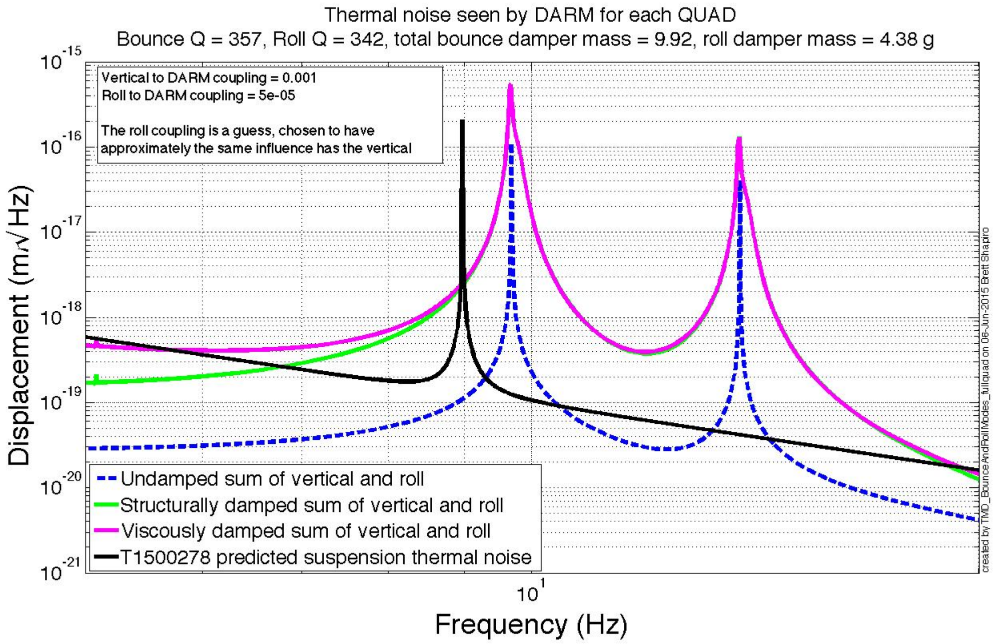

Figure 7: Thermal noise seen in DARM for each quad with 2.5×10^{-4} mass ratios. The dampers are detuned by $+1\%$ resulting in bounce and roll Qs of about 350. The thermal noise is about the same as the perfectly tuned case.

4.3 Trend Plots

Some trend plots are included here. Fig. 8 shows the curve for the ideal structural damping factor d given by Eq. (12). The factor results in the smallest possible Q of the bounce or roll mode. A correction of 1.15 has been applied to this equation. As mentioned before, the equation is an approximation that improves for larger mass ratios.

Fig. 9 is similar to Fig. 8 except that it shows viscous damping (e.g. eddy current). The solid and dashed blue lines show the damping coefficient c from Eq. (11) for the vertical and roll dampers respectively. Unlike the structural case, the vertical and roll dampers have different damping coefficients. However, this difference appears only because of the units on c. If we normalize it to a damping ratio (where 1 is critical damping) by dividing by $2m\Omega$ we see that the damping ratios are the same, and indeed correspond to the damping ratio of the structural case ($0.5d$).

Fig. 10 shows the curves for the minimum bounce and roll Qs as a function of mass ratio. The Qs for the 0.1% and 1% tuning errors are also plotted. A fit to the perfectly tuned case is included. This fit is $Q \approx 1.5/\sqrt{\mu}$.

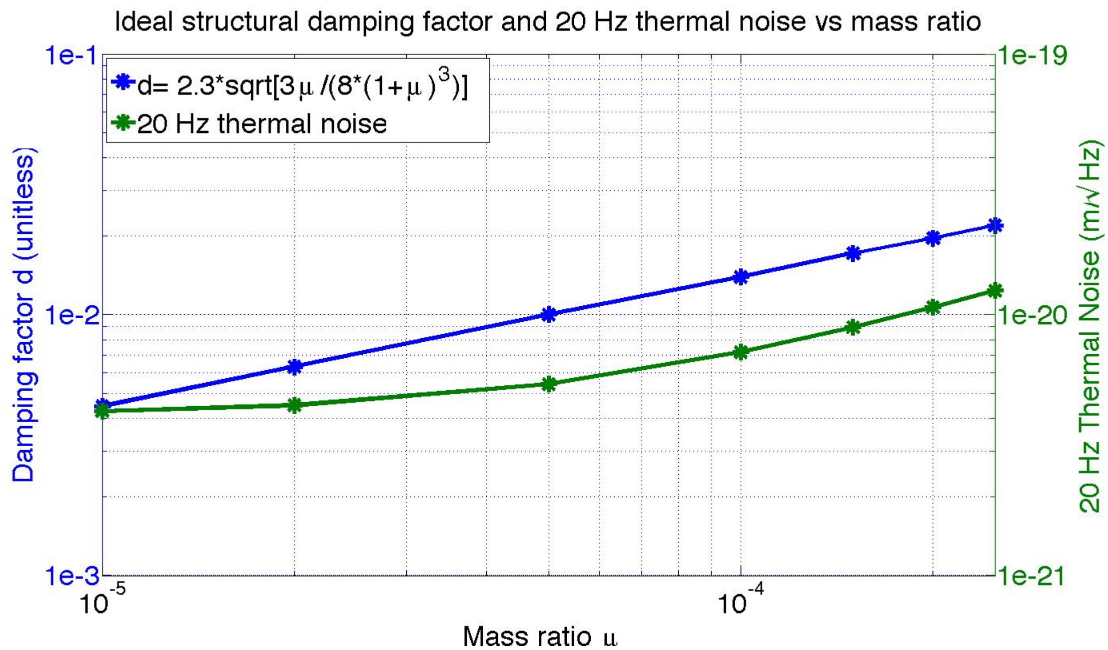

Figure 8: Blue: the ideal structural damping factor d that yields the smallest bounce and roll Qs, where the damper stiffness is given by $k(1 + id)$. Green: the sum of the vertical and roll 20 Hz thermal noise with the ideal damping factor d.

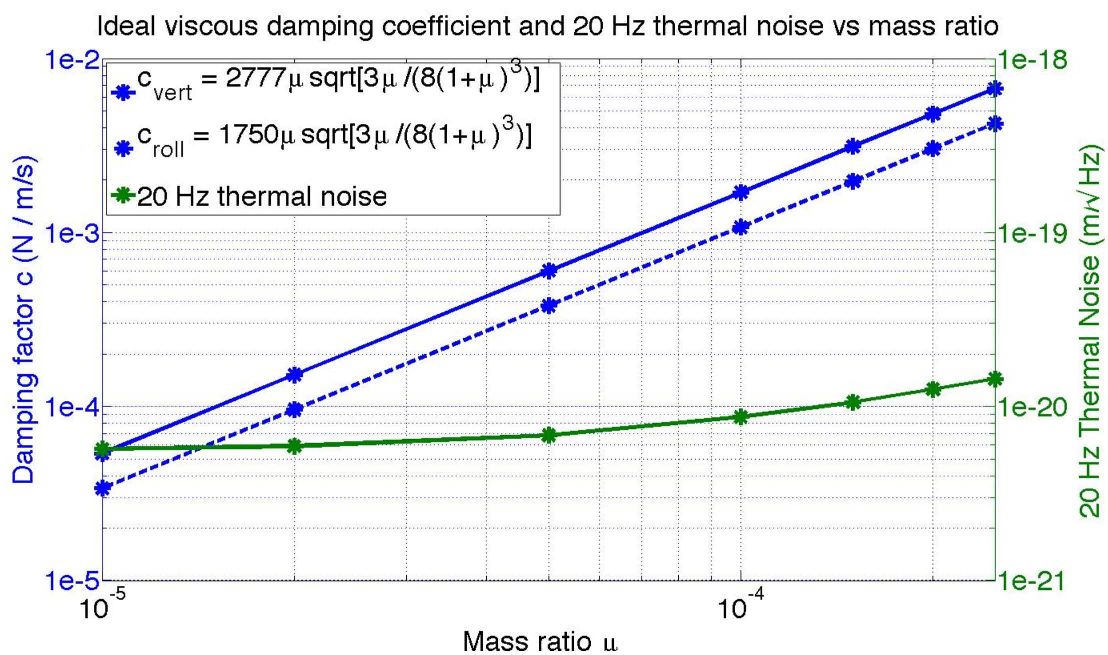

Figure 9: Solid blue: the ideal viscous damping coefficient c that yields the smallest bounce Q. Dashed blue: the ideal viscous damping coefficient c that yields the smallest roll Q. Green: the sum of the vertical and roll 20 Hz thermal noise with the ideal damping factor c.

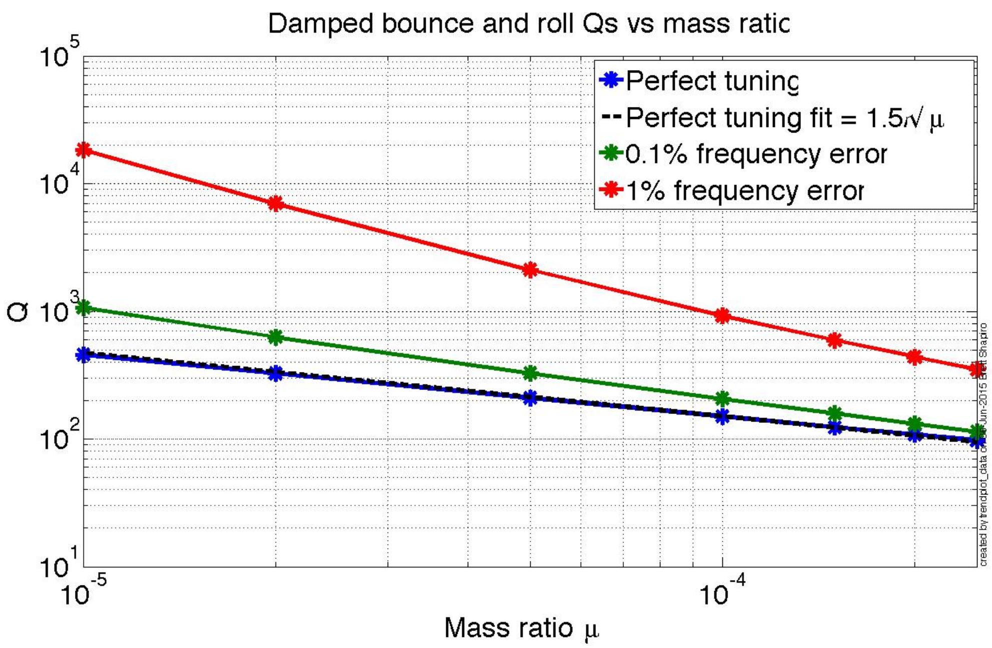

Figure 10: The smallest possible Qs of the bounce and roll modes. The blue line is the Q achieved with a perfectly tuned damper. The green and red lines show the Qs for dampers detuned by 0.1% and 1% respectively in frequency. The dotted black line is a fit to the perfectly tuned case.

5 Discussion

In general, larger dampers provide both greater damping and greater thermal noise. However, the damped Q is less sensitive to tuning errors for more massive dampers. It seems that the sensitivity of the damped Q is inversely related to the damper mass. That is, doubling the mass permits twice the percent error on the damper resonant frequency for the same damped Q.

While not shown here, I tried playing around with the mass and damping values to see if we could use larger dampers with less damping to improve tuning sensitivity for a given Q, without increasing thermal noise much. The result was that you can indeed make larger dampers that give the same Q as a smaller damper with less tuning sensitivity. However, the thermal noise is still worse than using the optimal smaller damper.

It is worth noting that the bounce damper contributes to the roll thermal noise, and vice versa. However, each damper dominates the thermal noise for its respective mode. If the roll coupling to DARM is smaller than what is guessed here, it is possible the vertical damper will dominate the roll thermal noise we observe.

We will need to think about what size damper we want to go with. The smaller one seems best in theory because a Q of 460 is already a huge improvement. Although the thermal noise is increased, it is not a huge amount. On the other hand, the tuning has to be better than 0.1% to keep the Q less than 1000. The suspensions at LHO have a spread in bounce frequency of 0.8%, so we would likely need to tune each damper to each suspension. The larger dampers will be easier to build and will be insensitive enough to tuning that we could put any damper on any suspension.

For the design approach, Figs. 8 and 10 should be very helpful. A good place to start is to decide what range of Qs is acceptable, and what kind of tuning tolerance seems reasonable. Find the mass ratio that gives those Qs in Fig. 10. Then, go up to Fig. 8 and use that mass ratio to tell you what damping factor you need and see if the thermal noise is acceptable. If those numbers do not work, then iterate.

For example, if we want Qs within 1000, and we think we can build to a tolerance of 0.1%, Fig 10 tells us we want a mass ratio of 1e-5. Fig. 8 then tells us to use a damping factor of about 0.0045. The 20 Hz thermal noise with will then be about 4e-21 m/$\sqrt{\mathrm{Hz}}$ at 20 Hz.

A Appendix - Class Notes

See the attached class notes for the tuned mass damper derivations.

The vibration absorber

Let K, M denote the stiffness and mass of a structure, modeled as a single dof system, and let k, m be the stiffness and mass of a vibration absorber or tuned-mass damper (TMD). The TMD is used to ameliorate the intensity of motion in the structure when the latter is subjected to dynamic forces F. We define

$\Omega = \sqrt{\frac{K}{M}}$ Frequency of structure alone (without the TMD)

$\omega_o = \sqrt{\frac{k}{m}}$ Frequency of TMD alone (without the structure)

$\omega_r = \sqrt{\frac{K}{m+M}}$ Frequency of structure with infinitely rigid oscillator

ω_1, ω_2 Frequencies of coupled system

$\mu = \frac{m}{M}$ Mass ratio

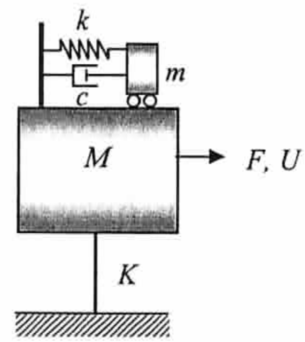

Fig. A1

The stiffness and mass matrices of the coupled system as well as the displacement and load vectors are

$$\mathbf{K} = \begin{Bmatrix} k & -k \\ -k & k+K \end{Bmatrix} \qquad \mathbf{M} = \begin{Bmatrix} m & \\ & M \end{Bmatrix} \qquad \mathbf{u} = \begin{Bmatrix} u \\ U \end{Bmatrix} \qquad \mathbf{p} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Solving the eigenvalue problem for this 2-DOF system, we obtain the coupled frequencies

$$\left(\frac{\omega_j}{\Omega}\right)^2 = \frac{1}{2}\left\{(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2 + 1 \mp \sqrt{\left[(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2 - 1\right]^2 + 4\mu\left(\frac{\omega_o}{\Omega}\right)^2}\right\} \qquad j=1,2$$

Next, we evaluate the dynamic response in the structure elicited by a harmonic force with amplitude F acting on the structural mass M. Neglecting damping in the structure (but not in the oscillator), the dynamic equilibrium equation is then

$$\begin{Bmatrix} k + \mathrm{i}\omega c - \omega^2 m & -(k+\mathrm{i}\omega c) \\ -(k+\mathrm{i}\omega c) & K + k + \mathrm{i}\omega c - \omega^2 M \end{Bmatrix} \begin{Bmatrix} u \\ U \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

Solving for the response in the structure U, we obtain after brief algebra

$$U = \frac{F\left[k - \omega^2 m + \mathrm{i}\omega c\right]}{\left[K - \omega^2(m+M)\right]\left[k - \frac{\left(K-\omega^2 M\right)\omega^2 m}{K - \omega^2(m+M)} + \mathrm{i}\omega c\right]} \qquad (1) \qquad \text{Eq. A1}$$

We shall show now that there exist two frequencies for which the response is independent of the damping constant c. Hence, all amplification functions have these points in common, no matter what the damping. These two points occur when the complex terms in the numerator and denominator cancel identically. This is satisfied if, and only if, the two real parts are equal, that is, if

$$\pm\left[k-\omega^2 m\right]=k-\frac{\left(K-\omega^2 M\right)\omega^2 m}{K-\omega^2(m+M)} \qquad (2) \qquad \text{Eq. A2}$$

The plus/minus sign on the left-hand side is to allow for frequencies greater than that of the tuned mass damper (a condition that would make the left-hand side term negative). If we consider first the positive sign, we find that it is satisfied only if $\omega=0$. This solution is not interesting, because it represents a static problem. On the other hand, if we consider the negative sign, we obtain the biquadratic equation

$$m(m+2M)\omega^4-2\left[k(m+M)+Km\right]\omega^2+2Kk=0$$

whose solution is

$$\left(\frac{\omega_{P,Q}}{\Omega}\right)^2=\frac{1}{2+\mu}\left\{(1+\mu)\left(\frac{\omega_o}{\Omega}\right)^2+1\mp\sqrt{\left[\left(\frac{\omega_o}{\Omega}\right)^2-1\right]^2+\mu(2+\mu)\left(\frac{\omega_o}{\Omega}\right)^2}\right\} \qquad (3) \qquad \text{Eq. A3}$$

The two frequencies given by this equation represent two points P, Q through which all transfer functions must pass, no matter what their damping should be. In particular, these two points must

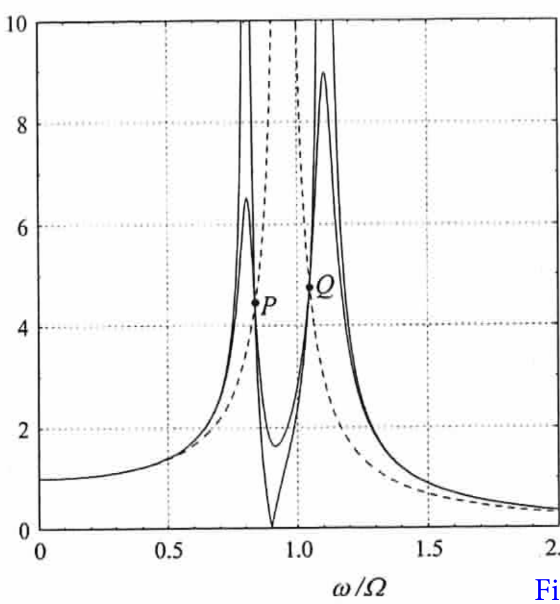

Fig. A2

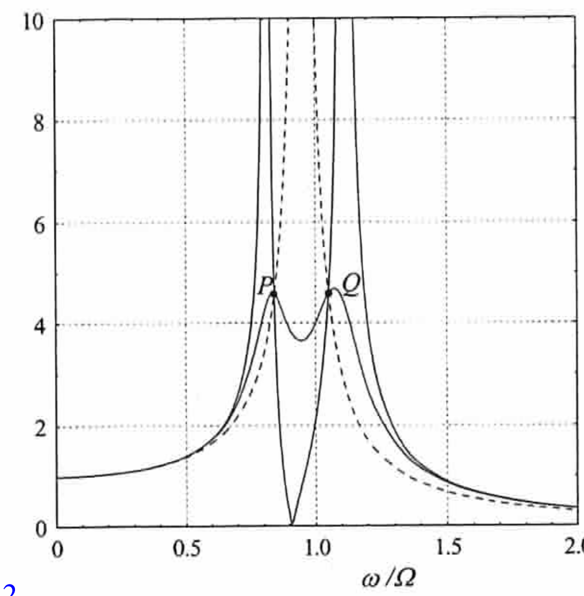

Fig. A3

also be traversed by the two transfer functions that correspond to zero damping and to infinite damping, i.e. $c=0$ and $c=\infty$. In the latter case, the system behaves the same as if the TMD was perfectly rigid. This in turn represents an undamped SDOF system with stiffness K, total mass $m+M$, and frequency ω_r as defined earlier. This frequency, and that of the oscillator alone (ω_o), are bracketed by the two natural frequencies of the coupled system, as can be shown by considering Rayleigh's quotient with an arbitrary vector $\mathbf{v}^T=\{a \quad b\}$. From the enclosure theorem, we have

$$\omega_1^2 \le R = \frac{\mathbf{v}^T\mathbf{K}\,\mathbf{v}}{\mathbf{v}^T\mathbf{M}\,\mathbf{v}} = \frac{(a-b)^2 k + b^2 K}{a^2 m + b^2 M} \le \omega_2^2 \qquad \text{Eq. A4}$$

in which a and b can be chosen arbitrarily; if we consider in turn the two choices a=b as well as b=0, we obtain the two inequalities

$$\omega_1^2 \le \frac{K}{m+M} \le \omega_2^2 \qquad \text{and} \qquad \omega_1^2 \le \frac{k}{m} \le \omega_2^2 \qquad \text{Eq. A5}$$

that is, $\omega_1 \le \omega_r \le \omega_2$ and $\omega_1 \le \omega_o \le \omega_2$. It follows that the two points P, Q lie somewhere in between the two natural frequencies at the intersection of the amplification function of the undamped coupled system and the undamped SDOF system with augmented mass m+M. The response amplitudes at these two frequencies are obtained by substituting the left-hand side of eq. 2 (negative sign case) into the denominator of eq. 1, and setting the dashpot constant to zero. The result is

$$U_{P,Q} = -\frac{F}{K - \omega_{P,Q}^2 (m+M)} \qquad \text{Eq. A6}$$

In general, the response amplitudes at the two frequencies for P, Q will not be equal. Optimal tuning of the mass damper can be achieved by enforcing these two amplitudes to be the same:

$$K - \omega_P^2 (m+M) = \pm\left[K - \omega_Q^2 (m+M) \right] \qquad \text{Eq. A7}$$

The case where both amplitudes are equal and have the same sign cannot be satisfied, since it implies equal frequencies for P and Q. Alternatively, if we consider equal amplitudes and opposite phase, we obtain

$$\omega_P^2 + \omega_Q^2 = \frac{2K}{m+M} = \frac{2\Omega^2}{1+\mu} \qquad \text{Eq. A8}$$

Equating this to the sum of the two roots in eq. 3, we obtain

$$\frac{\omega_P^2 + \omega_Q^2}{2\Omega^2} = \frac{1}{1+\mu} = \frac{(1+\mu)\left(\frac{\omega_O}{\Omega}\right)^2 + 1}{2+\mu} \qquad \text{Eq. A9}$$

From here, we obtain the optimal tuning condition

$$\boxed{\frac{\omega_o}{\Omega} = \frac{1}{1+\mu} = \sqrt{\frac{k\,M}{K\,m}}} \qquad \text{Eq. A10}$$

which relates the optimal frequency of the oscillator to the design mass ratio. This ratio ensures that the two points P, Q have the same height. The coupled frequencies observed with optimal tuning are

$$\boxed{\frac{\omega_j}{\Omega}=\sqrt{\frac{1+\frac{1}{2}\mu\mp\sqrt{\mu+\frac{1}{4}\mu^2}}{1+\mu}}}$$ Eq. A11

The optimal damping constant that should be assigned to an optimally tuned mass damper is the one that would cause the transfer function at the two points *P*, *Q* to have a horizontal slope. However, the analysis for this condition is rather cumbersome, and exact expressions are not available. A reasonably close approximation is given by the expression[9]

$$\boxed{\frac{c}{2m\Omega}\equiv\frac{\xi\,\omega_o}{\Omega}=\sqrt{\frac{3\mu}{8(1+\mu)^3}}}$$ Eq. A12

The transfer function for an optimally tuned mass damper is shown on the figure on the right. The maximum amplification for this case is $A_{\max}=\sqrt{1+2/\mu}$.

Lanchester damping

A Lanchester tuned mass damper is one in which the stiffness of the damper is zero (or nearly zero). The optimal parameters for this case are

Eq. A13 $$A_{\max}=1+2/\mu$$

Eq. A14 $$\xi=\frac{c}{2m\Omega}=\sqrt{\frac{1}{2(2+\mu)(1+\mu)}}$$

Eq. A15 $$\frac{\omega_Q}{\Omega}=\sqrt{1-\frac{\mu}{2(1+\mu)}}\approx\sqrt{\frac{1}{1+\frac{1}{2}\mu}}$$

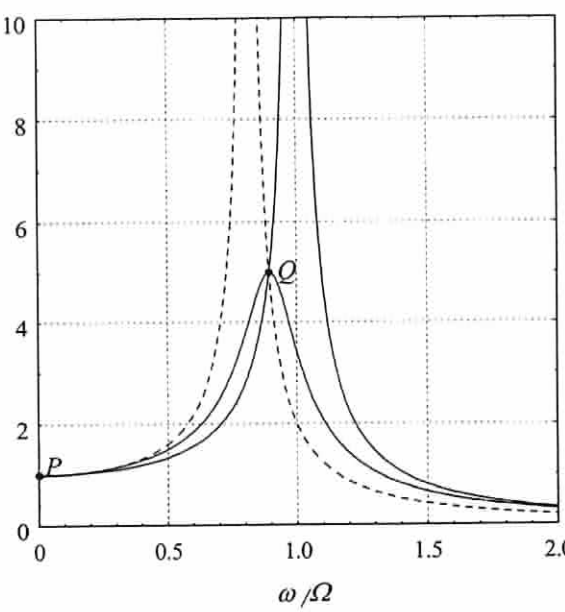

Fig. A4

Torsional Tuned Mass Damper

The torsional mass damper is a pendulum damper that automatically adjusts its resonant frequency to the rotational speed of wheels and shafts. It is used to ameliorate vibrations in these systems and dissipates energy through friction instead of viscous dashpots.

Consider a wheel to which two simple penduli of length *L* are attached. The penduli pivot about diametrically opposite points that are distant *a* from the axis, and are maintained in place by springs. As the wheel turns with rotational speed ω, the pivoting points experience a centripetal

[9] Den Hartog, J.P. : Mechanical Vibration (4th edition), McGraw-Hill, New York, 1956

acceleration that elicits a fictitious centrifugal gravity field $g' = \omega^2 (a+L) \gg g$, which in turn imparts on the penduli a resonant frequency

$$\omega_o = \sqrt{\frac{g'}{L}} = \omega \sqrt{1+\frac{a}{L}} \qquad \text{Eq. A16}$$

If $a \ll L$, then $\omega_o \approx \omega$, implying an oscillator that is tuned to the rotational speed.

As an example of application, consider a machine shaft whose flexural vibration mode is being excited by unavoidable eccentricities, and assume that the frequency of this mode is twice the operational speed of the shaft. To suppress this vibration, we must design a tuned mass damper that is tuned to that frequency, that is, having a natural frequency $\omega_o = 2\omega$. This can be accomplished by setting $4\omega^2 = \omega^2(1+a/L)$, which yields a=3L.

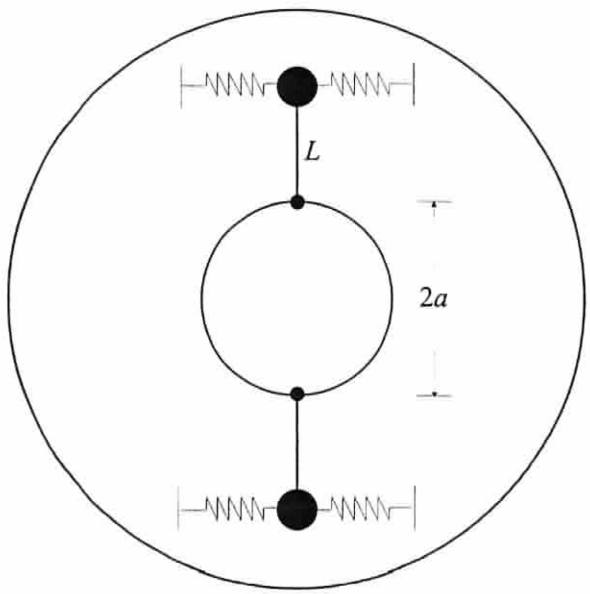

Fig. A5