The Geometry of Gravitational Wave Detection

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0 Preview

Most important formula in lecture: strain measured by detector

$$h = \overleftrightarrow{h} : \overleftrightarrow{d} = h_+ \underbrace{\overleftrightarrow{e}_+ : \overleftrightarrow{d}}_{F_+} + h_\times \underbrace{\overleftrightarrow{e}_\times : \overleftrightarrow{d}}_{F_\times} \tag{0.1}$$

where

$$\overleftrightarrow{e}_+ = \vec{\ell} \otimes \vec{\ell} - \vec{m} \otimes \vec{m} \tag{0.2a}$$
$$\overleftrightarrow{e}_\times = \vec{\ell} \otimes \vec{m} + \vec{m} \otimes \vec{\ell} \tag{0.2b}$$

and

$$\overleftrightarrow{d} = \frac{\vec{u} \otimes \vec{u} - \vec{v} \otimes \vec{v}}{2} \tag{0.3}$$

Note $\vec{\imath}, \vec{\jmath}, \vec{k}, \vec{\ell}, \vec{m}, \vec{u}, \vec{v}$ are all unit vectors.

1 Propagating Gravitational Waves

1.0 Reminders from General Relativity

Given a spacetime on which you've defined some coördinates $\{x^\mu\}$ where $\mu \in \{0,1,2,3\}$, the metric tensor can be written in terms of its components $\{g_{\mu\nu}\}$, and the spacetime interval is[1]

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1.1}$$

$\{g_{\mu\nu}\}$ are the components of a tensor field, which means they are functions of the spacetime coördinates $\{x^\mu\}$. We can consider a different set of coördinates $\{x^{\bar{\mu}}\}$, and as with any tensor, the metric tensor has a new set of components $\{g_{\bar{\mu}\bar{\nu}}\}$ associated with the corresponding basis, defined by

$$g_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\nu}{\partial x^{\bar{\nu}}} \tag{1.2}$$

which ensures that

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = g_{\bar{\mu}\bar{\nu}}\, dx^{\bar{\mu}}\, dx^{\bar{\nu}} \tag{1.3}$$

is the same no matter which coördinate system we use. Note that this notation considers $\bar{\mu}$ and μ to be different indices, and stresses that there are different sets of coördinates, so that there's no real connection between x^1 and $x^{\bar{1}}$. (They might for example be Cartesian coördinates $\{t,x,y,z\}$ and double-null coördinates $\{u,v,\theta,\phi\}$.) This does conceal, however, that as functions of four variables, $\{g_{\mu\nu}\}$ and $\{g_{\bar{\mu}\bar{\nu}}\}$ take different sets of arguments to describe the geometry at the same spacetime point. So we could write, more completely,

$$\bar{g}_{\alpha\beta}(\{\bar{x}^\gamma\}) = g_{\mu\nu}(\{x^\lambda\}) \frac{\partial x^\mu}{\partial \bar{x}^\alpha} \frac{\partial x^\nu}{\partial \bar{x}^\beta} \tag{1.4}$$

This form is especially useful when considering an infinitesimal coördinate transformation, in which $\bar{x}^\mu = x^\mu + \xi^\mu$ where ξ^μ is in some sense small. Then, to first order in $\{\xi^\mu\}$, the change in the components of the metric tensor can be shown to be

$$\bar{g}_{\mu\nu}(\{x^\lambda\}) - g_{\mu\nu}(\{x^\lambda\}) = -\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu \tag{1.5}$$

where

$$\nabla_\mu \xi_\nu = \frac{\partial \xi_\nu}{\partial x_\mu} - \Gamma^\lambda_{\mu\nu} \xi_\lambda \tag{1.6}$$

is the usual covariant derivative defined in terms of the Christoffel symbols

$$\Gamma^\lambda_{\mu\nu} = \frac{g^{\lambda\rho}}{2} \left(\frac{\partial g_{\mu\nu}}{\partial \rho} - \frac{\partial g_{\rho\nu}}{\partial \mu} - \frac{\partial g_{\mu\rho}}{\partial \nu} \right) \tag{1.7}$$

and $\{g^{\mu\nu}\}$ is the matrix inverse of $\{g_{\mu\nu}\}$, so that $g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu$.

The spacetime interval of special relativity is associated with the Minkowski metric, which can be written as

$$\eta_{\mu\nu}\, dx^\mu\, dx^\nu = -c^2\, dt^2 + \delta_{ij}\, dx^i\, dx^j \tag{1.8}$$

the linearized theory of gravity assumes that the metric tensor can be written as some background metric plus a small perturbation. Choosing Minkowski as the background metric, we have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1.9}$$

we can make a gauge transformation (small coördinate change)

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{1.10}$$

[1] We're using the Einstein summation convention: there is an implicit sum over repeated $\mu, \nu, \lambda, \ldots$ indices, e.g., $\sum_{\mu=0}^{3}$. In this linearized 3-space language, there's no strong distinction between subscripted and superscripted indices, but we should have one "upstairs" and one "downstairs" for the implicit sum, and have the same "unbalanced" indices in the same places on both sides of an equation.

and represent the same physical spacetime in slightly different coördinates. This gauge transformation is analogous to the transformation that allows us to change $\{A^\mu\} = \{\varphi, \vec{A}\}$ in electromagnetism and not change the physical electric and magnetic fields. One convenient gauge condition that we can enforce is the so-called transverse-traceless-temporal gauge, in which $h_{0\mu} = 0$, $\eta^{\mu\lambda}\partial_\lambda h_{\mu\nu} = 0$, and $\eta^{\mu\nu}h_{\mu\nu} = 0$. The temporal part of the gauge condition means that we can just talk about the spatial components of the metric perturbation, and in fact we won't need to talk about spacetime indices outside of this introductory review. Instead, we have spatial components $\{h_{ij}\}$ where $\delta^{ij}h_{ij} = 0$ and $\delta^{ik}\partial_k h_{ij} = 0$.

1.1 The polarization decomposition

If we describe linearized GR in the transverse-traceless-temporal gauge, the spacetime interval is replaced by[2]

$$ds^2 = -c^2\, dt^2 + (\delta_{ij} + h_{ij})\, dx^i\, dx^j \tag{1.11}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{1.12}$$

is the Kronecker delta and $\{h_{ij}\}$ are small perturbations. In this gauge, the components $\{h_{ij}\}$ all obey the wave equation

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right) h_{ij} = 0 \tag{1.13}$$

Because the metric (1.11) is the Minkowski metric plus a small spatial perturbation, we can do all of the important calculations for GW detection in the notation of vectors in a three-dimensional Euclidean space.

A wave coming from a single distant source can be treated as a plane wave propagating along a unit vector $\vec{k}$ which points from the source to the observer. If we choose our coördinate axes so that this unit vector has components

$$\{k_i\} \equiv \boldsymbol{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.14}$$

the components of the metric tensor perturbation are

$$\{h_{ij}\} \equiv \mathbf{h} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.15}$$

where the two independent degrees of freedom h_+ and $h_\times$ are functions of $t - \vec{k}\cdot\vec{r}/c$. We can also write this as

$$h_{ij} = h_+ e_{+\,ij} + h_\times e_{\times\,ij} \tag{1.16}$$

or

$$\mathbf{h} = h_+\mathbf{e}_+ + h_\times\mathbf{e}_\times \tag{1.17}$$

in terms of the matrices

$$\{e_{+\,ij}\} \equiv \mathbf{e}_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \{e_{\times\,ij}\} \equiv \mathbf{e}_\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.18}$$

It's useful, however, to be able to write things without referring to a specific coördinate system.[3] We think of a vector $\vec{v}$ as a physical object with magnitude and direction, not just as the collection of

[2]We're again using the Einstein summation convention: there is an implicit sum over repeated $i, j, k, \ldots$ indices, e.g., $\sum_{i=1}^{3}$.

[3]Our "coördinate-free" notation will only be covariant in the three-dimensional sense; since it assumes we're still in the TT gauge.

three numbers $\{v_i\}$. In fact, we can resolve the vector $\vec{v}$ in different bases, e.g., $\{\vec{e}_i\}$ and $\{\vec{e}_i'\}$. There will be different components $\{v_i\}$ and $\{v_{i'}\}$ defined by

$$v_i = \vec{e}_i \cdot \vec{v} \qquad \text{vs} \qquad v_{i'} = \vec{e}_i' \cdot \vec{v} \tag{1.19}$$

and we could collect them into different 3×1 matrices (column vectors)

$$\{v_i\} \equiv \boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \qquad \text{vs} \qquad \{v_{i'}\} \equiv \boldsymbol{v}' = \begin{pmatrix} v_{1'} \\ v_{2'} \\ v_{3'} \end{pmatrix} \tag{1.20}$$

The matrices $\boldsymbol{v}$ and $\boldsymbol{v}'$ contain different triples of numbers, but they are just different ways of describing the same object $\vec{v}$. The vector $\vec{v}$ is what's fundamental, because "physics is about things."[4] Sometimes we'll want to refer to a complicated object in a coördinate-independent way. For that purpose it's useful to introduce what's known as *abstract index notation*, using Latin indices from the front of the alphabet. So if I write v_a, this is the same as writing $\vec{v}$. It refers not to a particular component or set of components, but to the object which has components $\{v_i\}$ when resolved in the basis $\vec{e}_i$.

Now we return to consideration of equation (1.16) which describes the metric perturbations $\{h_{ij}\}$, or the corresponding matrix equation (1.17). We'd like to describe this in an abstract way which doesn't rely on a particular set of basis vectors. We can do this by considering the construction of the matrices $\mathbf{e}_+$ and $\mathbf{e}_\times$ from the components of unit vectors $\vec{\ell}$ and $\vec{m}$ which form an orthonormal triple with $\vec{k}$. In the coördinate system we're working in so far those vectors have components

$$\{\ell_i\} \equiv \boldsymbol{\ell} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \{m_i\} \equiv \boldsymbol{m} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{1.21}$$

and we can write the plus and cross polarization matrices as

$$\mathbf{e}_+ = \boldsymbol{\ell}\boldsymbol{\ell}^{\mathrm{T}} - \boldsymbol{m}\boldsymbol{m}^{\mathrm{T}} \tag{1.22a}$$
$$\mathbf{e}_\times = \boldsymbol{\ell}\boldsymbol{m}^{\mathrm{T}} + \boldsymbol{m}\boldsymbol{\ell}^{\mathrm{T}} \tag{1.22b}$$

Or, in terms of components,

$$e_{+\,ij} = \ell_i \ell_j - m_i m_j \tag{1.23a}$$
$$e_{\times\,ij} = \ell_i m_j + m_i \ell_j \tag{1.23b}$$

We know how to talk about a vector like $\vec{\ell}$ (in arrow notation) or ℓ^a (in abstract index notation). If we think about the matrix $\{e_{+\,ij}\}$ as making up the components, in a particular basis, of a tensor, we can describe that tensor in abstract index notation as $e_{+\,ab}$ and define it and its counterpart $e_{\times\,ab}$ by

$$e_{+\,ab} = \ell_a \ell_b - m_a m_b \tag{1.24a}$$
$$e_{\times\,ab} = \ell_a m_b + m_a \ell_b \tag{1.24b}$$

If we like to use arrow notation, we can define these two basis tensors equivalently using the tensor (dyad) product as

$$\overleftrightarrow{e}_+ = \vec{\ell} \otimes \vec{\ell} - \vec{m} \otimes \vec{m} \tag{1.25a}$$
$$\overleftrightarrow{e}_\times = \vec{\ell} \otimes \vec{m} + \vec{m} \otimes \vec{\ell} \tag{1.25b}$$

That then allows us to write the general plane wave propagating along $\vec{k}$ in covariant tensor notation as

$$\overleftrightarrow{h} = h_+ \overleftrightarrow{e}_+ + h_\times \overleftrightarrow{e}_\times \tag{1.26}$$

[4] T. Dray and C. A. Manogue, *The Geometry of Vector Calculus*, http://www.math.oregonstate.edu/BridgeBook/book/revper/things

where $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ are defined according to (1.25) from the orthonormal basis vectors $\vec{\ell}$ and $\vec{m}$ perpendicular to $\vec{k}$, and h_+ and $h_\times$ are functions of $t - \vec{k} \cdot \vec{r}/c$.

Examples of other tensors you may have seen are the inertia tensor $\overleftrightarrow{I}$ from rigid body motion, the stress tensor $\overleftrightarrow{T}$ from statics or electromagnetism, and the quadrupole moment tensor $\overleftrightarrow{Q}$ from a multipole expansion. These are all symmetric, second-rank tensors like $\overleftrightarrow{h}$.

1.2 Exercise: Normalization of Basis Tensors

Dot products involving tensors can be defined in straightforward ways using the abstract index notation as a guide. For example, $\overleftrightarrow{T} \cdot \vec{v}$ is the vector which can be written in abstract index notation[5] as $[\overleftrightarrow{T} \cdot \vec{v}]_a = T_{ab} v^b$, $\overleftrightarrow{S} \cdot \overleftrightarrow{T}$ is the tensor $[\overleftrightarrow{S} \cdot \overleftrightarrow{T}]_{ab} = S_{ac} T^c{}_b$ and the double dot product $\overleftrightarrow{S} : \overleftrightarrow{T}$ is the scalar which is the "trace" of this: $\overleftrightarrow{S} : \overleftrightarrow{T} = S_{ab} T^{ba}$

1. Use the abstract index notation to show that the double dot product of two dyads is $(\vec{u} \otimes \vec{v}) : (\vec{a} \otimes \vec{b}) = (\vec{v} \cdot \vec{a})(\vec{b} \cdot \vec{w})$

2. Calculate the four double dot products $\overleftrightarrow{e}_A : \overleftrightarrow{e}_B$, where A and B can each be $+$ or $\times$. (I.e., calculate $\overleftrightarrow{e}_+ : \overleftrightarrow{e}_+$, $\overleftrightarrow{e}_+ : \overleftrightarrow{e}_\times$, etc.)

1.3 Change of Basis

The propagation direction $\vec{k}$ does not uniquely specify the construction of basis tensors $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$; we also need to choose a vector $\vec{\ell}$ in the plane perpendicular to $\vec{k}$. (This then uniquely determines $\vec{m} = \vec{k} \times \vec{\ell}$.) For different types of sources and analyses, there may be a choice of polarization basis which is particularly convenient. It may also be desirable to convert between a convenient polarization basis and some canonical reference basis constructed only from the propagation direction and some absolute reference directions.

For example, suppose that we specify the source location in equatorial coördinates in terms of its right ascension α and declination δ. This is equivalent to specifying $\vec{k}$; we can assign a reference basis to each sky position by producing a prescription for defining additional unit vectors $\vec{\imath}$ and $\vec{\jmath}$ which, together with $\vec{k}$, form a an orthonormal set. One prescription is to require $\vec{\imath}$ to be parallel to the celestial equator, i.e., perpendicular to the direction of the Earth's axis. We choose $\vec{\imath}$ to point in the direction of *decreasing* right ascension, so that the third vector $\vec{\jmath} = \vec{k} \times \vec{\imath}$ points into the Northern celestial hemisphere. This is illustrated in figure 1. From these unit vectors $\vec{\imath}$ and $\vec{\jmath}$ we can construct a reference polarization basis for traceless symmetric tensors transverse to $\vec{k}$:

$$\overleftrightarrow{\varepsilon}_+ = \vec{\imath} \otimes \vec{\imath} - \vec{\jmath} \otimes \vec{\jmath} \tag{1.27a}$$

$$\overleftrightarrow{\varepsilon}_\times = \vec{\imath} \otimes \vec{\jmath} + \vec{\jmath} \otimes \vec{\imath} \tag{1.27b}$$

The basis vectors $\vec{\ell}$ and $\vec{m}$, from which the source's natural polarization basis is constructed, lie in the same plane as $\vec{\imath}$ and $\vec{\jmath}$, since they're all perpendicular to the propagation direction $\vec{k}$. The natural basis can be located relative to the reference basis by the angle from $\vec{\imath}$ to $\vec{\ell}$, measured counter-clockwise around $\vec{k}$, as shown in figure 2. As in the usual rotation of basis vectors, we can resolve $\vec{\ell}$ and $\vec{m}$ in terms of $\vec{\imath}$ and $\vec{\jmath}$:

$$\vec{\ell} = \quad \vec{\imath} \cos\psi + \vec{\jmath} \sin\psi \tag{1.28a}$$

$$\vec{m} = -\vec{\imath} \sin\psi + \vec{\jmath} \cos\psi \tag{1.28b}$$

We can substitute (1.28) into (1.25) to get $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ in terms of $\overleftrightarrow{\varepsilon}_+$ and $\overleftrightarrow{\varepsilon}_\times$. The one tricky thing is the tensor product, which is

[5] In an expression like $\vec{u} \cdot \vec{v} = u^a v_a = u_a v^a$, it doesn't matter which abstract index is written "upstairs" and which "downstairs", as long as there is one in each position.

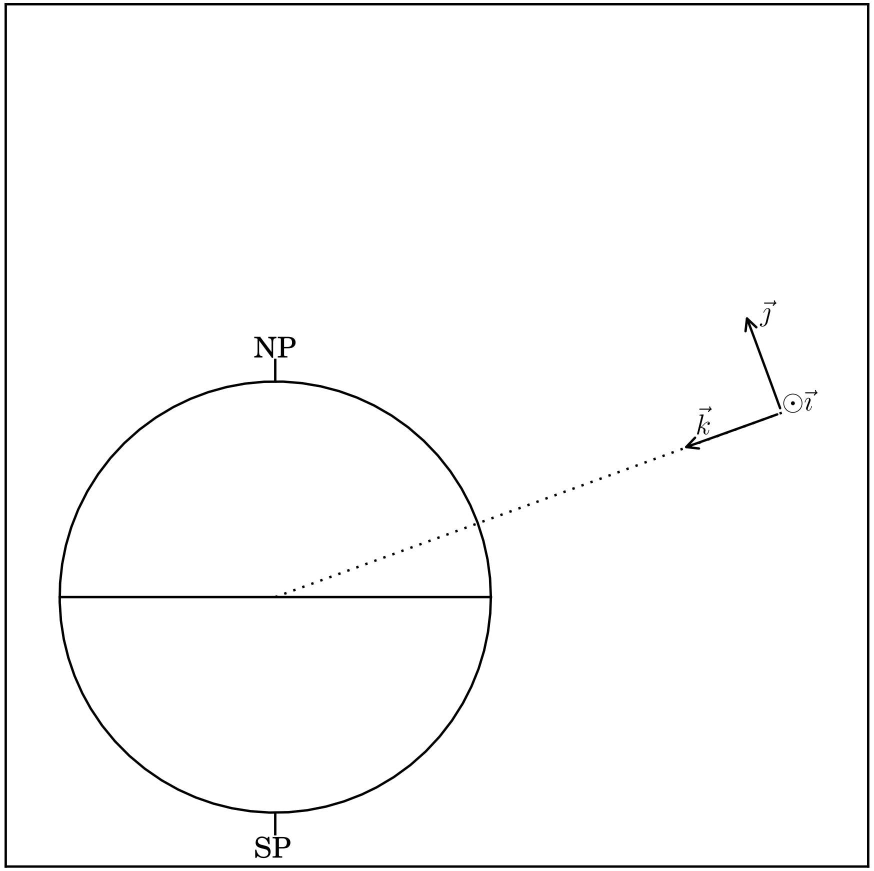

Figure 1: Definition of the unit vectors $\vec{\imath}$ and $\vec{\jmath}$, orthogonal to the propagation direction $\vec{k}$, used to define the reference polarization basis tensors $\overleftrightarrow{\varepsilon}_+$ and $\overleftrightarrow{\varepsilon}_\times$ via (1.27). The unit vector $\vec{\imath}$ is orthogonal both to $\vec{k}$ (which points from the source to the observer) and to the axis of the equatorial coördinate system. (I.e., it is parallel to the celestial equator.) The unit vector $\vec{\jmath}$ ($= \vec{k} \times \vec{\imath}$) points into the Northern hemisphere.

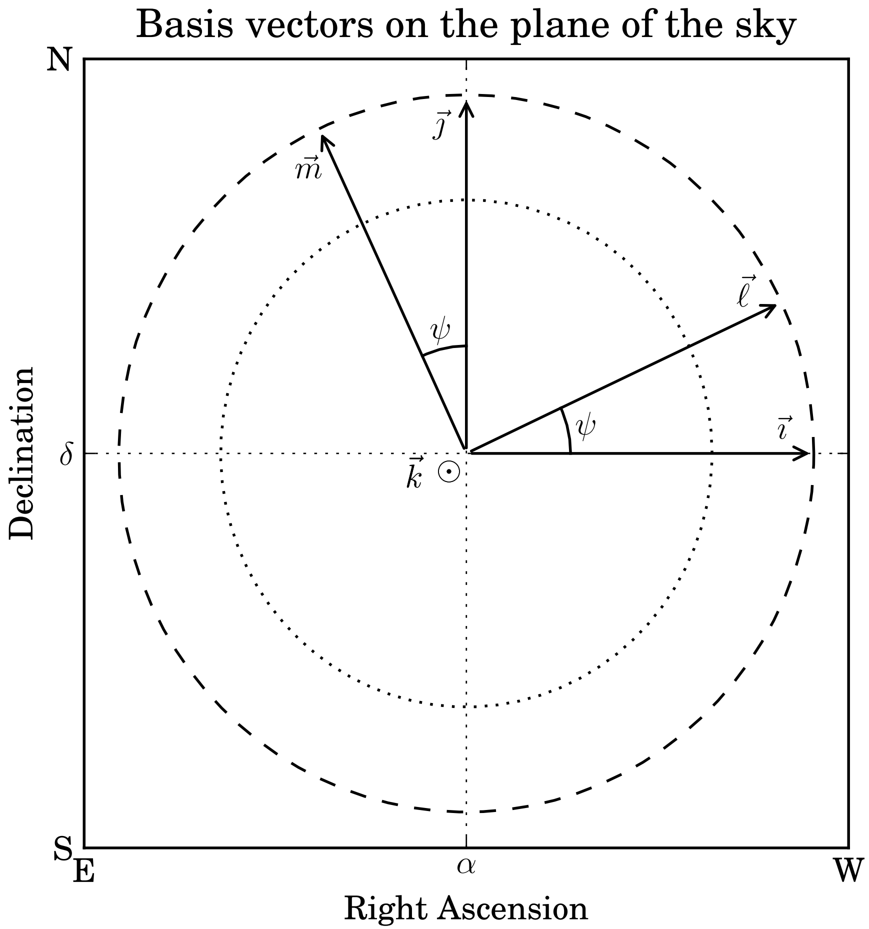

Figure 2: Rotation of basis. The natural polarization basis tensors $\overleftrightarrow{e}_+$ and $\overleftrightarrow{e}_\times$ are created from the unit vectors $\vec{\ell}$ and $\vec{m}$. The reference polarization basis tensors $\overleftrightarrow{\varepsilon}_+$ and $\overleftrightarrow{\varepsilon}_\times$ are created from the unit vectors $\vec{\imath}$ and $\vec{\jmath}$ via (1.25). The polarization angle ψ which completes the specification of $\vec{\ell}$ and therefore of the natural polarization basis, is measured from $\vec{\imath}$ to $\vec{\ell}$, counter-clockwise around $\vec{k}$. (For the example illustrated in this figure, ψ lies between 0 and $\pi/2$.)

not commutative. One example term is

$$\vec{\ell}\otimes\vec{m} = -\vec{\imath}\otimes\vec{\imath}\cos\psi\sin\psi+\vec{\imath}\otimes\vec{\jmath}\cos^2\psi+\vec{\jmath}\otimes\vec{\imath}\sin^2\psi+\vec{\jmath}\otimes\vec{\jmath}\sin\psi\cos\psi \tag{1.29}$$

1.4 Exercise: Change of Polarization Basis

Do the algebra and apply the double angle formulas to show that

$$\overleftrightarrow{e}_+ = \overleftrightarrow{\varepsilon}_+\cos 2\psi + \overleftrightarrow{\varepsilon}_\times \sin 2\psi \tag{1.30a}$$
$$\overleftrightarrow{e}_\times = -\overleftrightarrow{\varepsilon}_+\sin 2\psi + \overleftrightarrow{\varepsilon}_\times \cos 2\psi \tag{1.30b}$$

This shows that the specification of the polarization basis associated with a particular source requires three angles: the right ascension α and declination δ to specify the sky position and thus the propagation direction $\vec{k}$, and an additional *polarization angle* ψ to define the orientation of the preferred polarization basis $\{\overleftrightarrow{e}_+, \overleftrightarrow{e}_\times\}$ relative to some reference basis like $\{\overleftrightarrow{\varepsilon}_+, \overleftrightarrow{\varepsilon}_\times\}$. Note that, since (1.30) contains only trig functions of 2ψ, the polarization angle ψ can generally be taken to range over π rather than 2π. (Changing ψ to $\psi+\pi$ would flip both $\vec{\ell}$ and $\vec{m}$, and leave the polarization basis tensors unchanged.)

2 Interaction with a Detector

2.1 The Detector Tensor

The simplest description of an interferometric gravitational-wave detector (see figure 3) is to say it measures the difference between the lengths of its arms. That's a bit too simplistic, though, and it opens the trap of outsiders asking "how can a GW measure anything if both the spacetime and the detector are stretching?" So instead, let's note that it measures the difference in phase of

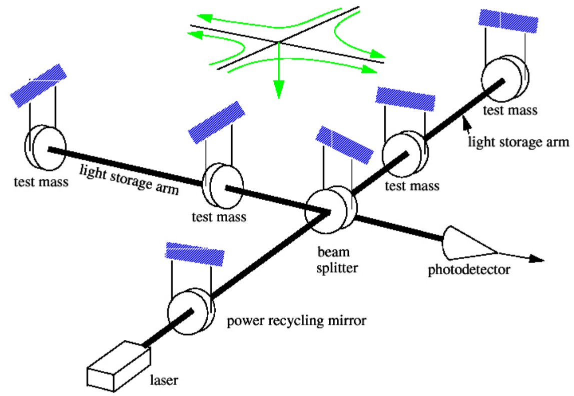

Figure 3: Schematic of an interferometric gravitational wave detector. Image by Ray Frey. We define the basis vectors $\vec{u}$ and $\vec{v}$ to lie parallel to the two arms.

light which has gone down and back one arm versus the other. This is equivalent to measuring the difference in the roundtrip travel time of photons down the two arms, a measurement which can be made entirely locally, and without worrying about the effects of the wave, since the time components of the spacetime metric are not changed in the transverse-traceless-temporal gauge. We also use the fact that points with constant coördinates in the TT gauge are in free fall, i.e., experience no non-gravitational forces, to note that if define coördinates so that the beam splitter is at the origin and the end mirror of an arm is at position $(x^1,x^2,x^3) = (L_0,0,0)$, those coördinates will not be changed by the passage of a gravitational wave. We can now consider the trajectory of a photon going down the arm from $(0,0,0)$ to $(L_0,0,0)$ and back. Its trajectory $x^1(t)$ will be given by solving the differential equation

$$ds^2 = -c^2\,dt^2 + (1+h_{11})(dx^1)^2 = 0 \tag{2.1}$$

or, working to first order in the perturbation h_{11},

$$dt = \frac{\sqrt{1+h_{11}}}{c}\left|dx^1\right| \approx \left(1+\frac{1}{2}h_{11}\right)\frac{\left|dx^1\right|}{c} \tag{2.2}$$

Now, in general, we have to worry about the fact that $h_{11}\left(t-\frac{\vec{k}\cdot\vec{r}}{c}\right)$ is a function of space and time, but if the travel time of the photon is short compared to the period of the gravitational wave, or equivalently if the gravitational wavelength is large compared to the length of the arm, we are in the so-called long-wavelength limit, and we can approximate h_{11} as a constant during the trajectory of the photon. Then the time the photon takes to go from $(0,0,0)$ to $(L_0,0,0)$ and back is

$$T_1 = \left(1+\frac{1}{2}h_{11}\right)\frac{2L_0}{c} =: \frac{2L_1}{c} \tag{2.3}$$

As usual, we would like to approach this problem in a coördinate-free way, so we define a unit vector $\vec{u}$ along the arm, which has components

$$\{u^i\} \equiv \boldsymbol{u} = \begin{pmatrix}1\\0\\0\end{pmatrix} \tag{2.4}$$

in the specialized coördinate system we've used to describe the detector arm. The metric perturbation component h_{11} appearing in (2.3) can be written as

$$h_{11} = u^i h_{ij} u^j = \boldsymbol{u}^{\mathrm{T}}\mathbf{h}\boldsymbol{u} \tag{2.5}$$

This quantity can be written in a basis independent way, using either abstract index or arrow notation, as

$$u^a h_{ab} u^b = \vec{u}\cdot\overleftrightarrow{h}\cdot\vec{u} \tag{2.6}$$

so that

$$L_{\vec{u}} = L_0\left(1+\frac{1}{2}h_{ab}u^a u^b\right) = L_0\left(1+\frac{1}{2}\vec{u}\cdot\overleftrightarrow{h}\cdot\vec{u}\right) \tag{2.7}$$

where we have written $L_{\vec{u}}$ to emphasize that the interferometer arm is parallel to the unit vector $\vec{u}$.

Now consider an interferometer with one arm along the unit vector $\vec{u}$ and the other along the unit vector $\vec{v}$.[6] The interferometer measures the difference in roundtrip times down the two arms; this divided by $2L_0/c$ is known as the gravitational wave strain $h(t)$:

$$\begin{aligned} h &= \frac{L_{\vec{u}}-L_{\vec{v}}}{L_0} = \frac{1}{2}\left(\vec{u}\cdot\overleftrightarrow{h}\cdot\vec{u}-\vec{v}\cdot\overleftrightarrow{h}\cdot\vec{v}\right) \\ &= h_{ab}\frac{u^a u^b - v^a v^b}{2} = h_{ab}d^{ab} = \overleftrightarrow{h}:\overleftrightarrow{d} \end{aligned} \tag{2.8}$$

where we have defined the detector tensor

$$d^{ab} = \frac{u^a u^b - v^a v^b}{2} \qquad \text{or} \qquad \overleftrightarrow{d} = \frac{\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}}{2} \tag{2.9}$$

Subject to the approximations of this section (primarily the long-wavelength limit), the detector response to a particular gravitational wave tensor $\overleftrightarrow{h}$ is determined by the detector tensor $\overleftrightarrow{d}$. (We also need to know the location of the detector, in order to tell what phase of the gravitational wave is hitting at what local time.)

2.2 Antenna response functions

If we recall the resolution (1.26) of $\overleftrightarrow{h}$ into a preferred polarization basis, the strain measured by a detector is

$$h = \overleftrightarrow{h}:\overleftrightarrow{d} = (h_+\overleftrightarrow{e}_+ + h_\times\overleftrightarrow{e}_\times):\overleftrightarrow{d} = h_+F_+ + h_\times F_\times \tag{2.10}$$

[6]Note that in this approach we do *not* need to assume the arms are perpendicular, as would be the case if we did everything in a Cartesian coördinate system where $\vec{u}$ had components $(1,0,0)$ and $\vec{v}$ had components $(0,1,0)$.

where the *antenna pattern factors* are given by

$$F_+ = \overleftrightarrow{d} : \overleftrightarrow{e}_+ = d^{ab} e_{+\,ab} \tag{2.11a}$$

$$F_\times = \overleftrightarrow{d} : \overleftrightarrow{e}_\times = d^{ab} e_{\times\,ab} \tag{2.11b}$$

For a given detector at a given time (i.e., for a fixed detector tensor $\overleftrightarrow{d}$), F_+ and $F_\times$ will depend on the three angles defining the sky position and polarization basis with respect to some reference system. E.g., using equatorial coördinates, they will depend on right ascension, declination and the polarization angle ψ. Note that we don't need to define F_+ and $F_\times$ as explicit complicated functions of detector latitude, longitude, etc. The fundamental conceptual definition of the antenna pattern functions is (2.11), and all the rest is just working out the dot products. In particular, we can separate out the dependence on the polarization angle ψ; if we know the sky position of the source, we can construct the reference polarization basis $\{\overleftrightarrow{\varepsilon}_+, \overleftrightarrow{\varepsilon}_\times\}$, and for a given detector at a given sidereal time, we can construct the combinations

$$a = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_+ = d^{ab} \varepsilon_{+\,ab} \tag{2.12a}$$

$$b = \overleftrightarrow{d} : \overleftrightarrow{\varepsilon}_\times = d^{ab} \varepsilon_{\times\,ab} \tag{2.12b}$$

from which we get

$$F_+(\alpha,\delta,\psi) = \quad a(\alpha,\delta)\,\cos 2\psi + b(\alpha,\delta)\,\sin 2\psi \tag{2.13a}$$

$$F_\times(\alpha,\delta,\psi) = -\,a(\alpha,\delta)\,\sin 2\psi + b(\alpha,\delta)\,\cos 2\psi \tag{2.13b}$$

2.3 Exercise: Invariant Combination

Consider the combination

$$F_+^2 + F_\times^2 = a^2 + b^2 \tag{2.14}$$

which is manifestly independent of the polarization angle ψ and therefore the same in any polarization basis. Define a transverse traceless projector

$$P^{\mathrm{TT}\vec{k}ab}{}_{cd} = \frac{1}{2} \sum_{A=+,\times} e_A{}^{ab} e_{A\,cd} = \frac{1}{2} \sum_{A=+,\times} \varepsilon_A{}^{ab} \varepsilon_{A\,cd} \tag{2.15}$$

and show that

$$F_+^2 + F_\times^2 = 2 d_{ab} P^{\mathrm{TT}\vec{k}ab}{}_{cd} d^{cd} \tag{2.16}$$

The projector $P^{\mathrm{TT}\vec{k}ab}{}_{cd}$ picks out the symmetric traceless tensor components transverse to $\vec{k}$. For the case of an interferometer with perpendicular arms, whose detector tensor is given by (2.9), the detector tensor is already traceless and transverse to a vector normal to the plane of the detector. Use this to calculate the maximum value of $F_+^2 + F_\times^2$, which occurs for waves coming from directly overhead or underfoot.

2.4 Earth-fixed basis vectors

A convenient basis for describing ground-based GW detectors is one fixed to the Earth: the unit vector $\vec{e}_3^{\,*}$ points parallel to the Earth's axis, from the center of the Earth to the North Pole. The unit vector $\vec{e}_1^{\,*}$ points from the center of the Earth to the point on the equator with 0° latitude and longitude. This then makes the remaining unit vector $\vec{e}_2^{\,*} = \vec{e}_3^{\,*} \times \vec{e}_1^{\,*}$ point from the center of the Earth to the point on the equator with latitude 0° and longitude 90°E. The asterisk has nothing to do with a complex conjugate, but rather stresses that the basis vectors are co-rotating with the Earth.

2.5 Exercise: Detector Tensor

Consider a detector located at a latitude of 30°N and a longitude of 90°W, with one arm pointing due West and the other arm pointing due South. (This is approximately the configuration of the LIGO Livingston detector, except that the angles are not quite so nice.) Write the components of the following along the co-rotating basis vectors $\{\vec{e}_i^*\}$:

1. The unit vector $\vec{u}$ along the West arm
2. The unit vector $\vec{v}$ along the South arm

Resolve the detector tensor $\overleftrightarrow{d} = \frac{\vec{u}\otimes\vec{u}-\vec{v}\otimes\vec{v}}{2}$ along the unit dyads $\{\vec{e}_i^* \otimes \vec{e}_j^*\}$.

2.6 Equatorial basis vectors

There is a corresponding basis, inertial with respect to the fixed stars: the unit vector $\vec{e}_3$ points to the Celestial North Pole (which means that $\vec{e}_3 = \vec{e}_3^*$). The unit vector $\vec{e}_1$ points to the point with declination $\delta = 0°$ and right ascension $\alpha = 0\,\text{hr}$, i.e., the intersection of the ecliptic with the celestial equator known as the Vernal (March) Equinox. The third unit vector $\vec{e}_2 = \vec{e}_3 \times \vec{e}_1$ thus points to the point with $\delta = 0°$ and right ascension $\alpha = 6\,\text{hr}$.

The relationship between these bases is shown in figure 4. In particular, if we define the angle γ to correspond to the Sidereal Time at the Greenwich Meridian (which increases by 24 hours, i.e., $360° = 2\pi$, every sidereal day of approximately 23 hours and 56 minutes, so that $\gamma = \Omega_\oplus(t - t_{\text{Gmid}})$, where $\Omega_\oplus$ is the Earth's rotation frequency and t_{Gmid} corresponds to sidereal midnight at 0° longitude), then

$$\vec{e}_1^* = \vec{e}_1 \cos\gamma + \vec{e}_2 \sin\gamma \qquad (2.17\text{a})$$
$$\vec{e}_2^* = -\vec{e}_1 \sin\gamma + \vec{e}_2 \cos\gamma \qquad (2.17\text{b})$$

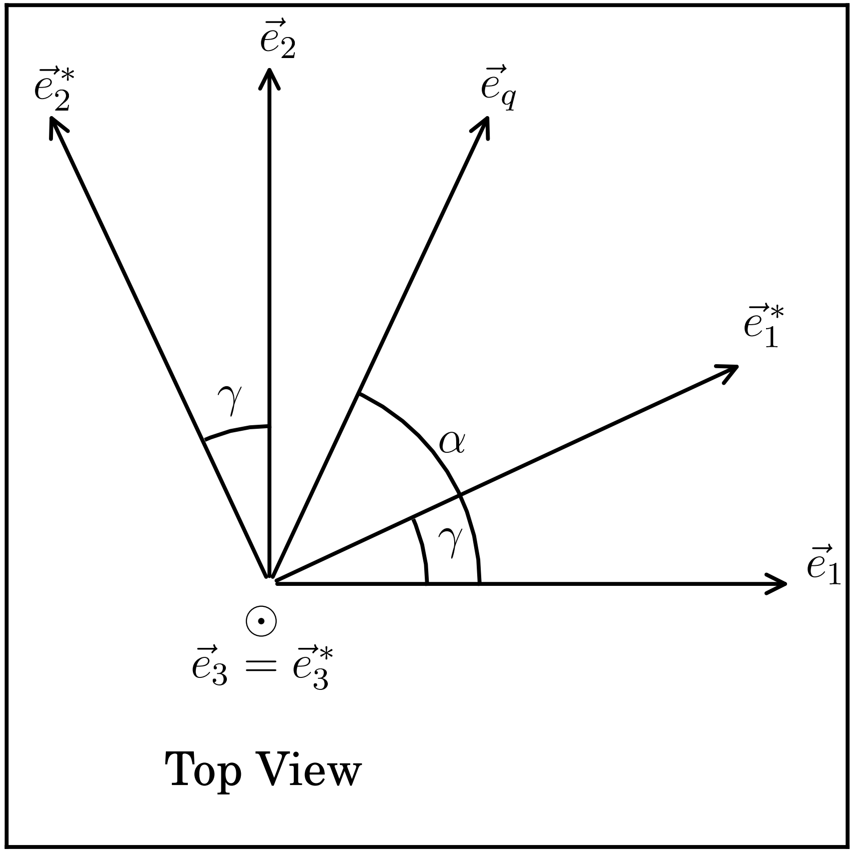

Figure 4: Relationship between the Earth-fixed and inertial bases, and illustration of right ascension. $\vec{e}_3 = \vec{e}_3^*$ points along the Earth's rotation axis towards the North Pole and the Celestial North Pole. $\vec{e}_1^*$ points to 0° latitude and longitude while $\vec{e}_1$ points to $\delta = 0°$ and right ascension $\alpha = 0\,\text{hr}$. As the Earth rotates, the starred unit vectors rotate relative the unstarred ones. $\vec{e}_1^*$ and $\vec{e}_1$ coincide when the sidereal time at Greenwich is midnight. We define the angle γ to be the Greenwich Sidereal Time (GST), which is the angle from $\vec{e}_1$ to $\vec{e}_1^*$, measured counterclockwise around $\vec{e}_3$. The unit vector $\vec{e}_q$ is the projection into the equatorial plane of the vector $\vec{k}$ from the observer to the source, as shown in figure 5.

2.7 Exercise: Basis associated with a sky position

Consider a potential source of gravitational waves with right ascension $\alpha = 0\,\mathrm{hr}$ and declination $\delta = +60°$. Work out the components in the equatorial basis $\{\vec{e}_i\}$ of the propagation vector $\vec{k}$, as well as the vectors $\vec{\imath}$ and $\vec{\jmath}$ which point "West on the sky" and "North on the sky" at that point. Take the dot products $\{\vec{\imath}\cdot\vec{e}_i^{\,*}\}$ and $\{\vec{\jmath}\cdot\vec{e}_i^{\,*}\}$ to find the components of $\vec{\imath}$ and $\vec{\jmath}$ in the Earth-fixed basis, as a function of Greenwich sidereal time γ. These can be used to find the amplitude modulation coëfficients a and b for this source and e.g., the detector of exercise 2.5.

2.8 Sketch of General Calculation

2.8.1 Polarization basis from α, δ, and ψ

To get the antenna response functions for an arbitrary sky point, we need to do the following: calculate the components of $\vec{k}$ in some basis, given α and δ; do likewise for the perpendicular unit vectors $\vec{\imath}$ and $\vec{\jmath}$; find $\vec{\ell}$ and $\vec{m}$ for the given ψ, if desired. We can then construct $\overleftrightarrow{\varepsilon}_{+,\times}$ from $\vec{\imath}$ and $\vec{\jmath}$ or $\overleftrightarrow{e}_{+,\times}$ from $\vec{\ell}$ and $\vec{m}$, as needed.

The first step is to resolve the propagation direction $\vec{k}$ for a given right ascension α and declination δ. First, consider the plane containing $\vec{e}_3$ and $\vec{k}$, as shown in figure 5 Define $\vec{e}_q$ to be the unit vector pointing towards the point on the Celestial Equator with right ascension α, which also lies in the same plane. Since δ is the angle measured up from the Celestial equator to the sky position of the source, we can resolve

$$\vec{k} = -\vec{e}_q \cos\delta - \vec{e}_3 \sin\delta \tag{2.18}$$

To get the components of $\vec{e}_q$ in the equatorial basis, we look at the equatorial plane in figure 4. The right ascension is the angle

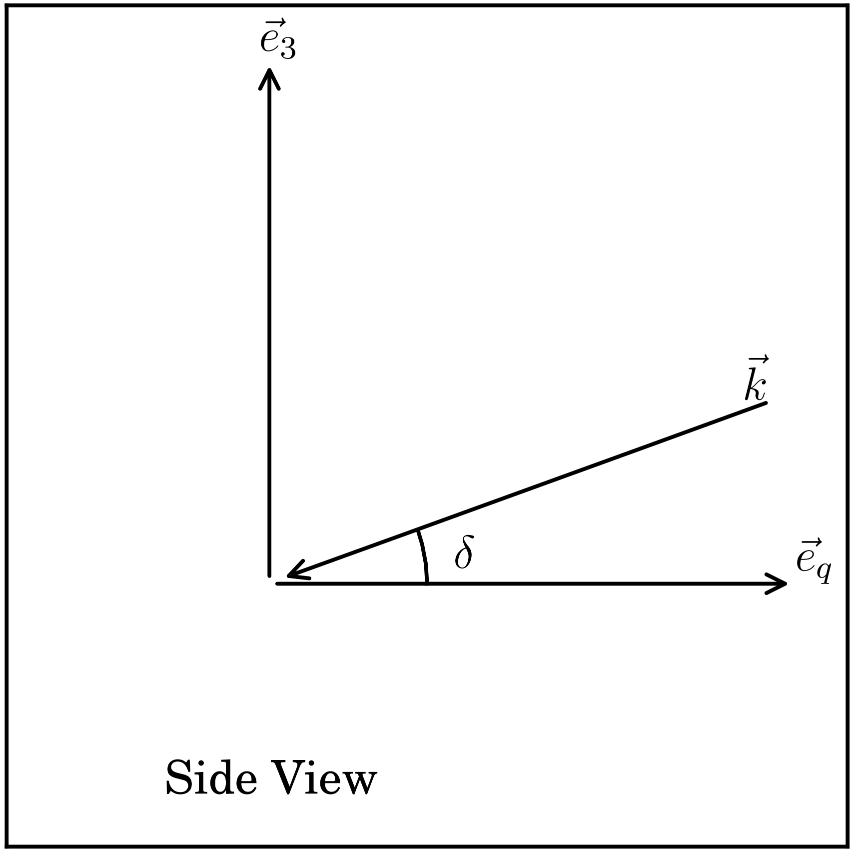

Figure 5: Illustration of declination, in the plane containing the propagation vector $\vec{k}$ and the unit vector $\vec{e}_3$ pointing along the Earth's rotation axis towards the Celestial North Pole. The unit vector $\vec{e}_q$ is the projection into the equatorial plane of the vector $\vec{k}$ from the observer to the source.

around from the Vernal Equinox ($\vec{e}_1$) to the sky position of the source, so

$$\vec{e}_q = \vec{e}_1 \cos\alpha + \vec{e}_2 \sin\alpha \tag{2.19}$$

and thus

$$\vec{k} = -\vec{e}_1 \cos\delta\cos\alpha - \vec{e}_2 \cos\delta\sin\alpha - \vec{e}_3 \sin\delta \tag{2.20}$$

The components along $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ will be constant for a given source. To get the components along the starred unit vectors, we just need to note that the angle from $\vec{e}_1^{\,*}$ to $\vec{e}_q$ is $\alpha - \gamma$.[7] In terms of the starred basis,

$$\vec{e}_q^{\,*} = \vec{e}_1^{\,*} \cos(\alpha-\gamma) + \vec{e}_2^{\,*} \sin(\alpha-\gamma) \tag{2.21}$$

and thus

$$\vec{k} = -\vec{e}_1^{\,*} \cos\delta\cos(\alpha-\gamma) - \vec{e}_2^{\,*} \cos\delta\sin(\alpha-\gamma) - \vec{e}_3^{\,*} \sin\delta \tag{2.22}$$

The calculations of $\vec{\imath}$ and $\vec{\jmath}$ in either basis proceed along similar lines.

Note that to calculate $F_{+,\times} = \overleftrightarrow{d} : \overleftrightarrow{e}_{+,\times}$ it may actually be easier to work out

$$F_+ = \overleftrightarrow{d} : \overleftrightarrow{e}_+ = \vec{\ell}\cdot\overleftrightarrow{d}\cdot\vec{\ell} - \vec{m}\cdot\overleftrightarrow{d}\cdot\vec{m} \tag{2.23a}$$
$$F_\times = \overleftrightarrow{d} : \overleftrightarrow{e}_\times = \vec{\ell}\cdot\overleftrightarrow{d}\cdot\vec{m} + \vec{m}\cdot\overleftrightarrow{d}\cdot\vec{\ell} \tag{2.23b}$$

rather than constructing the matrix representations of $\overleftrightarrow{e}_{+,\times}$ in a particular basis.

[7] Note that in astronomy one defines a quantity called "hour angle" associated with a point on an observer's sky so that the hour angle increases with increasing sidereal time. Since γ is the Greenwich Sidereal Time, $\gamma - \alpha$ is the Greenwich Hour Angle of a source with right ascension α. The combination $\alpha - \gamma$ can thus be referred to as "Minus Greenwich Hour Angle".

2.8.2 Detector tensor from coördinates of detector

Given the coordinates of an Earth-bound detector (latitude, longitude and elevation) and some angles to identify the directions of its arms (usually and azimuth measured clockwise from local North and an altitude angle above or below the local horizontal plane), we can work out the components in the Earth-fixed basis $\{\vec{e}_i^{\,*}\}$ of the unit vectors $\vec{u}$ and $\vec{v}$ along its arms. (Note that we often neglect both the elevation in meters relative to some reference shape of the Earth–sphere or ellipsoid–as well as the altitude angles of the arms relative to the horizontal, since these are both small for practical detectors.) Since there are only a few detectors on the Earth, it's actually usually easiest just to work out the components of $\overleftrightarrow{d}$ in the Earth-fixed basis, once and for all.

The calculation can be done by working out the components of vectors and tensors in the Earth-fixed (starred) or the non-rotating (unstarred) inertial basis by going in two steps:

1. Express the unit vectors $\vec{u}$ and $\vec{v}$ in terms of a local basis $\{\vec{E},\vec{N},\vec{U}\}$ of unit vectors pointing East (along a parallel of latitude), North (along a meridian) and Up (normal to the local reference tangent plane, using the azimuth and possibly altitude angle of each arm.

2. Express the basis vectors $\{\vec{E},\vec{N},\vec{U}\}$, which correspond to a particular latitude and longitude, in terms of the starred or unstarred reference basis.

3 Preferred polarization basis

3.1 The Quadrupole Formula

Gravitational waves from a particular direction can be resolved in different polarization bases, but some make the calculations more convenient than others. For a stochastic background, we're looking at a superposition of different sources, so the result is an unpolarized signal which can be described equally well in any basis. For unmodelled bursts, there's nothing special about the source, but the detector network can pick out a preferred basis for some search methods. For modelled signals such as nearly periodic continuous wave signals, or compact binary inspirals, the geometry of the system provides us with a preferred polarization basis in which the signal description is simple.

Most of the gravitational waves seen by a distant observer, from a typical system, are in the form of quadrupole radiation. The metric perturbation is given by the quadrupole formula as[8]

$$h_{ab} = \frac{2G}{c^4 d} P^{\mathrm{TT}\vec{k}}{}^{cd}_{ab} \ddot{I\!\!\!-}_{cd}(t-d/c) \tag{3.1}$$

where $\vec{k}$ is the direction from the source to the observer and d is the distance. We can accomplish the projection onto transverse traceless states by writing $\overleftrightarrow{h}$ in terms of its plus and cross components as usual:

$$\overleftrightarrow{h} = h_+ \overleftrightarrow{e}_+ + h_\times \overleftrightarrow{e}_\times \tag{3.2}$$

where (3.1) tells us that

$$h_A = \frac{2G}{c^4 d} \frac{\overleftrightarrow{e}_A}{2} : \frac{d^2}{dt^2} \overleftrightarrow{I\!\!\!-}(t-d/c) \qquad A = +, \times \tag{3.3}$$

The $t - d/c$ indicates that the observer is seeing the source as it was at a time in the past, when the gravitational waves now reaching the observer were emitted. Here $\overleftrightarrow{I\!\!\!-}$ is the reduced quadrupole moment defined in MTW equation (36.3):

$$\overleftrightarrow{I\!\!\!-} = \iiint \rho \left(\vec{r} \otimes \vec{r} - \overleftrightarrow{1} \frac{r^2}{3} \right) d^3V \tag{3.4}$$

[8]See for example equation (36.20) of Misner, Thorne and Wheeler, *Gravitation* (1973)

If we recall the standard definition of the moment of inertia tensor $\overleftrightarrow{I}$ from mechanics

$$\overleftrightarrow{I} = \iiint \rho \left(\overleftrightarrow{1} r^2 - \vec{r} \otimes \vec{r} \right) d^3V \tag{3.5}$$

we see that the two are related by

$$I\!\!\!-_{ab} = -P^{\mathrm{T}}{}^{cd}_{ab} I_{cd} \tag{3.6}$$

(note the minus sign!), where $P^{\mathrm{T}}{}^{cd}_{ab}$ is the projection operator onto traceless symmetric tensors:

$$P^{\mathrm{T}}{}^{cd}_{ab} = \frac{1}{2}\left(\delta^c_a \delta^d_b + \delta^d_a \delta^c_b\right) - \frac{1}{3}\delta_{ab}\delta^{cd} \tag{3.7}$$

3.2 Geometry of a non-precessing quasiperiodic source

Consider a mass distribution which is rigidly rotating with constant angular velocity about one of its principal axes of inertia. This could be the nearly-periodic signal given off by a rotating neutron star, or the signal from a binary system where the inspiral is occurring slowly. We can expand the inertia tensor about its principal axes like this:

$$\overleftrightarrow{I} = I_1 \vec{u}_1 \otimes \vec{u}_1 + I_2 \vec{u}_2 \otimes \vec{u}_2 + I_3 \vec{u}_3 \otimes \vec{u}_3 \tag{3.8}$$

In this approximation, I_1, I_2, and I_3 are all constant; if it's rotating about $\vec{u}_3$ with angular speed Ω, then the principal axes can be written with respect to some non-rotating axes $\{\vec{u}^0_i\}$ as

$$\vec{u}_1 = \vec{u}^0_1 \cos\Omega(t-t_0) + \vec{u}^0_2 \sin\Omega(t-t_0) \tag{3.9a}$$
$$\vec{u}_2 = -\vec{u}^0_1 \sin\Omega(t-t_0) + \vec{u}^0_2 \cos\Omega(t-t_0) \tag{3.9b}$$
$$\vec{u}_3 = \vec{u}_3 \tag{3.9c}$$

It's not hard to work out the time derivatives of the basis vectors along the principal axes due to the rotation:

$$\frac{d\vec{u}_1}{dt} = \Omega\vec{u}_3 \times \vec{u}_1 = \Omega\vec{u}_2 \tag{3.10a}$$
$$\frac{d\vec{u}_2}{dt} = \Omega\vec{u}_3 \times \vec{u}_2 = -\Omega\vec{u}_1 \tag{3.10b}$$
$$\frac{d\vec{u}_3}{dt} = \Omega\vec{u}_3 \times \vec{u}_3 = \vec{0} \tag{3.10c}$$

This means that

$$\begin{aligned}\frac{d}{dt}\overleftrightarrow{I} &= \Omega I_1(\vec{u}_1 \otimes \vec{u}_2 + \vec{u}_2 \otimes \vec{u}_1) - \Omega I_2(\vec{u}_2 \otimes \vec{u}_1 + \vec{u}_1 \otimes \vec{u}_2) \\ &= \Omega(I_1 - I_2)(\vec{u}_1 \otimes \vec{u}_2 + \vec{u}_2 \otimes \vec{u}_1)\end{aligned} \tag{3.11}$$

and

$$\frac{d^2}{dt^2}\overleftrightarrow{I} = -2\Omega^2(I_1 - I_2)(\vec{u}_1 \otimes \vec{u}_1 - \vec{u}_2 \otimes \vec{u}_2) \tag{3.12}$$

Since this is already traceless, (3.6) tells us that

$$\frac{d^2}{dt^2}\overleftrightarrow{\not{I}} = 2\Omega^2(I_1 - I_2)(\vec{u}_1 \otimes \vec{u}_1 - \vec{u}_2 \otimes \vec{u}_2) \tag{3.13}$$

To get the explicit time dependence of (3.13), we could substitute the explicit time-dependent forms of $\vec{u}_1$ and $\vec{u}_2$ into (3.13), but it's easier to note that the combination $\vec{u}_1 \otimes \vec{u}_1 - \vec{u}_2 \otimes \vec{u}_2$ appearing in (3.13) is a traceless tensor transverse to $\vec{u}_3$ and so if we define basis tensors[9]

$$\overleftrightarrow{E}_+ = \vec{u}_1^0 \otimes \vec{u}_1^0 - \vec{u}_2^0 \otimes \vec{u}_2^0 \tag{3.14a}$$
$$\overleftrightarrow{E}_\times = \vec{u}_1^0 \otimes \vec{u}_2^0 + \vec{u}_2^0 \otimes \vec{u}_1^0 \tag{3.14b}$$

[9]Note that these are traceless tensors transverse to $\vec{u}_3$ and *not* to the propagation direction $\vec{k}$.

then, by analogy to the polarization rotation in section 1.3, we have

$$\vec{u}_1 \otimes \vec{u}_1 - \vec{u}_2 \otimes \vec{u}_2 = \overleftrightarrow{E}_+ \cos 2\Omega(t - t_0) + \overleftrightarrow{E}_\times \sin 2\Omega(t - t_0) \tag{3.15}$$

and

$$\frac{d^2}{dt^2}\overleftrightarrow{\not{I}} = 2\Omega^2(I_1 - I_2)\left(\overleftrightarrow{E}_+ \cos[\Phi(t) + \phi_0] + \overleftrightarrow{E}_\times \sin[\Phi(t) + \phi_0]\right) \tag{3.16}$$

where

$$\Phi(t) = 2\Omega(t - d/c) \tag{3.17}$$

and

$$\phi_0 = -2\Omega t_0 \tag{3.18}$$

This means that the polarization components of the gravitational wave travelling in a particular direction are

$$h_A = \frac{4G\Omega^2(I_1 - I_2)}{c^4 d}\left[\frac{\overleftrightarrow{e}_A : \overleftrightarrow{E}_+}{2}\cos[\Phi(t) + \phi_0] + \frac{\overleftrightarrow{e}_A : \overleftrightarrow{E}_\times}{2}\sin[\Phi(t) + \phi_0]\right] \tag{3.19}$$

So far, we haven't specified the non-rotating basis vectors $\vec{u}_1^0$ and $\vec{u}_2^0$, perpendicular to $\vec{u}_3 = \vec{u}_3^0$ (which, incidentally, determine ϕ_0), nor the basis vectors $\vec{\ell}$ and $\vec{m}$, perpendicular to $\vec{k}$, which define the polarization basis. We can do this by picking $\vec{\ell} = \vec{u}_1^0$ along the line of nodes, which is perpendicular to both the propagation direction $\vec{k}$ and the system angular momentum direction $\vec{u}_3$. If ι is the inclination angle between the angular momentum direction $\vec{u}$ and the propagation vector $\vec{k}$, as shown in figure 6, the dot products between the basis vectors defining $\{\overleftrightarrow{e}_A\}$ and $\{\overleftrightarrow{E}_A\}$ are

$$\vec{\ell} \cdot \vec{u}_1^0 = 1 \qquad \vec{m} \cdot \vec{u}_1^0 = 0 \qquad \vec{\ell} \cdot \vec{u}_2^0 = 0 \qquad \vec{m} \cdot \vec{u}_2^0 = \cos\iota \tag{3.20}$$

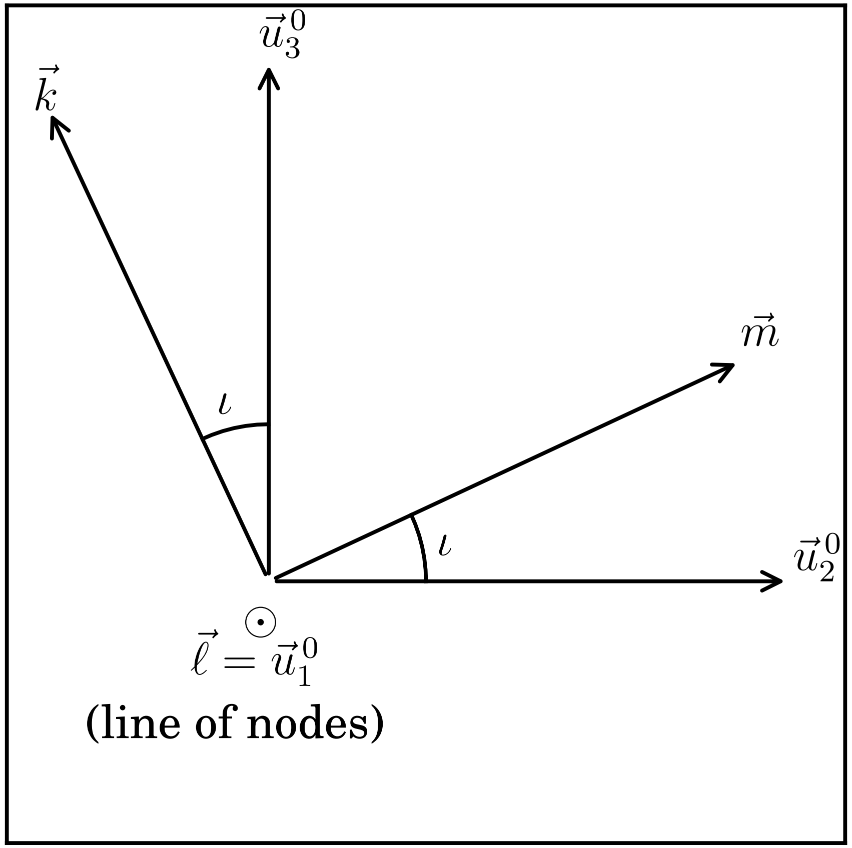

Figure 6: Illustration of bases associated with a rotating gravitational-wave source and its propagation. The unit vector $\vec{k}$ points from the source to the observer, and $\vec{u}_3^0$ points along the axis of rotation; the angle between these is the inclination ι. The preferred polarization basis (see figure 2) is constructed by choosing $\vec{\ell}$ to be along the line of nodes, perpendicular to both $\vec{k}$ and $\vec{u}_3^0$.

3.3 Exercise: Projection from source to propagation basis

Use the inner products (3.20) to show that

$$\overleftrightarrow{e}_+ : \overleftrightarrow{E}_+ = 1 + \cos^2\iota \quad \text{and} \quad \overleftrightarrow{e}_+ : \overleftrightarrow{E}_\times = 0 \tag{3.21a}$$

$$\overleftrightarrow{e}_\times : \overleftrightarrow{E}_+ = 0 \quad \text{and} \quad \overleftrightarrow{e}_\times : \overleftrightarrow{E}_\times = 2\cos\iota \tag{3.21b}$$

3.4 Waveform in preferred basis

This then means that, in the preferred basis,

$$h_+ = h_0 \frac{1+\cos^2\iota}{2} \cos[\Phi(t) + \phi_0] \tag{3.22a}$$

$$h_\times = h_0 \cos\iota \, \sin[\Phi(t) + \phi_0] \tag{3.22b}$$

where the GW amplitude is

$$h_0 = \frac{4G\Omega^2(I_1 - I_2)}{c^4 d} \tag{3.23}$$

Placing the basis vector $\vec{\ell}$ along the line of nodes means that this preferred polarization basis has ψ as the angle from "West on the sky" to the line of nodes, i.e., if we're talking about objects moving in circular orbits, this is the angle from "West on the sky" to the long axis of the projected orbit. The nice feature of the preferred basis is that h_+ and $h_\times$ are a quarter-cycle out of phase, as illustrated in `http://ccrg.rit.edu/~whelan/gwmovie/`

3.5 Exercise: slow inspiral

The waveform (3.22) also applies to the early stages of a binary inspiral. Consider two objects of mass m_1 and m_2, total mass M, and reduced mass $m_1 m_2/M$. If the trajectories are

$$\vec{r}_1(t) = \frac{m_2}{M}\vec{r}(t) \tag{3.24a}$$
$$\vec{r}_2(t) = -\frac{m_1}{M}\vec{r}(t) \tag{3.24b}$$

where

$$\vec{r}(t) = r(t)\cos\phi(t)\vec{u}_1^0 + r(t)\sin\phi(t)\vec{u}_2^0 \tag{3.25}$$

show that

$$\frac{d^2}{dt^2}\overleftrightarrow{I} \approx -\mu r^2\dot{\phi}^2\left(\vec{u}_1\otimes\vec{u}_1 - \vec{u}_2\otimes\vec{u}_2\right) \tag{3.26}$$

What conditions are needed on $\dot{r}$, $\ddot{r}$, and $\ddot{\phi}$ in order to make this approximation valid?

3.6 "Effective distance" for inspiral signals

If we compare the form (3.26) to (3.13), we see that we've just replaced $2\Omega^2(I_1 - I_2)$ with $-\mu r^2\dot{\phi}^2$ and therefore by comparison to (3.22) we can write down

$$h_+ \approx -\frac{A(t)}{d}\frac{1+\cos^2\iota}{2}\cos\Phi(t) \tag{3.27a}$$
$$h_\times \approx -\frac{A(t)}{d}\cos\iota\,\sin\Phi(t) \tag{3.27b}$$

where we've collected together the part of the amplitude

$$A(t) = \left(\frac{4G\mu[r(t-d/c)\dot{\phi}(t-d/c)]^2}{c^4}\right) \tag{3.28}$$

which depends only on properties of the source like masses and trajectory.[10] If we think about the signal generated in a detector with antenna pattern factors F_+ and $F_\times$, we get

$$\begin{aligned} h(t) &= \frac{A(t)}{d}\left[-F_+\frac{1+\cos^2\iota}{2}\cos\Phi(t) - F_\times\cos\iota\sin\Phi(t)\right] \\ &= \frac{A(t)}{d}\left(\sqrt{F_+^2\frac{(1+\cos^2\iota)^2}{4} + F_\times^2\cos^2\iota}\right)\cos[\Phi(t)-\Psi] \end{aligned} \tag{3.29}$$

where we have used the usual trick of rewriting

$$\alpha\cos\varphi + \beta\sin\varphi = \gamma\cos(\varphi - \psi) \tag{3.30}$$

where

$$\alpha = \gamma\cos\psi \qquad \text{and} \qquad \beta = \gamma\sin\psi \tag{3.31}$$

so

$$\gamma = \sqrt{\alpha^2+\beta^2} \tag{3.32}$$

So we see that the overall amplitude is determined by the distance to the source, but in this slow-inspiral approximation, that distance appears together with the observing geometry in a combination known as *effective distance*:

$$d_{\text{eff}} = \frac{d}{\sqrt{F_+^2\frac{(1+\cos^2\iota)^2}{4} + F_\times^2\cos^2\iota}} \tag{3.33}$$

Note that the factor in the square root is a maximum (and therefore the effective distance corresponding to a given physical distance) when $|\cos\iota| = 1$, i.e., we are seeing the binary face on ($\iota = 0$ or $\iota = \pi$). Using the result of section 2.3, that $F_+^2 + F_\times^2 \leq 1$, we have

$$\left(\frac{d}{d_{\text{eff}}}\right)^2 = F_+^2\frac{(1+\cos^2\iota)^2}{4} + F_\times^2\cos^2\iota \leq F_+^2 + F_\times^2 \leq 1 \tag{3.34}$$

which means that $d_{\text{eff}} \geq d$. The effective distance equals the physical distance if the binary is seen face-on, and is at the detector's zenith or nadir.

[10] There is still some dependence on distance to the source, of course, but only in terms of when the signal arrives, not in the overall magnitude.