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Modeling mirror shape to reduce substrate Brownian Noise in interferometric gravitational wave detectors

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Abstract

This paper is a report on the effect of varying mirror shape upon Brownian noise is the test mass substrate. Using finite element analysis, it was determined that by using frustum shaped test masses with a ratio between the opposing radii of about 0.7 the frequency of the principle real eigenmodes of the test mass can be shifted into higher frequency ranges. For a fused silica test mass this shape modification could increase this value from 5951 Hz to 7210 Hz, and in a silicon test mass it would increase the value from 8491 Hz to 10262 Hz, in both cases moving the principle real eigenmode to a frequency further from LIGO bands, reducing slightly the noise seen by the detector. The Brownian thermal noise in the optimized substrate was calculated in order to demonstrate that it is lower than in a cylindrical substrate.

Motivation

The direct detection of gravitational waves has been a goal of physics since their existence was first predicted by Einstein's general theory of relativity[1]. Moreover, gravitational waves, once detected, could be used to make measurements of a number of cosmic systems, particularly compact binary systems[2]. In interforometric gravitational wave detectors such as advanced LIGO[3], gravitational waves passing through the detector change the length traveled by light passing through the orthogonally oriented detector arms. This causes the beams to become slightly out of phase, causing the detector to register a signal[3]. This is an incredibly sensitive measurement, so minimizing noise from various sources is a significant issue[4].

According to Visscher[**?**] for certain cone geometries, there is a peak in the frequency of the first real eigenmode. If a similar peak could be found for viable substrate geometries, this would shift the lowest eigenfrequency further from the measurement bandwidth, resulting in a decrease in substrate Brownian noise. A frustrum shaped test mass was selected as it is a close analogue to a cone which retains the two flat surfaces, nessesary for use as a test mass in an interferometric gravitational wave detector.

Theory

Brownian volume fluctuations such as these in LIGO's mirrors lead to phase changes in the reflected beam, which can be of similar amplitude to a gravitational wave signal[5]. The approach of Levin[5] utilizes the fluctuation dissipation theorum to calculate the Brownian noise sensed by the detector through its relation to mechanical loss. The fluctuation dissipation theorem demonstrates that the cause of fluctuations is the same as the cause of dissipation in the system, so by probing the dissipation of the system one probes the fluctuations as well[6]. Levin's method of calculation provides the following formula:

$$S_x(f) = \frac{2k_BT}{\pi^2 f^2}\frac{W_{diss}}{F_0^2}. \tag{1}$$

In formula 1, k_B is the Boltzmann constant, T is the temperature, and f is the frequency of interest. F_0 is a notional amplitude of the oscillating force applied to the surface of the mirror, which is equivalent to the integral of the pressure over the surface of the mirror. This force can be thought of as the laser probing the face of the mirror. W_{diss} is the time averaged power dissipation while the oscillating pressure is applied. This dissipation can be computed from the substrate's loss angle and the maximum energy of elastic deformation in the mirror, U_{max} giving $W_{diss} = 2\pi f U_{max}\phi(f)$. Calculation of U_{max} by Levin yields

$$U_{max} = \frac{F_0^2}{\pi^2 E_0 r_0}(1-\sigma^2)I\left[1 + O(\frac{r_0}{R})\right] \tag{2}$$

where E_0 is the Young's modulus of the material, r_0 is the Gaussian beam radius of the laser, or the distance at which the beam intensity falls off to e^{-1} of its maximal value, σ is the Poisson ratio of the material, $I \backsimeq 1.87322$, and $O(\frac{r_0}{R})$ is an indication that the result may be off by a small factor due to the finite size of the mirror. So, combining equations 1 and 2 the formula one obtains a more applicable form:

$$S_x(f) = \frac{4k_BT}{f}\frac{1-\sigma^2}{\pi^3 E_0 r_0}I\phi[1 - O(\frac{r_0}{R})]. \tag{3}$$

Equations 2 and thus 3 assume the test mass to be an infinite half-space[5], but Liu and Thorne demonstrate that the correction factor which must be applied results in a modification of results of only a few percent[7].

The Eigenfrequency Model

A simple model was constructed representing the test mass mirrors using COMSOL Multiphysics designed to determine how the principle real eigenfrequencies of the test mass change with modifications to the test mass shape. Analysis was performed for test masses constructed out of fused silica, which is used in current gravitational wave detectors, and silicon, which is expected to be used in future gravitational wave detectors. The model was constructed based on proposed specifications for the ETM in a future gravitational wave detector with a test mass radius of $R = 0.17m$ and height of $h = 0.2m$. The test mass front face's radius was held constant, while the back face could be modified by a multiplier, $ratioR$. The height of the mirror was modified in order to ensure that it retained its initial mass. So, for $ratioR = 1$ the mirror becomes a cylinder with the currently proposed shape, while for $ratioR > 1$ the back face becomes larger than the front face and the mirror becomes shorter, and for $ratioR < 1$ the back face becomes smaller than the front face and the mirror becomes longer.

In order to verify the model by an eigenfrequency study was performed to determine that the principle real eigenfrequcny of the test mass was at 5951 Hz which agrees with known values.

The model was tested to ensure that it returned convergent values which did not differ by a significant amount, more than about 3%, when the meshing was made increasingly fine. It was also nessesary to demonstrate that different types of eigenmodes would respond in such a way that the principle real eigenmode could be increased by modifying $ratioR$. There are several relevent types of eigenmodes, primarily the drumhead and butterfly modes shown in figure 2.

As shown by figure 3, different types of eigenmodes are shifted to either higher or lower frequencies. The value of $ratioR$ which maximized the frequency of the principle real eigenmode was desired. The results are presented in figure 4 which demonstrates that the frequency of the principle real eigenmode of either a fused silica or silicon test mass is maximized for $ratioR \approx 0.74$ which results in a shift of the principle real eigenmode from 5951 Hz to 7210 Hz for a fused silica test mass, and from 8491 Hz to 10262 Hz for a silicon test mass. Both of these shift the principle real eigenmode of the test mass to higher frequencies further from the LIGO band, which should result in lower noise.

The Stationary Model

While the shifting of the principle eigenfrequencies found using the eigenfrequency model indicates that a $ratioR$ value of about 0.74 will minimize the substrate Brownian noise, it was decided that a calculation which would allow us to determine how much of a reduction would be expected and verify that that is the optimal value for $ratioR$ should be performed. So, a more complicated model was constructed, implementing additional constraints and symmetries present in the physical system which allowed us to numerically determine the strain energy, U_{max} required in equation 1. Several modifications were made to the eigenfrequency model, first a notional force of magnitude F_0 applied to the front face of the mirror with the same Gaussian spatial profile as the laser probing the test mass face was imlemented. The Gaussian beam size of the laser, and thus the of the applied force, was set to be $rBeam = 0.0156m$, the value given by the ETM strawman. Boundary conditions were needed to prevent the bulk motion of the substrate, so a force of equal magnitude and opposite direction to the one applied to the mirror's front face distributed over the body of the test mass as suggested by Liu and Thorne[7] was also added. In order to ensure that the model was stable to minor perterbations caused by numerical rounding errors, additional constraints which prevented the sheer motion and rotation of the test mass were implemented. Additionally, a meshing scheme which contained more nodes in the central regions where the applied force was greatest was implemented to improve accuracy and decrease the amount of time required to run simulations.

In order to veryify the model, the results it gave in the case where $ratioR = 1$ were compared to analytical results. A volume integration of the strain energy in the substrate using the results of a stationary study was perfomed to give the simulated result in table . Values were also caluculated using Levin[5] as well as Liu and Thorne's[7] calculation methods. The model was also tested to ensure that it returned convergent values which did not differ by a significant amount, more than about 3%, when the meshing was made increasingly fine.

height$\sqrt{S_x}$ Simulated	$\sqrt{S_x}$	% Difference	$\sqrt{S_x}$ Liu and Thorne	% Difference
$2.83 * 10^{-20} \frac{m}{\sqrt{Hz}}$	$2.95 * 10^{-20} \frac{m}{\sqrt{Hz}}$	4.1%	$2.79 * 10^{-20} \frac{m}{\sqrt{Hz}}$	1.4%

U_{max} was plotted in figure 5 instead of the spectral noise profile since a reduction of U_{max} directly reduces the spectral noise profile, and the spectral noise profile is frequency dependent, while U_{max} is not. As shown in figure 5 the optimal value of $ratioR$ to reduce substrate Brownian noise is about 0.71 for a fused silica test mass and about 0.66 for a silicon test mass. At these values of $ratioR$, $\sqrt{S_x}$ is reduced by 2% for a fused silica test mass and 2.2% reduction for a silicon test mass compared to using a standard cylindrical test mass.

Discussion

The results obtained from our two models agree, but provide slightly different optimal values of $ratioR$. The eigenfrequency model demonstrated that a change in mirror shape to a frustum with $ratioR = 0.74$ could shift the principle real eigenmode for test masses constructed of either fused silica or silicon to higher frequencies, resulting in a reduction of substrate Brownian noise when probing in the LIGO bands. Because not all types of modes are probed as strongly by the laser, in particular the laser couples less strongly to butterfly than drumhead modes, our stationary model demonstrates that the optimal $ratioR$ value to reduce substrate Brownian noise is smaller than would be predicted from the eigenfrequency analysis.

Some readers may note that a modification to the mirror mass by changing the height of the test mass would result in a shifting of the frequencies of the principle eigenmodes of the test mass. This also changes the optimal value of $ratioR$ since a decrease in mirror mass will result in a reduction of the frequency of the principle butterfly mode, resulting in both greater noise due to substrate Brownian noise and a decrease in the optimal $ratioR$ value to maximize the frequency of the principle eigenmode. Similarly, increasing the mirror mass will increase the frequency of the principle drumhead mode resulting in a reduction of substrate Brownian noise and an increase in the optimal $ratioR$ value. These effects are shown in figure 6.

References

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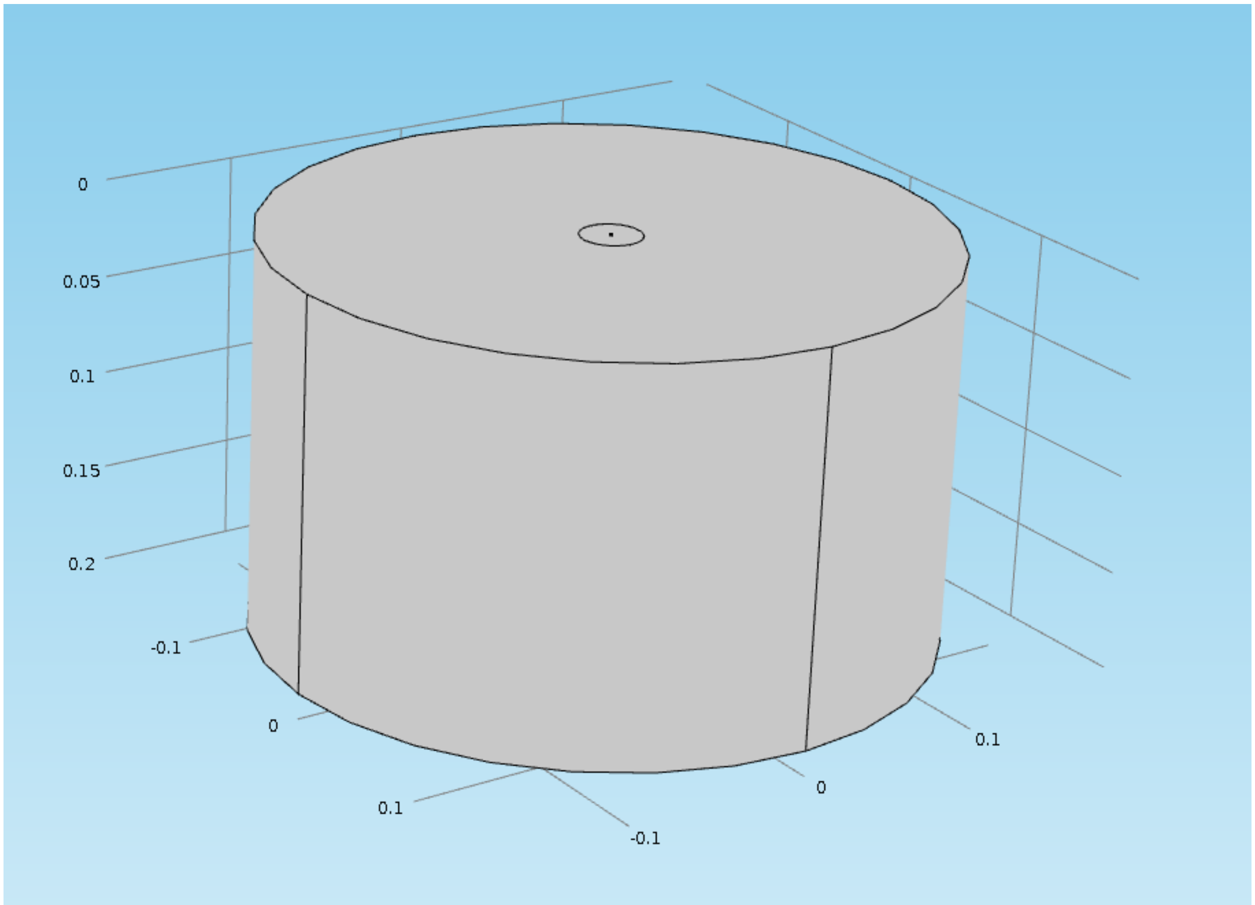
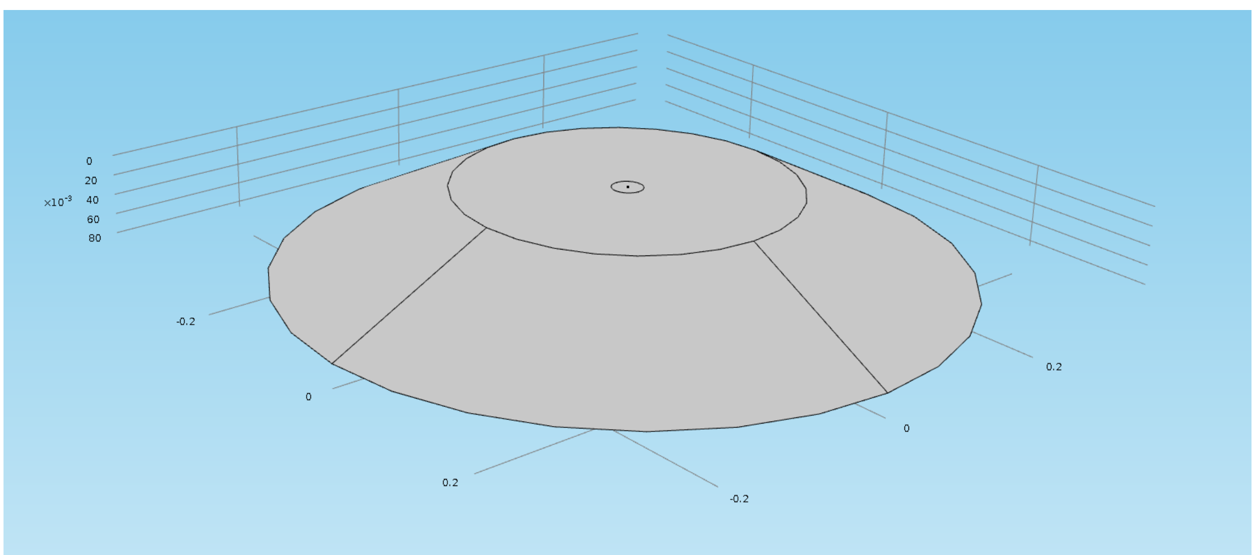
Figure 1: Mirror shape for different values of *ratioR*

(a) $ratioR = 1$



(b) $ratioR = 2$



(c) $ratioR = 0.5$

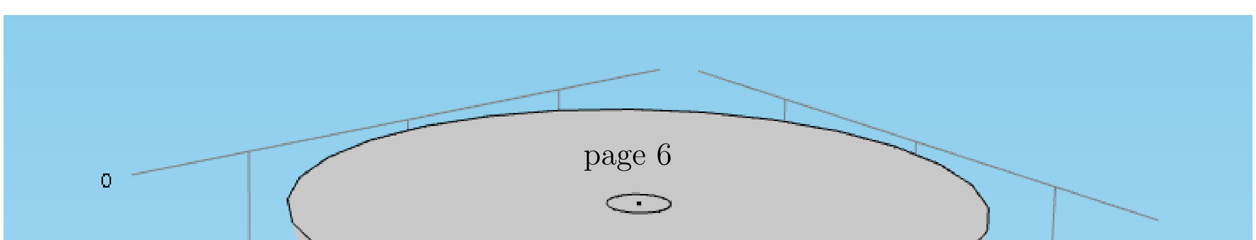
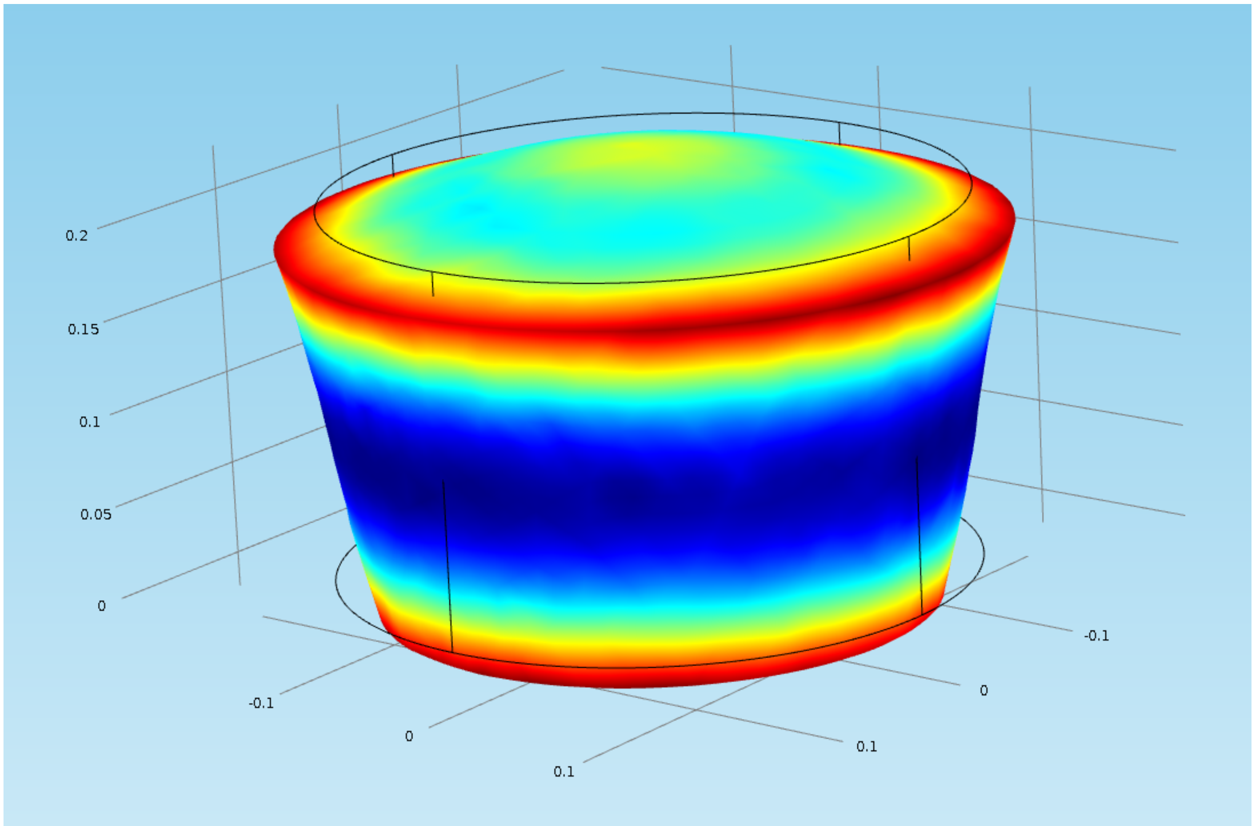


Figure 2: Mode Shapes

(a) A drumhead mode



(b) A butterfly mode

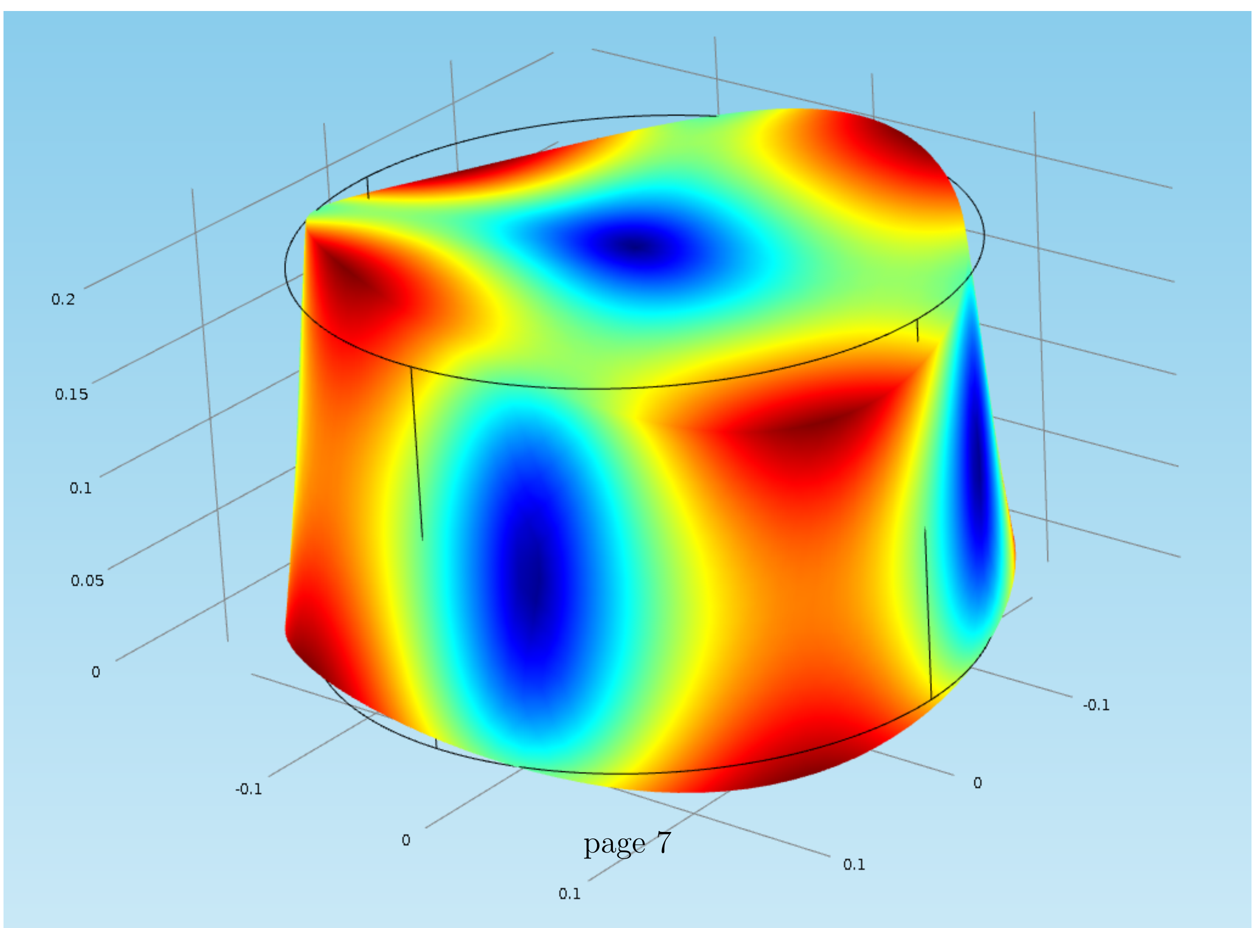


Figure 3: Frequency shifting of lowest eigenmodes against *ratioR*

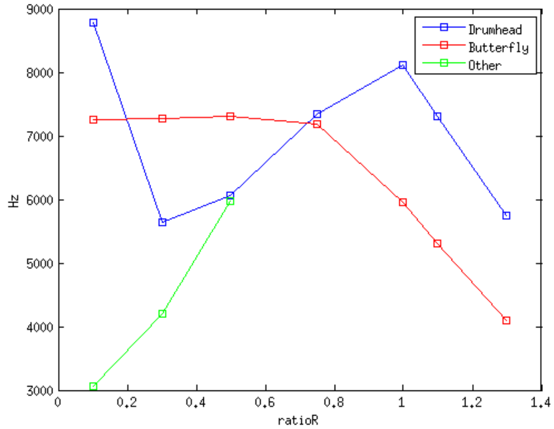


Figure 4: Frequency of the principle eigenmode for varying *ratioR*

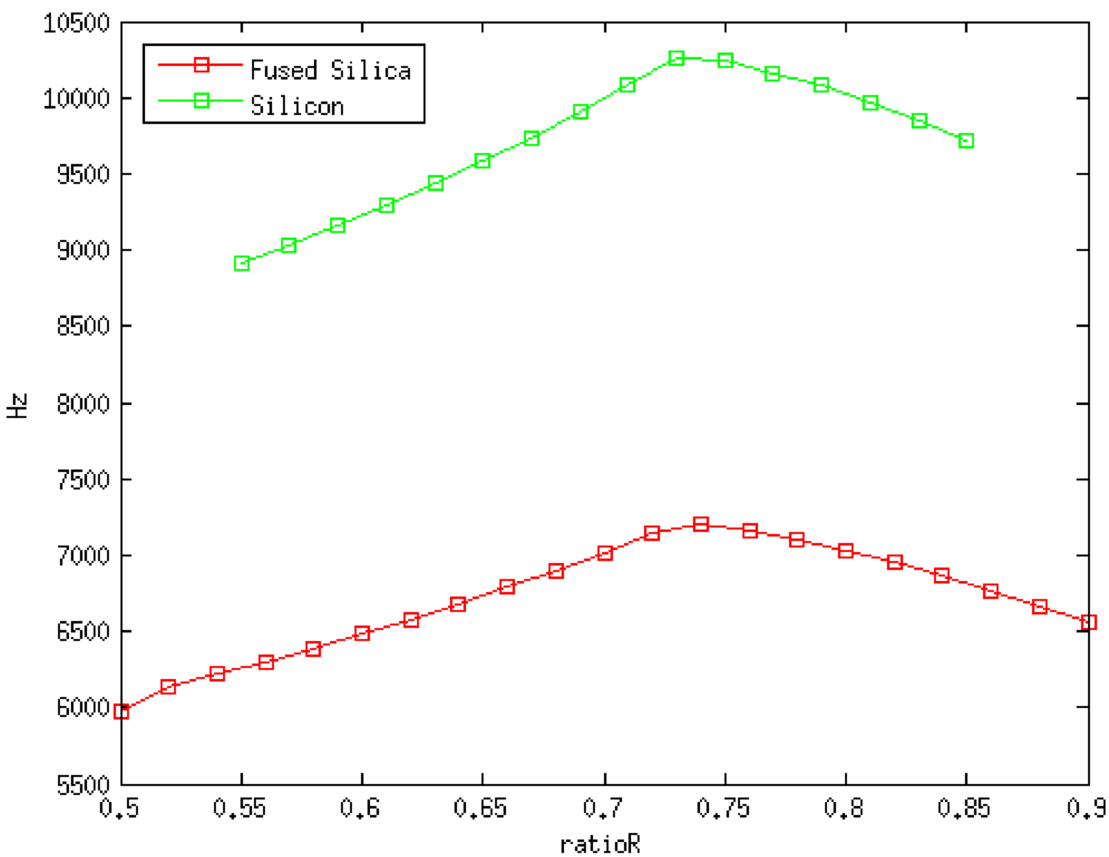
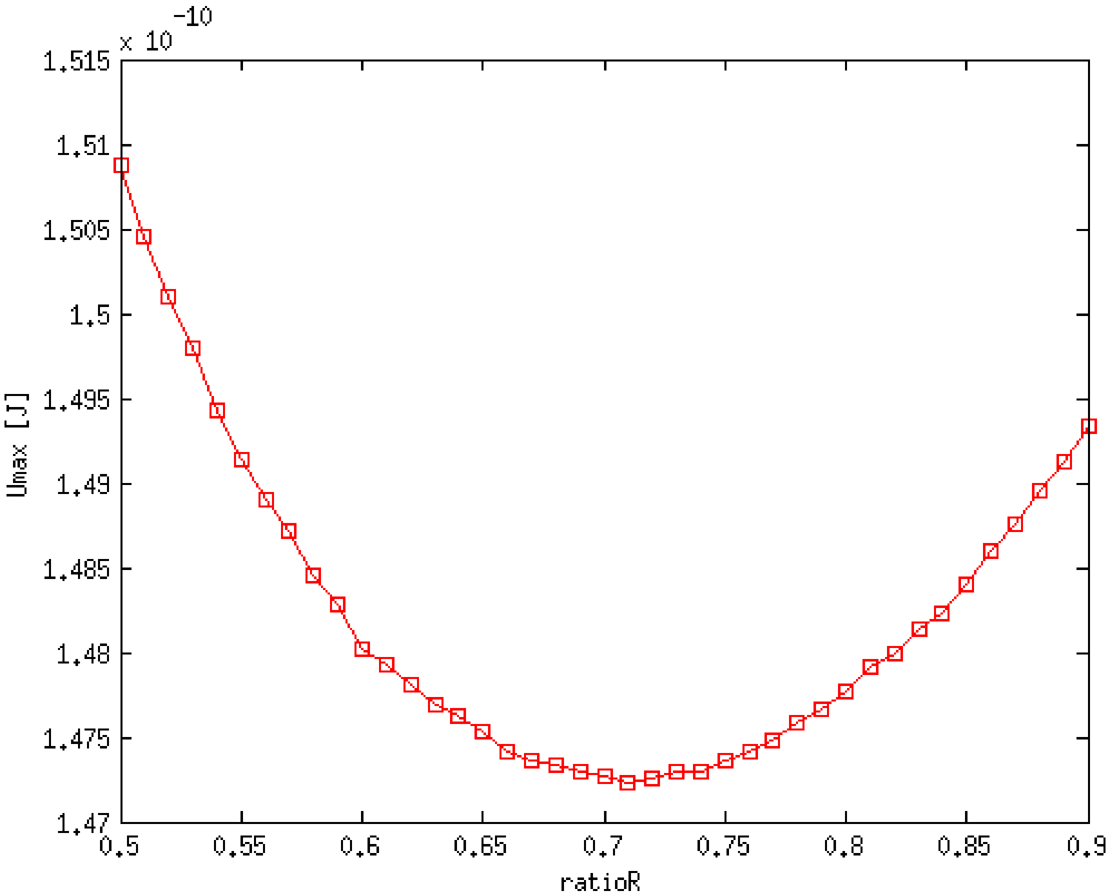


Figure 5: Stain energy for varying *ratioR*

(a) Fused silica



(b) SiliconUmax

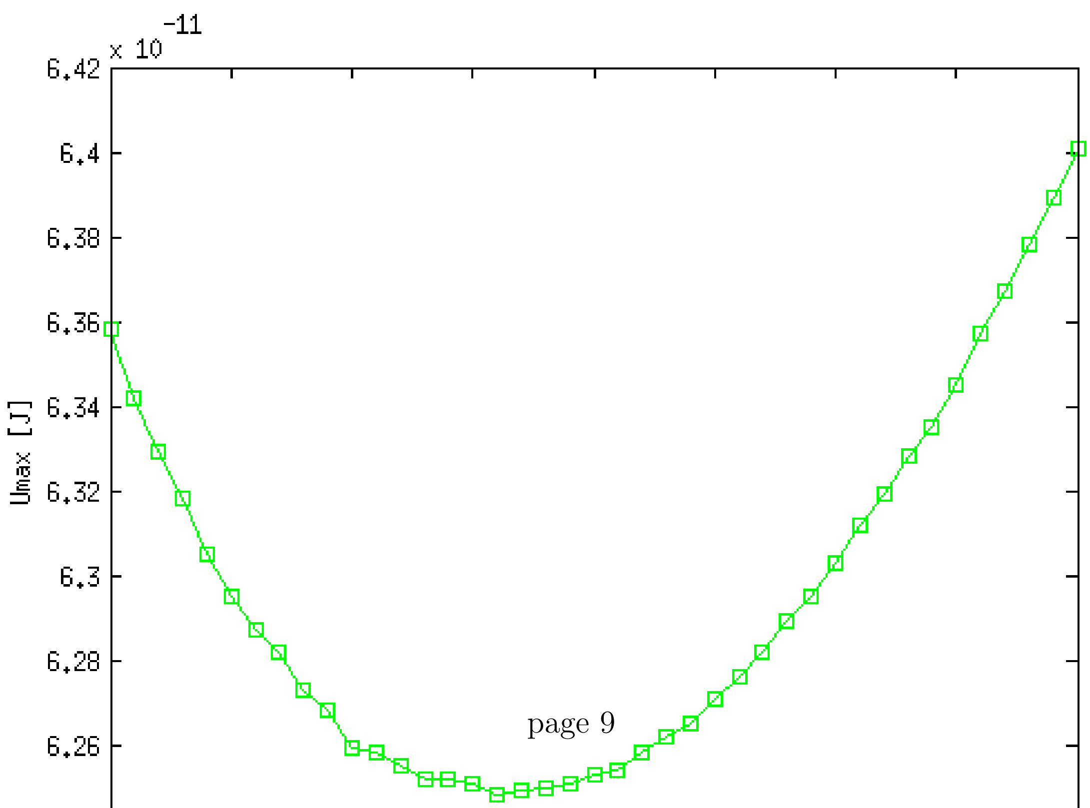


Figure 6: Strain energy for varying *ratioR* for varying mass mirrors