# Dynamics and Electromagnetic Wave Signatures of Magnetized Neutron Stars near Black Holes

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We derive the dipole tensor for a magnetic dipole in a curved space-time, and expand $J^\mu$ in vector harmonics assuming a spherically symmetric spacetime. We then consider the specific case of a stationary, precessing magnetic dipole in a Schwarzschild background. For this case, we set up the equations that the electromagnetic fields satisfy, and derive expressions for them on the horizon. We solve this system numerically.

## I. INTRODUCTION

In this research project, we aim at studying the electromagnetic field of a precessing magnetic dipole near a black hole. This is meant to model a neutron star (which is typically between 1 and 3 $M_\odot$) falling into a much more massive black hole (for example with mass around or greater than $10\,M_\odot$, we also assume the neutron star will not be tidally disturbed). We would like to understand the distribution of the magnetic field, as well as the induced electric field — due to the orbital motion and precession of the neutron star, and (if time permits) due to the spin of the black hole. Our project is not only of interest for theoretical reasons, but also for observational reasons. In presence of a plasma around the black hole, as proposed by McWilliams and Levin, the induced electric field mentioned above may drive a current through the plasma, the neutron star, and across the event horizon of the black hole — and cause emission of transient electromagnetic waves that may be visible from the earth during the merger process [**?** **?** ].

From considerations based on energetic and time scale estimates, black-hole–neutron-star coalescence is believed to be a good candidate for short gamma-ray bursts (duration about less than 2 s) — the most luminous events in the universe. Moreover, Advanced LIGO will become operative next year, and will be searching for gravitational waves from neutron-star–black hole binary mergers [**?** ]. If EM counterparts of gravitational waves from neutron-star–black-hole binaries are found and characterized, one will be able to make the connection between the mergers and the bursts — yet the mechanism of EM emission will still have to be clarified. There is also the possibility that the EM counterparts will be detectable but not in the form of short gamma-ray bursts.

This interesting possibility of an electromagnetic counterpart for Advanced LIGO's detections of such mergers motivates our more careful study of the electromagnetic field of a magnetic dipole near a black hole. The problem can be approached hierarchically: because gravitational interaction in this system dominates, we ignore the back reaction of the EM field onto the orbital motion. We can simply treat the neutron star as moving along its own trajectory, as governed by pure gravitational physics, and sources an EM field that propagates on the space-time of the black hole.

The organization of this paper is as follows: in section 2 we describe the set-up and set-up all of the equations. In section 3 we solve Maxwell's equations. In section 4 we show our results, and we discuss our results and conclude in section 5.

## II. MAXWELL EQUATIONS ON SCHWARZSCHILD GEOMETRY

The background that we will be considering is a stationary, spherically symmetric background. The metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \text{(II.1)}$$

where in the case of Schwarzschild,

$$f(r) = 1 - \frac{2M}{r} \quad \text{(II.2)}$$

Throughout we work in units $G = c = 1$.

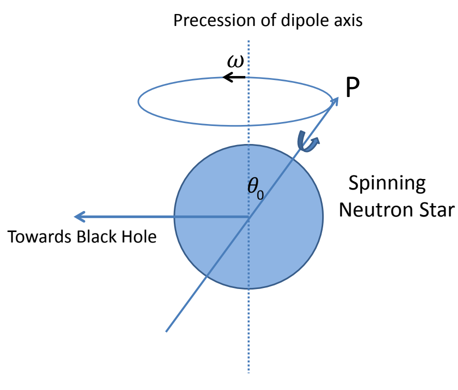



FIG. 1: Diagram of set-up. Stationary, neutron star precesses with angular frequency of $\omega$ at an angle $\theta_0$ with respect to the $z$ axis, and the dipole moment is $p$.

### A. Maxwell's equations

We would like to solve the Maxwell's equations

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu \quad \text{(II.3)}$$
$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \quad \text{(II.4)}$$

on the Schwarzschild spacetime. Here $F_{\mu\nu}$ is the Maxwell tensor, and the second line of the Maxwell equation (II.4) allows us to write

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu\,, \quad \text{(II.5)}$$

with $A_\mu$ the vector potential.

Spherical symmetry of the Schwarzschild background makes it preferable to expand the four-current $J^\mu$ and the vector potential $A_\mu$ in scalar and vector harmonics [**?** ]:

$$4\pi J_t = \sum_{\ell,m} \Psi^{\ell m}(t,r) Y^{lm}\,, \quad \text{(II.6)}$$

$$4\pi J_r = \sum_{\ell,m} \eta^{\ell m}(t,r) Y^{lm}\,, \quad \text{(II.7)}$$

$$4\pi \begin{pmatrix} J_\theta \\ J_\phi \end{pmatrix} = \sum_{\ell,m} \chi^{lm} \begin{pmatrix} Y^{\ell m}_{,\theta} \\ Y^{\ell m}_{,\phi} \end{pmatrix} + \sum_{\ell,m} \alpha^{\ell m} \begin{pmatrix} Y^{\ell m}_{,\phi}/\sin\theta \\ -\sin\theta Y^{\ell m}_{,\theta} \end{pmatrix} \quad \text{(II.8)}$$

and

$$A_t = \sum_{\ell,m} n^{\ell m}(t,r) Y^{lm}\,, \quad \text{(II.9)}$$

$$A_r = \sum_{\ell,m} h^{\ell m}(t,r) Y^{lm}\,, \quad \text{(II.10)}$$

$$\begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = \sum_{\ell,m} k^{lm} \begin{pmatrix} Y^{\ell m}_{,\theta} \\ Y^{\ell m}_{,\phi} \end{pmatrix} + \sum_{\ell,m} a^{\ell m} \begin{pmatrix} Y^{\ell m}_{,\phi}/\sin\theta \\ -\sin\theta Y^{\ell m}_{,\theta} \end{pmatrix} \quad \text{(II.11)}$$

In both expressions, the first sum is the terms of $(-)^\ell$ parity, which we will call odd, and the second sum is the terms of $(-)^{\ell+1}$ parity, which we will call even. From (II.3), we find that for each $\ell, m$, we have the following pair of differential equations for the radial functions

$$f(fa')' - \ddot{a} + f\frac{\ell(\ell+1)}{r^2} a = -f\alpha \quad \text{(II.12a)}$$

$$f(fb')' - \ddot{b} + f\frac{\ell(\ell+1)}{r^2} b = \frac{f}{\ell(\ell+1)}[(r^2\Psi)' - r^2\dot{\eta}], \quad \text{(II.12b)}$$

where $b$ is defined to be

$$b \equiv \frac{r^2}{\ell(\ell+1)}(\dot{h} - n'). \quad \text{(II.13)}$$

Here for simplicity we have dropped the $\ell m$ dependence of fields. If we define the tortoise coodinate $r_*$ as

$$r_* = r + 2M \log\left[\frac{r}{2M} - 1\right] \quad \text{(II.14)}$$

with

$$\frac{dr_*}{dr} = \frac{1}{f(r)}, \quad \text{(II.15)}$$

then the wave equation for $a$ and $b$ can be written as

$$\frac{\partial^2 u}{\partial r_*^2} - \frac{\partial^2 u}{\partial t^2} + \left(1 - \frac{2M}{r}\right)\frac{\ell(\ell+1)}{r^2} u = 0\,. \quad \text{(II.16)}$$

The charge conservation equation

$$\nabla_\mu J^\mu = 0 \quad \text{(II.17)}$$

is satisfied by the constraint

$$\frac{1}{r^2}(r^2 f\eta)' - \frac{\dot{\Psi}}{f} = \frac{\ell(\ell+1)}{r^2}\chi. \quad \text{(II.18)}$$

That is why $\chi$ does not appear in (II.12).

In terms of $a$ and $b$, the odd parity components of $F^{\mu\nu}$ are given by

$$(F^{t\theta}, F^{t\phi}) = -\sum_{\ell m} \frac{\dot{a}_{\ell m}}{r^2 f \sin\theta}(Y^{\ell m}_{,\phi}, -Y^{\ell m}_{,\theta}) \quad \text{(II.19)}$$

$$(F^{r\theta}, F^{r\phi}) = +\sum_{\ell m} \frac{f a'_{\ell m}}{r^2 \sin\theta}(Y^{\ell m}_{,\phi}, -Y^{\ell m}_{,\theta}) \quad \text{(II.20)}$$

and the even parity components by

$$(F^{t\theta}, F^{t\phi}) = -\sum_{\ell m} \frac{b'_{\ell m}}{r^2}(Y^{\ell m}_{,\theta}, Y^{\ell m}_{,\phi}/\sin^2\theta) \quad \text{(II.21)}$$

$$(F^{r\theta}, F^{r\phi}) = +\sum_{\ell m} \frac{\dot{b}_{\ell m}}{r^2}(Y^{\ell m}_{,\theta}, Y^{\ell m}_{,\phi}/\sin^2\theta) \quad \text{(II.22)}$$

At this point in our discussion, we find it helpful to define a tetrad of tangent vectors:

$$\vec{e}_{\hat{t}} = \frac{\vec{\partial}_t}{\sqrt{f}}, \quad \vec{e}_{\hat{r}} = \sqrt{f}\vec{\partial}_r, \quad \vec{e}_{\hat{\theta}} = \frac{\vec{\partial}_\theta}{r}, \quad \vec{e}_{\hat{\phi}} = \frac{\vec{\partial}_\phi}{r\sin\theta}\,. \quad \text{(II.23)}$$

We can define a set of fiducial observers (FIDOs), who stay at fixed spatial locations in the Schwarzschild coordinate system, with four velocity field $\vec{e}_{\hat{t}}$. Let us assume that they each carry the above tetrad, then in the proper reference frame of these observers (which are accelerating), we measure the following electric and magnetic fields:

$$E^{\hat{r}} = F^{\hat{t}\hat{r}}\,, \quad E^{\hat{\theta}} = F^{\hat{t}\hat{\theta}}\,, \quad E^{\hat{\phi}} = F^{\hat{t}\hat{\phi}}\,, \tag{II.24}$$

$$B^{\hat{r}} = F^{\hat{\phi}\hat{\theta}}\,, \quad B^{\hat{\theta}} = F^{\hat{r}\hat{\phi}}\,, \quad B^{\hat{\phi}} = F^{\hat{\theta}\hat{r}}\,. \tag{II.25}$$

### B. Boundary Conditions

We need to impose boundary conditions for $a$ and $b$ at the spatial boundaries of our computational domain, which are located at $r_* \to \pm\infty$, where the source vanishes and the wave equation simplifies to

$$\frac{\partial^2 u}{\partial r_*^2} - \frac{\partial^2 u}{\partial t_*^2} = 0\,. \tag{II.26}$$

where $u$ stands for either $a$ or $b$. For $r_* \to +\infty$, it is natural to impose

$$u(t, r_*) \sim u(t - r_*)\,, \quad r_* \to +\infty\,, \tag{II.27}$$

which is the out-going boundary condition. On the other hand, for $r_* \to -\infty$, near the horizon, we should impose

$$u(t, r_*) \sim u(t + r_*)\,, \quad r_* \to -\infty\,. \tag{II.28}$$

This means wave must go down the horizon. This condition is required in order the freely falling observers experience finite field strengths.

### C. Poynting flux

The electromagnetic stress-energy tensor is given by

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right). \tag{II.29}$$

Since we have a static spacetime with a time-like Killing vector $\vec{\partial}_t$, we can define the energy flux vector by contracting the Killing vector and the stress-energy tensor, obtaining as $-T_t{}^{\mu}$. Here the minus sign is used to maintain the usual sign convention for energy.

Here we calculate the Poynting flux across a constant $r$ surface, which has a surface element 1-form of

$$d\Sigma_\mu = -\epsilon\left(\vec{\partial}_\mu, dt\vec{\partial}_t, d\theta\vec{\partial}_\theta, d\phi\vec{\partial}_\phi\right)\,, \tag{II.30}$$

with $\epsilon$ the Levi-Civita tensor; the only non-vanishing component of this 1-form is

$$d\Sigma_r = \sqrt{-g}dtd\theta d\phi = r^2\sin\theta dtd\theta d\phi\,. \tag{II.31}$$

The energy flow across a patch of the constant $r$ surface will then be given by

$$\mathcal{E} = -\int T_t{}^{\mu}d\Sigma_\mu = \int\left(-T_t{}^{r}\right)r^2\sin\theta d\theta d\phi dt\,. \tag{II.32}$$

Here we are interested in $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $t$ from $t_1$ to $t_2$. Note from Eq. (II.31) that the area element $d\Sigma_r$ has the same form as in Euclidean space — and this provides $-T_t{}^r$ usual physical interpretation of energy per unit area and per unit (universal) time. More specifically, we have

$$-T_t{}^r = \frac{r^2 f}{4\pi}\left(F^{t\theta}F^{r\theta} + F^{t\phi}F^{r\phi}\sin^2\theta\right) \tag{II.33}$$

Expanding in spherical harmonics, we can write

$$\begin{aligned}&\left[T_t{}^r(t,r,\theta,\phi)\right]_{\rm odd}\\ &= -\sum_{\substack{\ell_1,m_1\\ \ell_2,m_2}}\frac{f\dot{a}^*_{\ell_1 m_1}a'_{\ell_2 m_2}}{4\pi r^2}\\ &\qquad\left[\left(Y^{\ell_1 m_1}_{,\theta}\right)^* Y^{\ell_2 m_2}_{,\theta} + \frac{\left(Y^{\ell_1 m_1}_{,\phi}\right)^* Y^{\ell_2 m_2}_{,\phi}}{\sin^2\theta}\right].\end{aligned} \tag{II.34}$$

Here we have taken the complex conjugate of $F^{t\theta}$ and $F^{t\phi}$; since they are both real-valued, this does not change the value of the $T_t{}^r$, but simply makes the expression more compact.

Integrating over the angular directions, and noting the orthogonality between the scalar spherical harmonics, we obtain

$$P^{\rm odd}_{lm}(t,r) = -\frac{f\ell(\ell+1)\dot{a}^*_{\ell m}a'_{\ell m}}{4\pi}, \tag{II.35}$$

Note that near the horizon, we will have $a \sim a(t + r_*)$, which means $P < 0$, because energy flows toward increasing $r$; on the other hand, at $r \to +\infty$, we have $P > 0$. Similarly,

$$P^{\rm odd}_{\ell m}(t,r) = -\frac{f\ell(\ell+1)\dot{b}^*_{\ell m}b'_{\ell m}}{4\pi}. \tag{II.36}$$

We can then obtain the total fluxes,

$$P^{\rm tot}_{\ell m}(t,r) = -\frac{f\ell(\ell+1)\left[\dot{a}^*_{\ell m}a'_{\ell m} + \dot{b}^*_{\ell m}b'_{\ell m}\right]}{4\pi}, \tag{II.37}$$

and

$$P^{\rm tot}(t,r) = -\sum_{\ell m}\frac{f\ell(\ell+1)\left[\dot{a}^*_{\ell m}a'_{\ell m} + \dot{b}^*_{\ell m}b'_{\ell m}\right]}{4\pi}, \tag{II.38}$$

## III. CURRENT DENSITY OF A MOVING ELECTROMAGNETIC DIPOLE

We consider a neutron star near a black hole, and model its EM field as being produced by a magnetic dipole. We shall first discuss more broadly the EM field generated by a moving particle that has both a electric and a magnetic dipole, with the conclusion that the EM field of a moving magnetic dipole can be obtained from the dual of a moving electric dipole. We will then work out the spherical harmonic decomposition of the electric current density of a moving electric dipole — which will then be used in the next section to drive the Maxwell equation.

### A. Duality Relation

For antisymmetric rank-2 tensor $X^{\mu\nu}$, we define its dual, $\tilde{X}^{\alpha\beta}$, as

$$\tilde{X}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}X_{\alpha\beta}\,, \tag{III.1}$$

where $\epsilon$ is the Levi-Civita tensor. It is easy to check that

$$\tilde{\tilde{X}}^{\mu\nu} = -X^{\mu\nu}\,. \tag{III.2}$$

Using the dual of the Maxwell tensor, the Maxwell equations can be generalized to

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu_e \tag{III.3}$$

$$\nabla_\nu \tilde{F}^{\mu\nu} = 4\pi J^\mu_m \tag{III.4}$$

where $J^\mu_e$ is the electric charge current, while $J^\mu_m$ is the magnetic charge current. In absence of magnetic charges, $J^\mu_m$ vanishes.

If $F^{\mu\nu}$ solves the charge-free Maxwell's equation (i.e., $J^\mu = 0$), then its dual form, solves the charge-free Maxwell's equation, as well. Under this transformation, once we define the electric and magnetic fields in a local Lorentz frame, we have

$$\tilde{\mathbf{E}} = \mathbf{B}\,,\quad \tilde{\mathbf{B}} = -\mathbf{E} \tag{III.5}$$

where $\mathbf{E}$ and $\mathbf{B}$ are 3-vectors in the spatial slice. Moreover, in presence of charge, Maxwell's equation can be generalized to

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu_e \tag{III.6}$$

$$\nabla_\nu \tilde{F}^{\mu\nu} = 4\pi J^\mu_m \tag{III.7}$$

where $J^\mu_e$ is the electric charge current, while $J^\mu_m$ is the magnetic charge current. In absence of magnetic charges, $J^\mu_m$ vanishes. Using the fact that $-\tilde{\tilde{F}}^{\mu\nu} = F^{\mu\nu}$, we find that $\tilde{F}^{\mu\nu}$ also solves the generalized Maxwell equation, with

$$\tilde{J}^\mu_e = J^\mu_m\,,\quad \tilde{J}^\mu_m = -J^\mu_e\,. \tag{III.8}$$

### B. Electric Dipole in a General Space-time

Let us now show that the 4-current $J^\alpha$ for a point electric dipole source is given by (see [? ] for flat case)

$$J^\mu(x^\rho) = -\int \frac{Q^{\mu\alpha}(\tau)\partial_\alpha\delta^{(4)}[x^\rho - z^\rho(\tau)]}{\sqrt{-g(x^\rho)}}d\tau, \tag{III.9}$$

Here $z^\rho(\tau)$ is the trajectory of the dipole, with $\tau$ the proper time, and

$$V^\alpha = \frac{dz^\alpha(\tau)}{d\tau} \tag{III.10}$$

is the 4-velocity. The antisymmetric dipole tensor $Q^{\mu\nu}$ is defined to be

$$Q^{\mu\nu}(\tau) \equiv V^\mu p^\nu - p^\mu V^\nu \tag{III.11}$$

We present the argument below for an electric dipole. Suppose we have one object, then it is quite easy to argue that

$$J^\mu(x^\rho) = \int d\tau\, V^\mu(\tau)\frac{\delta^{(4)}\left[x^\rho - z^\rho(\tau)\right]}{\sqrt{-g(x^\rho)}} \tag{III.12}$$

Note here that we have taken the liberty to use $x^\rho$ in $g$ instead of $z^\rho(\tau)$. Let us also note that here the existence of $V^\alpha$ in the integrand means that $\tau$ does not need to be proper time. One can prove that

$$\nabla_\mu J^\mu = 0\,. \tag{III.13}$$

Now, a dipole is simply the difference of $J^\mu$ with two particles located at $z^\rho(\tau) \pm \epsilon p^\rho(\tau)$, divided by 2. We have

$$J^\mu_{\rm dipole} = \frac{1}{2\epsilon}\left[J^\mu_{+\epsilon} - J^\mu_{-\epsilon}\right] \tag{III.14}$$

with

$$J^\mu_\epsilon = \int d\tau\;\left[V^\alpha(\tau) + \epsilon\dot{p}^\alpha(\tau)\right]\frac{\delta^{(4)}\left[x^\rho - z^\rho(\tau) - \epsilon p^\rho(\tau)\right]}{\sqrt{-g(x^\rho)}} \tag{III.15}$$

Here we have the $\dot{p}^\alpha$ term because the particle has a slightly different 4-velocity. In fact, $V^\alpha + \epsilon\dot{p}^\alpha$ is not necessarily the 4-velocity, since $\tau$ may no longer be the proper time — which as we argued does not affect the validity of Eq. (III.15). In Taylor-expanding $J^\mu_\epsilon$ we obtain two terms, corresponding to the two terms in $Q^{\mu\alpha}$ in Eq. (III.9).

$$Q^{\mu\nu}(\tau) = V^\mu p^\nu - p^\mu V^\nu \tag{III.16}$$

where $V^\mu$ is instantaneous 4-velocity of the source, $p^\nu$ is the electric dipole moment such that $\vec{V}\cdot\vec{p} = 0$.

### C. Magnetic Dipole in a General Spacetime

In Minkowski space, it is well known that by taking a duality transformation, the EM field of a moving electric dipole **p** becomes the EM field of a magnetic dipole with **m** = −**p** moving along the same world line. We shall generalize this relation to an arbitrary spacetime. This is obviously true if we also allow the duality between electric and magnetic currents. However, in reality, the magnetic dipole must be generated by an electric current, and we will need to explicitly show that, the dual field of a electric dipole (away from the source) can in fact be generated by a purely electric current, as well.

Suppose we have an antisymmetric dipole tensor, $Q^{\mu\nu}(\tau)$, defined along the particle's world line, and suppose $F^{\mu\nu}[Q^{\alpha\beta}]$, is the unique EM field sourced by $Q^{\alpha\beta}$, given the no-in-going-wave boundary condition, it satisfies:

$$\nabla_\nu F^{\mu\nu}[Q^{\alpha\beta}] = -4\pi \int \frac{Q^{\mu\alpha}(\tau)\partial_\alpha \delta^{(4)}[x^\rho - z^\rho(\tau)]}{\sqrt{-g(x^\rho)}} d\tau, \tag{III.17}$$

$$\nabla_\nu \tilde{F}^{\mu\nu}[Q^{\alpha\beta}] = 0\,. \tag{III.18}$$

Let us try to find $F^{\mu\nu}[\tilde{Q}^{\alpha\beta}]$, where

$$\tilde{Q}^{\alpha\beta}(\tau) \equiv \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu}\,[z^\rho(\tau)]\, Q^{\mu\nu}(\tau)\,, \tag{III.19}$$

is the dual of $Q^{\alpha\beta}$ along the particle's world line. In order to do so, let us define

$$G^{\mu\nu}(x^\rho) = \tilde{F}^{\mu\nu}[Q^{\alpha\beta}](x^\rho) + \frac{4\pi}{\sqrt{-g(x^\rho)}} \int \tilde{Q}^{\mu\nu}(\tau)\delta^{(4)}[x^\rho - z^\rho(\tau)] d\tau\,. \tag{III.20}$$

which is equal to the dual of $F^{\mu\nu}$ away from the source. It is straightforward to show that

$$\begin{aligned}\nabla_\nu G^{\mu\nu}(x^\rho) &= \frac{1}{\sqrt{-g(x^\rho)}}\partial_\nu\left[\sqrt{-g(x^\rho)}G^{\mu\nu}\right] \\ &= 4\pi \int \frac{\tilde{Q}^{\mu\alpha}(\tau)\partial_\alpha\delta^{(4)}[x^\rho - z^\rho(\tau)]}{\sqrt{-g(x^\rho)}} d\tau,\end{aligned} \tag{III.21}$$

and that

$$\nabla_\nu \tilde{G}^{\mu\nu}(x^\rho) = 0\,, \tag{III.22}$$

therefore we can write

$$G^{\mu\nu} = F^{\mu\nu}[-\tilde{Q}^{\alpha\beta}]\,, \tag{III.23}$$

which means the electric current generated by $-\tilde{Q}^{\alpha\beta}$ produces a field that agrees with $\tilde{F}^{\alpha\beta}[Q^{\alpha\beta}]$ away from the source.

Returning to the issue of a moving magnetic dipole $\vec{\mathfrak{m}}$, withe $\vec{\mathfrak{m}} \cdot \vec{V} = 0$, and we need to find $F^{\mu\nu}[\vec{\mathfrak{m}}]$. According to relation between electric and magnetic dipole fields, which remains true for general spacetimes, we can write

$$\begin{aligned}&F^{\mu\nu}[\text{magnetic dipole} = \vec{\mathfrak{m}}] \\ =&-\tilde{F}^{\mu\nu}[\text{electric dipole} = \vec{\mathfrak{m}}] \\ =&\tilde{F}^{\mu\nu}[Q^{\alpha\beta} = -V^\alpha \mathfrak{m}^\beta + \mathfrak{m}^\alpha V^\beta] \\ =&\tilde{F}^{\mu\nu}[Q^{\alpha\beta} = \epsilon^{\alpha\beta}{}_{\mu\nu}V^\mu \mathfrak{m}^\nu]\,,.\end{aligned} \tag{III.24}$$

More generally, the current density of a particle with electric dipole moment vector $p^\mu$ and magnetic dipole moment vector $\mathfrak{m}^\mu$ (with $\vec{V}\cdot\vec{p} = \vec{V}\cdot\vec{\mathfrak{m}} = 0$) should be given by

$$Q^{\alpha\beta}(\tau) = V^\alpha p^\beta - p^\alpha V^\beta + \epsilon^{\alpha\beta}{}_{\mu\nu}V^\mu \mathfrak{m}^\nu \tag{III.25}$$

with all quantities on the right-hand side evaluated at $z^\rho(\tau)$.

## IV. PRECESSING, STATIONARY MAGNETIC DIPOLE IN SCHWARZSCHILD SPACETIME

In this section, we shall obtain the EM field for a precessing magnetic dipole located at a fixed spatial location in Schwarzschild coordinate system.

### A. Source Terms

the current density operator of a magnetic dipole that moves along $\vec{\partial}_t$, therefore staying at a constant spatial location in the Schwarzschild coordinate system, with $r = R$, $\theta = \pi/2$ and $\phi = 0$. The 4-velocity is given by

$$V^\mu = \left(1/\sqrt{f(R)}, 0, 0, 0\right) \tag{IV.1}$$

Let us assume that the dipole $\vec{\mathfrak{m}}$ has an inclination angle of $\iota$ away from the $\vec{e}_\theta$ direction, and precesses with a frequency $\omega$. This means the time-dependent part of the dipole moment is given by

$$\vec{\mathfrak{m}}(t) = \mathfrak{m}\left(\vec{e}_{\hat{r}}\cos\omega t + \vec{e}_{\hat{\phi}}\sin\omega t\right) \tag{IV.2}$$

where $\mathfrak{m}$ is the magnitude of the component of $\vec{\mathfrak{m}}$ in the equatorial plane. The static component simply causes a static field distribution that does not radiate. Note that $\omega$ here is frequency measured at infinity, which is related to the local frequency $\Omega$ via gravitational redshift:

$$\omega = \sqrt{f(R)}\Omega\,. \tag{IV.3}$$

In the complex notation, we have

$$\tilde{\mathfrak{m}}^\mu = \mathfrak{m}\left(0, \sqrt{f(R)}, 0, i/R\right) \tag{IV.4}$$

with

$$\mathfrak{m}^\mu(t) = \mathrm{Re}\left[\tilde{\mathfrak{m}}^\mu e^{-i\omega t}\right] \tag{IV.5}$$

Using Eq. (III.25) and Eq. (III.9), we obtain

$$\tilde{J}_r(r,\theta,\phi) = \frac{i\mathfrak{m}}{R^3 f(R)\sin\theta}\partial_\theta\delta^{(3)} \tag{IV.6}$$

$$\tilde{J}_\theta(r,\theta,\phi) = +\frac{\mathfrak{m}}{R^2\sqrt{f(R)}\sin\theta}\partial_\phi\delta^{(3)} - \frac{i\mathfrak{m}}{R}\partial_r\delta^{(3)} \tag{IV.7}$$

$$\tilde{J}_\phi(r,\theta,\phi) = -\frac{\mathfrak{m}\sin\theta}{R^2\sqrt{f(R)}}\partial_\theta\delta^{(3)} \tag{IV.8}$$

where we have defined

$$\delta^{(3)} \equiv \delta(r-R)\delta(\theta-\pi/2)\delta(\phi). \tag{IV.9}$$

We can now expand these into spherical harmonics

$$\begin{aligned}\tilde{\eta}^{\ell m}(r) &= \int 4\pi\tilde{J}_r(r,\theta,\phi)Y^*_{\ell m}(\theta,\phi)\sin\theta d\theta d\phi \\ &= -\frac{4\pi i\mathfrak{m}}{R^3 f(R)}\partial_\theta Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\end{aligned} \tag{IV.10}$$

this leads to

$$\begin{aligned}&f(fb')' + \omega^2 b + f\frac{\ell(\ell+1)}{r^2}b \\ &= \frac{4\pi\mathfrak{m}\omega}{\ell(\ell+1)R}\delta(r-R)\partial_\theta Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\end{aligned} \tag{IV.11}$$

Using orthogonality condition of vector harmonics, we have

$$\begin{aligned}\tilde{\alpha}_{\ell m} &= \frac{4\pi}{\ell(\ell+1)}\int d\theta d\phi\left[\tilde{J}_\theta\left(Y^{\ell m}_{,\phi}\right)^* - \tilde{J}_\phi\left(Y^{\ell m}_{,\theta}\right)^*\right] \\ &= \frac{4\pi\mathfrak{m}}{R}\left[\frac{\delta(r-R)}{R\sqrt{f(R)}} - \frac{m\delta'(r-R)}{\ell(\ell+1)}\right]Y^*_{\ell m}\left(\frac{\pi}{2},0\right).\end{aligned} \tag{IV.12}$$

This leads to

$$\begin{aligned}&f(fa')' + \omega^2 a + f\frac{\ell(\ell+1)}{r^2}a \\ =&4\pi\mathfrak{m}\left[\frac{mf(r)\delta'(r-R)}{R\ell(\ell+1)} - \frac{\sqrt{f(R)}}{R^2}\delta(r-R)\right]Y^*_{\ell m}\left(\frac{\pi}{2},0\right)\end{aligned} \tag{IV.13}$$

### B. Junction conditions for $a$ and $b$

In both even and odd parity cases, we have an equation of the following form:

$$\left[\frac{\partial^2}{\partial r_*^2} + \omega^2 + f\frac{\ell(\ell+1)}{r^2}\right]u = S_1\frac{\partial\delta(r-R)}{\partial r_*} + S_0\delta(r-R) \tag{IV.14}$$

where $u$ stands for either $a$ or $b$, while $S_0$ and $S_1$ are functions of $R$ but not $r$. Note that the source term vanishes except near $r = R$ — therefore the field $u$ would be a freely propagating wave at $r > R$ and $r < R$ respectively. Noting that for $r > R$, we can use boundary condition at $r_* \to +\infty$, while for $r < R$, we can use boundary condition for $r_* \to -\infty$, we will only need to obtain the junction conditions for $u$ and $\partial u/\partial r_*$ at $r = R$.

In order to do so, let us imagine $\delta(x)$ as a function with a finite but very narrow width around $x = 0$, and take an $\epsilon$ which is greater than that width. We can integrate Eq. (IV.14) with $\int_{R_*-\epsilon}^{y} dr_*$, and obtain

$$\left.\frac{\partial u}{\partial r_*}\right|_{R_*-\epsilon}^{y} = S_1\delta(r-R)\Big|_{r_*=R_*-\epsilon}^{r_*=y} + S_0\int_{R_*-\epsilon}^{y} dr_*\delta(r-R) \tag{IV.15}$$

Taking substituting $y \to R_* + \epsilon$ in Eq. (IV.15) we already obtain

$$\left.\frac{\partial u}{\partial r_*}\right|_{R_*-\epsilon}^{R_*+\epsilon} = \frac{S_0}{f(R)}, \quad \epsilon \to 0. \tag{IV.16}$$

On the other hand, if we further integrate Eq. (IV.15) with $\int_{R_*-\epsilon}^{R_*+\epsilon} dy$, we will obtain

$$u\Big|_{R_*-\epsilon}^{R_*+\epsilon} = \frac{S_1}{f(R)}, \quad \epsilon \to 0. \tag{IV.17}$$

Applying the above calculation to odd parity, we obtain

$$\left.\frac{\partial a}{\partial r_*}\right|_{R_*-0}^{R_*+0} = -\frac{4\pi\mathfrak{m}}{R^2\sqrt{f(R)}}Y^*_{\ell m}(\pi/2,0) \tag{IV.18}$$

$$a\Big|_{R_*-0}^{R_*+0} = \frac{4\pi m\mathfrak{m}}{Rf(R)\ell(\ell+1)}Y^*_{\ell m}(\pi/2,0) \tag{IV.19}$$

while for even parity, we obtain

$$\left.\frac{\partial b}{\partial r_*}\right|_{R_*-0}^{R_*+0} = \frac{4\pi\mathfrak{m}\omega}{\ell(\ell+1)Rf(R)}\partial_\theta Y^*_{\ell m}\left(\frac{\pi}{2},0\right), \tag{IV.20}$$

$$b\Big|_{R_*-0}^{R_*+0} = 0. \tag{IV.21}$$

### C. Numerical solutions for $a$ and $b$

We can obtain numerical solutions for $u$ by first constructing the normalized homogeneous solutions of the wave equation (IV.14):

$$u_R \sim e^{-i\omega(t-r_*)}\,, \quad r_* \to +\infty\,, \tag{IV.22}$$
$$u_L \sim e^{-i\omega(t+r_*)}\,, \quad r_* \to -\infty\,. \tag{IV.23}$$

Here both $u_L$ and $u_R$ are defined along the entire axis of $-\infty < r_* < +\infty$. While $u_R$ satisfies the out-going boundary condition at infinity, it does not necessarily satisfy the down-going boundary condition at the horizon, while the converse is true for $u_L$. The actual solution to Eq. (IV.14) that satisfies both the boundary conditions and the junction conditions is

$$u(r) = \begin{cases} Au_R(r), & r > R, \\ Bu_L(r), & r < R. \end{cases} \tag{IV.24}$$

with the coefficients $A$ and $B$ provided by

$$A = \frac{-S_0 u_L + f S_1 u_L'}{f^2(u_L' u_R - u_L u_R')} \tag{IV.25}$$
$$B = \frac{-S_0 u_R + f S_1 u_R'}{f^2(u_L' u_R - u_L u_R')} \tag{IV.26}$$

where all functions are evaluated at $r = R$.

## V. RESULTS

In this section I define the physical parameters used, and show the plots that we obtained, as well as calculations for the Poynting flux computed at both the horizon and at spatial infinity. For simplicity, for these simulations, we used the electromagnetic duality discussed in the paper and replaced the magnetic dipole of the neutron star with an electric dipole

### A. Physical parameters

For the values in our numerics, we assume

$$M = 10M_\odot = 1.99 \times 10^{34}\ \text{g}$$
$$R = 25\frac{2GM}{c^2} = 7.38 \times 10^7\ \text{cm}$$
$$\omega = 4 \times 10^3\ \text{rad/s}$$
$$\mathfrak{m} = 10^{30}\ \text{gauss cm}^3$$

Here $M$ is the mass of the black hole, $R$ is the distance from the neutron star to the black hole, $\omega$ is the frequency measured at infinity, and $p$ is the magnitude of the dipole moment projected onto the equatorial plane. The total power $P$ is given by

$$P = \frac{4\mathfrak{m}^2\omega^4}{3c^3} = 1.26 \times 10^{43}\ \text{erg/s}, \tag{V.1}$$

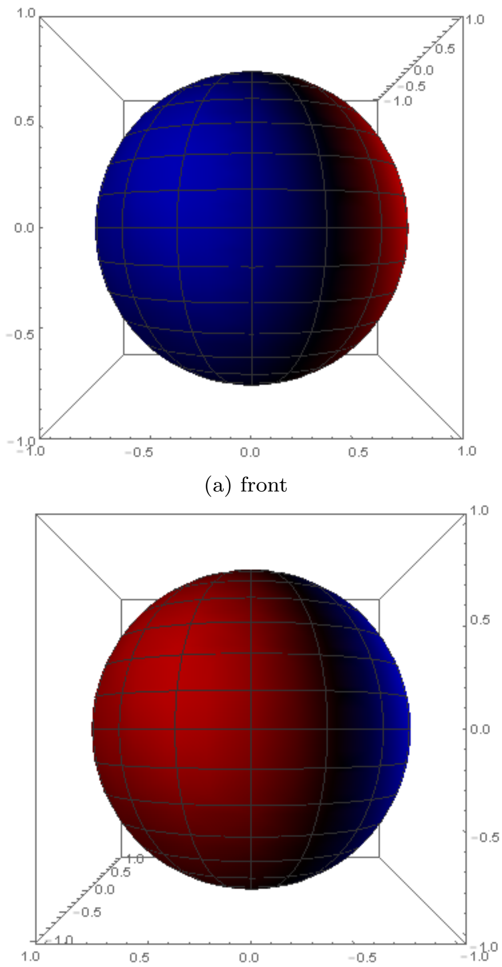
In the geometrical units with $M = G = c = 1$, we will have

$$\omega = \frac{GM\omega}{c^3} = 0.2 \tag{V.2}$$

and

$$\mathfrak{m} = \frac{\mathfrak{m}c^2}{M^2 G^{3/2}} = 2.63 \times 10^{26} \tag{V.3}$$

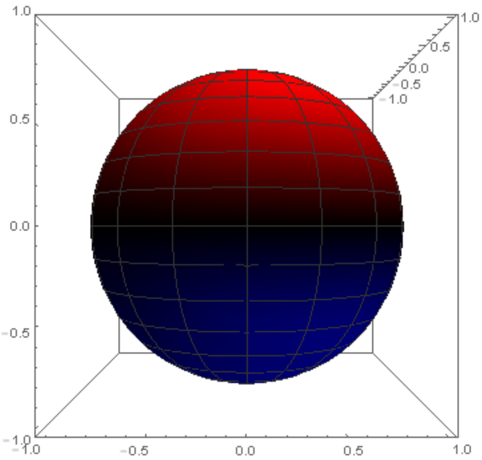
For the plots of the EM components, we represented the black hole as a sphere of radius 1, and normalized the strength in terms of the color, where red signifies high positive value, black represents 0, and blue represents high negative value. And by front, we mean facing the black hole, from the perspective of the neutron star, and by back, we mean the other side.



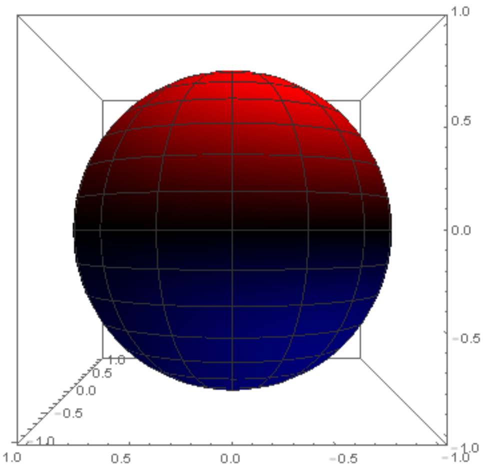
(a) front

(b) back

FIG. 2: Plots of $E_r$ on the horizon for a neutron star located at $R = 25$ and precession frequency of .2

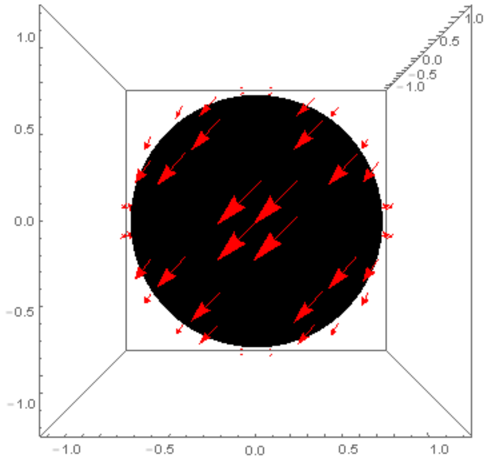



(a) front

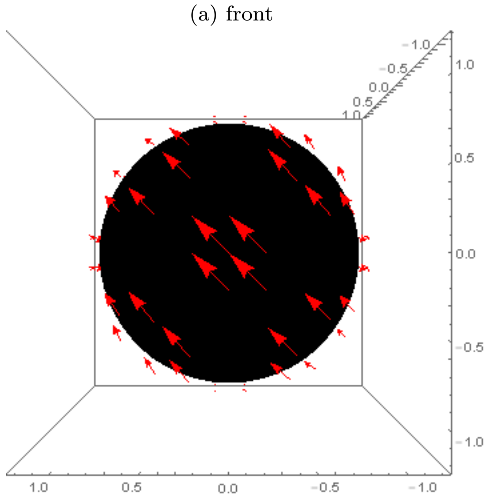



(b) back

FIG. 3: Plots of $B_r$ on the horizon for a neutron star located at $R = 25$ and precession frequency of .2

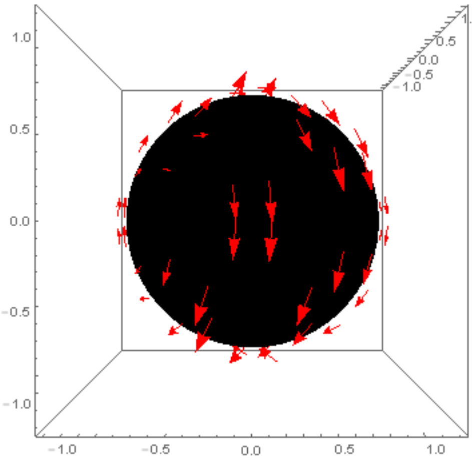



(a) front

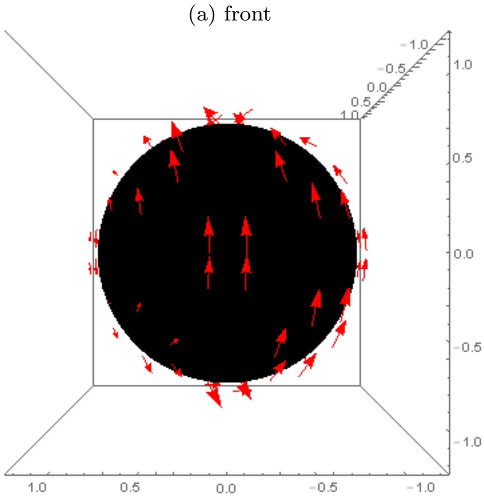



(b) back

FIG. 4: Plots of tangential components of $\vec{E}$ on the horizon for a neutron star located at $R = 25$ and precession frequency of .2



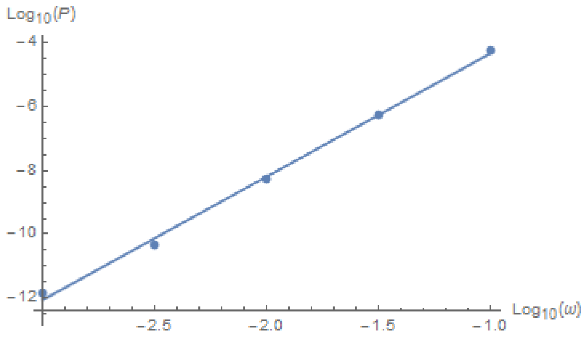

(a) front



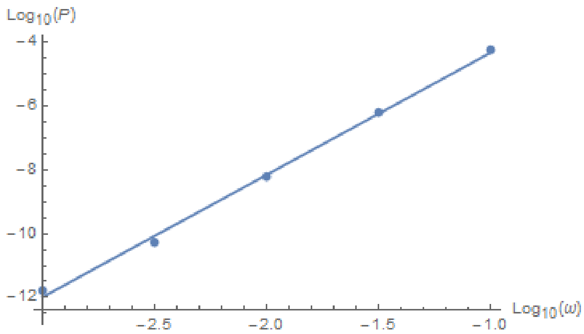

(b) back

FIG. 5: Plots of tangential components of $\vec{B}$ on the horizon for a neutron star located at $R = 25$ and precession frequency of .2

### B. Poynting flux




(a) $R = 25$, slope is 3.85




(b) $R = 50$, slope is 3.84

FIG. 6: Log-log plots of power vs $\omega$. We see that it supports power law $P \propto \omega^4$.

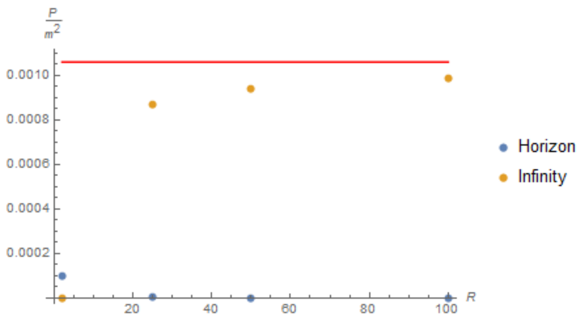



FIG. 7: Plot of power/$\mathfrak{m}^2$ for $\omega = .2$ and various values of $R$. We see that as $R \to \infty$, power approaches flat space-time value of .00106, represented by the red line. In physical units, flat space-time value of the power corresponds to $1.26 \times 10^{43}$ ergs/s. The flux through the horizon rapidly drops as $R$ increases, and in fact, even at $R = 25$, is $2 \times 10^{-6}$.

## VI. DISCUSSION

From these plots, we notice a few things. We notice that the power $P \propto \omega^x$, where $x$ is clsoe to the flat value of 4, which is what we expected. We also notice that as the neutron star approaches infinity, the flux at infinity approaches the flat space-time value and the flux at the horizon approaches 0, and the reverse as the neutron star approaches the black hole. This is what we expect.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