Searches for a Stochastic Gravitational-Wave Background

John T. Whelan

School of Gravitational Waves, Warsaw, 2013 July 5

Contents

1 Stochastic GW Backgrounds

Consider a superposition of many weak gravitational-wave sources. This may be of cosmological origin, associated with events in the early universe (inflation, phase transition, primordial gravitons, ...), or some unresolved astrophysical source, such as millions of white-dwarf binaries in our galaxy. The individual signals may not be detectable, but their combined effect would produce a random signal in gravitational wave detectors, analogous to the cosmic microwave background first observed by Penzias and Wilson.[1] Unlike Penzias and Wilson, we can't "point our detectors away from the sky", but we can distinguish a random GW signal from random instrumental noise, because of the correlations it would produce between the outputs of different detectors.

We can describe any superposition of gravitational waves by expanding it as a superposition of plane waves along each propagation vector $\vec{k}$:

$$\overleftrightarrow{h}(\vec{r},t) = \sum_{A=+,\times} \int_{-\infty}^{\infty} df \iint d^2\Omega_{\vec{k}}\, h_A(f,\vec{k})\, \overleftrightarrow{e}_A(\vec{k}) \exp\left(i2\pi f\left[t-\frac{\vec{k}\cdot\vec{r}}{c}\right]\right) \tag{1.1}$$

The statistical properties of $h_A(f,\vec{k})$ describe the nature of the stochastic background. Since it's a superposition of many individual signals, it's reasonable to assume it's Gaussian, stationary, and unpolarized, so that it's defined by its mean

$$E\left[h_A(f,\vec{k})\right] = 0 \tag{1.2}$$

and variance

$$E\left[h_A^*(f,\vec{k})\, h_{A'}(f',\vec{k}')\right] = \delta^2(\vec{k},\vec{k}')\,\delta_{AA'}\,\delta(f-f')\,H(f,\vec{k}) \tag{1.3}$$

[1] A. A. Penzias and R. W. Wilson, *ApJ* **142**, 419 (1965)

Specifically,

$$P(h) \propto \exp\left(-\frac{1}{2}\sum_{A=+,\times}\int_{-\infty}^{\infty} df \iint d^2\Omega_{\vec{k}}\, \frac{|h_A(f,\vec{k})|^2}{H(f,\vec{k})}\right) \tag{1.4}$$

If we consider the stochastic signal $h^X(t) = \overleftrightarrow{h}(t) : \overleftrightarrow{d}^X$ appearing in a detector X, it will also be Gaussian with zero mean; the covariance between data in detectors X and Y will be

$$\begin{aligned} E\left[\widetilde{h}^X(f')^*\widetilde{h}^Y(f)\right] = \sum_{A=+,\times}\iint d^2\Omega_{\vec{k}}\, H(f,\vec{k})\, F_A^X(\vec{k})F_A^Y(\vec{k}) \\ \times \exp\left(i2\pi f\left[\frac{-\vec{k}\cdot(\vec{r}^X-\vec{r}^Y)}{c}\right]\right)\delta(f-f') \end{aligned} \tag{1.5}$$

We can gain some insight into this if we write

$$\begin{aligned} \sum_{A=+,\times} F_A^X(\vec{k})F_A^Y(\vec{k}) &= \sum_{A=+,\times} d^X_{ab}e_A^{ab}(\vec{k})\, e_A^{cd}(\vec{k})\, d^Y_{cd} \\ &= 2d^X_{ab}\, P^{\mathrm{TT}\vec{k}ab}{}_{cd}\, d^{Y\,cd} \end{aligned} \tag{1.6}$$

where

$$P^{\mathrm{TT}\vec{k}ab}{}_{cd} = \frac{1}{2}\sum_{A=+,\times} e_A^{ab}(\vec{k})\, e_{A\,cd}(\vec{k}) \tag{1.7}$$

is an operator which projects onto the subspace of traceless, symmetric tensors transverse to the unit vector $\vec{k}$.

1.1 Exercise

Show that this is a projection operator, i.e., $P^{\mathrm{TT}\vec{k}ab}{}_{ef}P^{\mathrm{TT}\vec{k}ef}{}_{cd} = P^{\mathrm{TT}\vec{k}ab}{}_{cd}$, by using the normalization $e_A^{ab}(\vec{k})\, e_{B\,ab}(\vec{k}) = 2\delta_{AB}$ of the standard polarization basis tensors (see yesterday's lecture). What is the trace $P^{\mathrm{TT}\vec{k}ab}{}_{ab}$?

1.2 Spatial Distributions

The simplest signal geometry for a stochastic background is isotropic, so that $H(f,\vec{k}) = H(f)$. A slightly less specific assumption (although not fully general) is that we're interested in a background distributed in some way across the sky, whose spectrum is the same in each direction, i.e., $H(f,\vec{k}) = H(f)\mathcal{P}(\vec{k})$. There are several different strategies that are taken to address the direction dependence of a stochastic background:

- Search only for an isotropic background.
- Search for a background with a specified sky distribution, e.g., spread across a nearby galaxy cluster, or concentrated at a point.
- Attempt to reconstruct a sky map, e.g., by measuring the power in different spherical harmonics.

In this lecture we'll focus on the isotropic case, although the notes will include some formulas involving $\mathcal{P}(\vec{k})$.

If the spectrum factors, the correlation between different detectors is

$$E\left[\widetilde{h}^X(f')^*\widetilde{h}^Y(f)\right] = \gamma^{XY}(f)\frac{S_{\mathrm{gw}}(f)}{2}\delta(f-f') \tag{1.8}$$

where

$$\gamma^{XY}(f) = d^X_{ab}\, d^{Y\,cd}\, \frac{5}{4\pi}\iint d^2\Omega_{\vec{k}}\, \mathcal{P}(\vec{k})\, P^{\mathrm{TT}\vec{k}ab}{}_{cd}\, e^{-i2\pi f\vec{k}\cdot(\vec{r}^X-\vec{r}^Y)/c} \tag{1.9}$$

is known as the overlap reduction function, and

$$S_{\mathrm{gw}}(f) = \frac{16\pi}{5}H(f) \tag{1.10}$$

This normalization is chosen because, in the isotropic case where

$$\gamma^{XY}(f) = d^X_{ab}\, d^{Y\,cd}\, \frac{5}{4\pi} \iint d^2\Omega_{\vec{k}}\, P^{\mathrm{TT}\vec{k}ab}{}_{cd}\, e^{-i2\pi f \vec{k}\cdot(\vec{r}^X - \vec{r}^Y)/c} \tag{1.11}$$

the overlap reduction function for a detector with itself is

$$\gamma^{XX}(f) = d^X_{ab}\, d^{X\,cd}\, \frac{5}{4\pi} \iint d^2\Omega_{\vec{k}}\, P^{\mathrm{TT}\vec{k}ab}{}_{cd} = 2 d^X_{ab}\, d^{X\,ab} \tag{1.12}$$

For a standard interferometer with perpendicular arms, this is just 1, and so $S_{\mathrm{gw}}(f)$ is the contribution to the power spectrum in the detector due to the stochastic gravitational wave background.

1.3 Exercise

Show that

$$\frac{5}{4\pi} \iint d^2\Omega_{\vec{k}} P^{\mathrm{TT}\vec{k}ab}{}_{cd} = 2 P^{\mathrm{T}ab}{}_{cd} \tag{1.13}$$

where $P^{\mathrm{T}ab}{}_{cd}$ is a projector onto transverse symmetric tensors, in the following way

1. Argue by symmetry that it must be a constant times $P^{\mathrm{T}ab}{}_{cd}$.
2. Show that that constant has to be 2 by taking the trace of both sides of the equation

$$\frac{5}{4\pi} \iint d^2\Omega_{\vec{k}} P^{\mathrm{TT}\vec{k}ab}{}_{cd} \propto P^{\mathrm{T}ab}{}_{cd} \tag{1.14}$$

1.4 Aside: $\Omega_{\mathrm{gw}}(f)$

The spectrum of a gravitational wave background is often described in terms of the contribution to the cosmological parameter $\Omega = \rho/\rho_{\mathrm{crit}}$ where

$$\rho_{\mathrm{crit}} = \frac{3H_0^2}{8\pi G} \tag{1.15}$$

using the fact that the energy density in gravitational waves is

$$\begin{aligned}\rho_{\mathrm{gw}} &= \frac{c^2}{32\pi G} \langle \dot{h}_{ab}(t,\vec{r}) \dot{h}^{ab}(t,\vec{r}) \rangle = \frac{\pi c^2}{G} \int_0^\infty f^2 \iint d^2\Omega_{\vec{k}}\, H(f,\vec{k}) \\ &= \frac{4\pi^2 c^2}{G} \int_0^\infty f^2\, H(f)\, df\end{aligned} \tag{1.16}$$

The usual definition is the logarithmic energy density

$$\Omega_{\mathrm{gw}}(f) = \frac{f}{\rho_{\mathrm{crit}}} \frac{d\rho_{\mathrm{gw}}}{df} = \frac{32\pi^2}{3H_0^2} f^3 H(f) = \frac{10\pi}{3H_0^2} f^3 S_{\mathrm{gw}}(f) \tag{1.17}$$

This is of interest because some cosmological models (e.g., slow-roll inflation) predict a spectrum which corresponds to a constant $\Omega_{\mathrm{gw}}(f)$, but we will work in terms of $S_{\mathrm{gw}}(f)$ in this lecture because

1. The equations are simpler in terms of $S_{\mathrm{gw}}(f)$
2. $\Omega_{\mathrm{gw}}(f)$ depends on the (experimentally uncertain) value of the Hubble constant H_0

1.5 Overlap Reduction Function

Restricting attention now to an isotropic background, consider the overlap reduction function This normalization is chosen because, in the isotropic case where

$$\gamma^{XY}(f) = d^X_{ab}\, d^{X\,cd}\, \frac{5}{4\pi} \iint d^2\Omega_{\vec{k}}\, P^{\mathrm{TT}\vec{k}ab}{}_{cd}\, e^{-i2\pi f \vec{k}\cdot(\vec{r}^Y - \vec{r}^X)/c} \tag{1.18}$$

We've seen that it is equal to unity when X and Y refer to the same interferometric detector (as long as the arms are perpendicular). For any pair of detectors, it's a specific function of frequency, and in particular won't depend on when the observation is done. (Isotropy means that the rotation of the Earth doesn't change the observing geometry.) There are thus only $N_{\text{sites}}(N_{\text{sites}}-1)/2$ different functions to be worked out, where N_{sites} is the number of detector sites (e.g., in the initial detector era, $N_{\text{sites}}=4$: LIGO Hanford, LIGO Livingston, GEO and Virgo[2]; for the advanced detector era, we can add KAGRA and LIGO India.) It might seem like each of these just requires a numerical integration over the sky (propagation direction $\vec{k}$), but its even easier than that, since it's possible to work out explicit formulas for $\gamma^{XY}(f)$ in terms of the detector tensors and the separation vectors of the detectors.[3] Some examples are plotted in figure 1.

2 Data Analysis Method

2.1 Likelihood Ratio

We've described a signal model in which the signal contribution $\widetilde{h}^X(f)$ in the Fourier domain to detector X's output is Gaussian, with $E\left[\widetilde{h}(f)\right]=0$ and

$$E\left[\widetilde{h}^X(f')^*\widetilde{h}^Y(f)\right]=\gamma^{XY}(f)\frac{S_{\text{gw}}(f)}{2}\delta(f-f') \tag{2.1}$$

If we want to talk about breaking the data up into intervals of duration T labelled by I before Fourier transforming, this becomes

$$E\left[\widetilde{h}^X_I(f')^*\widetilde{h}^Y_J(f)\right]=\delta_{IJ}\gamma^{XY}(f)\frac{S_{\text{gw}}(f)}{2}\delta(f-f') \tag{2.2}$$

Note that for an isotropic background, the overlap reduction function does not change with time and therefore no additional I subscript is needed on $\gamma^{XY}(f)$.

[2] Also TAMA, depending on when you define the era. Resonant bar detectors can also be added to the picture, and were.

[3] See e.g., B. Allen and J. D. Romano *PRD* **59**, 102001 (1999) or J. T. Whelan *CQG* **23**, 1181 (2006).

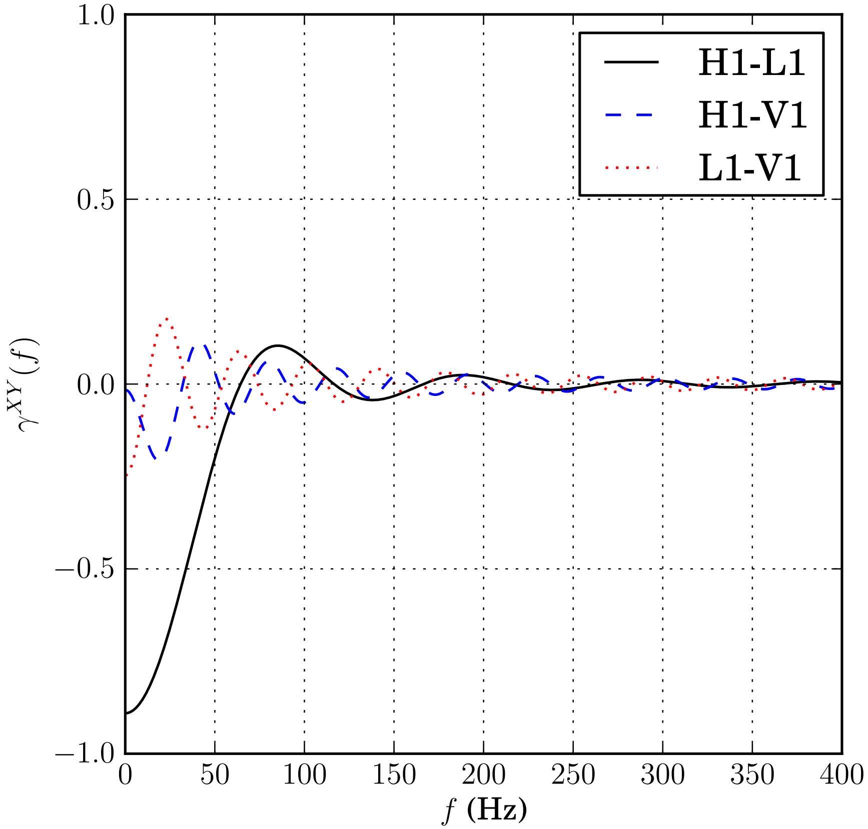

Figure 1: Plots of the isotropic overlap reduction function $\gamma^{XY}(f)$ for pairs of detectors among LIGO Hanford (H1), LIGO Livingston (L1) and Virgo. The overlap reduction function tends to oscillate and decrease in amplitude with increasing frequency, as correlations and anti-correlations of waves from different directions cancel. See Cella et al, *CQG*, **24**, S639 (2007) for more discussion.

If we assume that the data $\widetilde{x}_I^X(f)$ consist of this signal plus instrumental noise $\widetilde{n}_I^X(f)$ which is assumed to be zero-mean, independent between detectors and Gaussian with a one-sided power spectrum $S_I^X(f)$, the detector output will also be Gaussian, with $E\left[\widetilde{x}_I^X(f)\right] = 0$, and

$$E\left[\widetilde{x}_I^X(f')^* \widetilde{x}_J^Y(f)\right] = \delta_{IJ} \frac{S_I^{XY}(f)}{2} \delta(f - f') \tag{2.3}$$

with

$$S_I^{XY}(f) = \delta^{XY}\, S_I^X(f) + \gamma^{XY}(f)\, S_{\text{gw}}(f) \tag{2.4}$$

We'd like to construct a likelihood ratio $\frac{P(x|\mathcal{H}_s)}{P(x|\mathcal{H}_g)}$ between a model consisting of a stochastic signal plus noise to one with just Gaussian noise. For a given spectrum $S_{\text{gw}}(f)$, our assumptions tell us

$$P(x|\mathcal{H}_s, S_{\text{gw}}(f))) \propto \exp\left(-2 \int_0^{f_{\text{Ny}}} df \sum_I \sum_X \sum_Y \widetilde{x}_I^X(f)^*\, S^{-1}_{I\,XY}(f)\, \widetilde{x}_J^Y(f)\right) \tag{2.5}$$

where $S^{-1}_{I\,XY}(f)$ is the matrix inverse of $S_I^{XY}(f)$. If the spectrum in each detector is dominated by the noise, i.e., $S_{\text{gw}}(f) \ll S_I^X(f)$, we can use

$$S_I^{XY}(f) = \sqrt{S_I^X(f)} \left(\delta^{XY} + \frac{\gamma^{XY}(f)\, S_{\text{gw}}(f)}{\sqrt{S_I^X(f) S_I^Y(f)}}\right) \sqrt{S_I^Y(f)} \tag{2.6}$$

to approximate

$$\begin{aligned} S^{-1}_{I\,XY}(f) &\approx \frac{1}{\sqrt{S_I^X(f)}} \left(\delta^{XY} - \frac{\gamma^{XY}(f)\, S_{\text{gw}}(f)}{\sqrt{S_I^X(f) S_I^Y(f)}}\right) \frac{1}{\sqrt{S_I^Y(f)}} \\ &= \frac{\delta^{XY}}{S_I^X(f)} - \frac{\gamma^{XY}(f) S_{\text{gw}}(f)}{S_I^X(f) S_I^Y(f)} \end{aligned} \tag{2.7}$$

In this approximation, the logarithm of the likelihood ratio is

$$\ln \frac{P(x|\mathcal{H}_s, S_{\text{gw}}(f))}{P(x|\mathcal{H}_g)} \approx 2 \sum_I \sum_X \sum_Y \int_0^{f_{\text{Ny}}} df\, \frac{\gamma^{XY}(f) S_{\text{gw}}(f)}{S_I^X(f) S_I^Y(f)}\, \widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f) \tag{2.8}$$

We might want to consider a whole bunch of signal hypotheses with different spectra $S_{\text{gw}}(f)$, but for simplicity we know the shape of the spectrum so that $S_{\text{gw}}(f) = S_R\, \mathcal{S}(f)$ where $\mathcal{S}(f)$ is some known function over the frequencies of interest (e.g., constant or proportional to f^{-3}, the latter corresponding to constant $\Omega_{\text{gw}}(f)$). Then

$$\ln \frac{P(x|\mathcal{H}_s, S_R)}{P(x|\mathcal{H}_g)} \approx 2 S_R \sum_I \sum_X \sum_Y \int_0^{f_{\text{Ny}}} df\, \frac{\gamma^{XY}(f) \mathcal{S}(f)}{S_I^X(f) S_I^Y(f)}\, \widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f) \tag{2.9}$$

2.2 Cross-Correlation Statistic

The standard stochastic background search doesn't quite use this as a detection statistic, though. That's because the log likelihood ratio includes terms with $X = Y$. These autocorrelation terms represent the contribution of the stochastic GW background to the power in each detector. While including them would make the search more sensitive if all of the assumptions that went into constructing (2.9) were true, it relies on the assumption that the instrumental noise in each detector is Gaussian. Rather than try to construct a more sophisticated statistic involving a more realistic noise hypothesis, in practice we just leave out those autocorrelation terms and construct a cross-correlation statistic

$$\mathcal{Y} = \sum_{X>Y} \sum_Y \sum_I \mathcal{Y}_{IXY} \tag{2.10}$$

with

$$\mathcal{Y}_{IXY} = 2 \int_0^{f_{\text{Ny}}} df\, \frac{\gamma^{XY}(f) \mathcal{S}(f)}{S_I^X(f) S_I^Y(f)}\, \widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f) \tag{2.11}$$

The analysis method then constructs the posterior pdf for the stochastic background strength S_R according to Bayes's theorem

$$P(S_R|\mathcal{Y}) = \frac{P(\mathcal{Y}|S_R)P(S_R)}{P(\mathcal{Y})} \tag{2.12}$$

where the prior $P(S_R)$ is taken to be uniform for $0 < S_R < S_{\max}$, with $S_{\max}$ large enough not to influence the construction, and the normalization $\frac{1}{P(\mathcal{Y})}$ calculated by requiring that $\int_0^{S_{\max}} P(S_R|\mathcal{Y}) = 1$.

Rather than worry about the actual distribution $P(\mathcal{Y}|S_R)$, we use the fact that $\mathcal{Y}$ is a sum of contributions from many times and frequencies to invoke the central limit theorem, and approximate it as Gaussian, so we just need its mean and variance. We calculate the mean using

$$E\left[\mathcal{Y}_{IXY}\right] = 2\int_0^{f_{\mathrm{Ny}}} df\, \frac{\gamma^{XY}(f)\mathcal{S}(f)}{S_I^X(f)S_I^Y(f)}\, E\left[\widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f)\right] \tag{2.13}$$

This seems like a bit of a problem, because taking (2.3) literally (and recalling $X \neq Y$ would seem to indicate

$$E\left[\widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f)\right] = \gamma^{XY}(f)\, \frac{S_{\mathrm{gw}}(f)}{2}\, \delta(0) \tag{2.14}$$

which is infinite. But a careful treatment notes that since $\widetilde{x}_I^X(f)$ is not literally a continuous Fourier transform, only a finite-time approximation of one, we really should have put in the finite-time approximation of the dirac delta, and we should replace $\delta(0)$ with the time baseline for the Fourier transform, $\frac{1}{\delta f} = T$. Thus

$$E\left[\mathcal{Y}_{IXY}\right] = T\, S_R \int_0^{f_{\mathrm{Ny}}} df\, \frac{[\gamma^{XY}(f)\mathcal{S}(f)]^2}{S_I^X(f)S_I^Y(f)} \tag{2.15}$$

The calculation of the variance is simpler if we assume as usual that $S_{\mathrm{gw}}(f) \ll S_I^X(f)$, leaving us with only

$$\begin{aligned} E\left[(\mathcal{Y}_{IXY})^2\right] &\approx T\int_0^{f_{\mathrm{Ny}}} df\, \left(\frac{\gamma^{XY}(f)\mathcal{S}(f)}{S_I^X(f)S_I^Y(f)}\right)^2\, S_I^X(f)S_I^Y(f) \\ &= T\int_0^{f_{\mathrm{Ny}}} df\, \frac{[\gamma^{XY}(f)\mathcal{S}(f)]^2}{S_I^X(f)S_I^Y(f)} \end{aligned} \tag{2.16}$$

where we have used the fact that

$$E\left[\widetilde{x}_I^X(f)^*\, \widetilde{x}_I^Y(f)\widetilde{x}_I^X(f')\, \widetilde{x}_I^Y(f')^*\right] = \frac{S_I^X(f)}{2}\delta(f-f')\frac{S_I^Y(f)}{2}\delta(f-f') \tag{2.17}$$

The mean and variance of the statistic $\mathcal{Y}$ are thus

$$E\left[\mathcal{Y}\right] = S_R\mathcal{I} \qquad E\left[\mathcal{Y}^2\right] = \mathcal{I} \tag{2.18}$$

where

$$\mathcal{I} = \sum_I \sum_{X>Y} \sum_Y T\int_0^{f_{\mathrm{Ny}}} df\, \frac{[\gamma^{XY}(f)\mathcal{S}(f)]^2}{S_I^X(f)S_I^Y(f)} \tag{2.19}$$

This integral encapsulates the sensitivity of the search, since

$$P(\mathcal{Y}|S_R) = \frac{1}{\sqrt{2\pi\mathcal{I}}}\exp\left(-\frac{1}{2}\frac{(\mathcal{Y}-S_R\mathcal{I})^2}{\mathcal{I}}\right) \tag{2.20}$$

and the posterior becomes

$$P(S_R|\mathcal{Y}) = \frac{e^{-\mathcal{I}(S_R-\mathcal{Y}/\mathcal{I})^2/2}}{\int_0^\infty dS'_R\, e^{-\mathcal{I}(S'_R-\mathcal{Y}/\mathcal{I})^2/2}} \tag{2.21}$$

2.3 Exercise: Relation to standard formulas

This is not actually the way things are usually written; instead one normalizes a statistic $\mathcal{E}_{IXY} = \mathcal{N}_{IXY}\mathcal{Y}_{IXY}$ so that $E\left[\mathcal{E}_{IXY}\right] = S_R$ and $E\left[\mathcal{E}_{IXY}\right]^2 = \sigma_{IXY}^2$ and then constructs an optimal estimator

$$\mathcal{E} = \frac{\sum_I \sum_{X>Y} \sum_Y \sigma_{IXY}^{-2}\mathcal{E}_{IXY}}{\sum_I \sum_{X>Y} \sum_Y \sigma_{IXY}^{-2}} \tag{2.22}$$

Show that $\mathcal{E} = \mathcal{Y}/\mathcal{I}$, so that the two prescriptions are equivalent.