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MARGINALISATION OF THE TIME AND PHASE PARAMETERS IN CBC PARAMETER ESTIMATION

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ABSTRACT

I show how to marginalise over the time parameter or the time and phase parameters semi-analytically in gravitational-wave parameter estimation.

The likelihood[1] for a gravitational wave signal in the presence of coloured, Gaussian noise is

$$\log \mathcal{L} = -\frac{1}{2} \langle d - h \,|\, d - h \rangle , \tag{1}$$

where d is the detector data stream, h the waveform, and the inner product is defined in the frequency domain as

$$\langle a \,|\, b \rangle \equiv 4 \Delta f \sum_{i=0}^{N/2} \frac{\tilde{a}_i^* \tilde{b}_i}{S_i}, \tag{2}$$

with S_i the ith frequency component of the one-sided noise PSD.

Expanding the inner product in Eq. (1), we obtain

$$\log \mathcal{L} = -\frac{1}{2} \left[\langle d \,|\, d \rangle + \langle h \,|\, h \rangle - \langle d \,|\, h \rangle - \langle h \,|\, d \rangle \right]. \tag{3}$$

The time-dependence of the frequency domain waveform is

$$\tilde{h}_j = \tilde{h}_j^{(0)} \exp\left[-2\pi i j \Delta f t\right], \tag{4}$$

where $h^{(0)}$ is the waveform evaluated at a reference time, $t = 0$. From this expression, we can immediately see that $\langle d \,|\, d \rangle$ and $\langle h \,|\, h \rangle$ are independent of time, while

$$\langle d \,|\, h \rangle (t) = 4 \Delta f \sum_{j=0}^{N/2} \frac{\tilde{d}_j^* \tilde{h}_j^{(0)}}{S_j} \exp\left[-2\pi i j \Delta f t\right], \tag{5}$$

and similarly for $\langle h \,|\, d \rangle$.

If we are willing to evaluate $\langle d \,|\, h \rangle$ at integer timesteps, $t = k \Delta t$, then we can write

$$\langle d \,|\, h \rangle (k \Delta t) = 4 \Delta f \sum_{j=0}^{N/2} \frac{\tilde{d}_j^* \tilde{h}_j^{(0)}}{S_j} \exp\left[-2\pi i \frac{jk}{N}\right], \tag{6}$$

where we have exploited that $\Delta f \Delta t = 1/N$. Expanding the sum to all frequency components yields a factor of 1/2:

$$\langle d \,|\, h \rangle (k \Delta t) = 2 \Delta f \sum_{j=0}^{N} \frac{\tilde{d}_j^* \tilde{h}_j^{(0)}}{S_j} \exp\left[-2\pi i \frac{jk}{N}\right]. \tag{7}$$

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[1] The likelihood up to a waveform-independent constant.

We can evaluate this expression efficiently for all $k = 0, 1, \ldots, N-1$ using the FFT:

$$\langle d \,|\, h \rangle (k \Delta t) = 2 \Delta f \, \mathrm{FFT}_k \left(\frac{\tilde{d}^* \tilde{h}}{S} \right). \tag{8}$$

Since $\langle h \,|\, d \rangle = \langle d \,|\, h \rangle^*$, we have

$$\langle d \,|\, h \rangle + \langle h \,|\, d \rangle = 4 \Delta f \Re \left[\mathrm{FFT} \left(\frac{\tilde{d}^* \tilde{h}}{S} \right) \right]. \tag{9}$$

Now that we can evaluate $\log \mathcal{L}(k \Delta t)$, a quadrature rule can be applied to approximate

$$\log \langle \mathcal{L} \rangle = \log \int_0^T dt \, \exp\left(\log \mathcal{L}(t)\right) p(t), \tag{10}$$

where $p(t)$ is the time prior. The time parameter has been numerically marginalised out of the likelihood.

TODO: Incorporate Ilya's argument about the necessary sample rate.

If we want to also marginalise over phase, Eq. (4) becomes

$$\tilde{h}_j = \tilde{h}_j^{(0)} \exp\left[-2\pi i j \Delta f t\right] \exp\left[i \phi\right]. \tag{11}$$

The time shifting can be accomplished by FFT exactly as above, but we arrive at a modified Eq. (8):

$$\langle d \,|\, h \rangle + \langle h \,|\, d \rangle = 4 \Delta f \Re \left[\exp(i\phi) \, \mathrm{FFT} \left(\frac{\tilde{d}^* \tilde{h}}{S} \right) \right]. \tag{12}$$

It is convenient to perform the integral over phase before integrating in time. The following definition is useful:

$$\int_0^{2\pi} d\theta \exp\left(A \cos\theta + B \sin\theta\right) = 2\pi I_0 \left(\sqrt{A^2 + B^2} \right), \tag{13}$$

where I_0 is a modified Bessel function of the first kind.

We have

$$\log \langle \mathcal{L} \rangle = \log \int_0^T dt \int_0^{2\pi} \exp\left(\log \mathcal{L}(t,\phi)\right) p(t) p(\phi)$$
$$\approx \log \left[\Delta t \sum_{k=0}^{N} I_0 \left(4 \Delta f \left| \mathrm{FFT}_k \left(\frac{\tilde{d}^* \tilde{h}}{S} \right) \right| \right) \right]$$
$$- \frac{1}{2} \left[\langle d \mid d \rangle + \langle h \mid h \rangle \right], \quad (14)$$

assuming that the prior on the phase, ϕ, is uniform on $[0, 2\pi)$.