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Quantum-noise reduction schemes for future advanced gravitational-wave detectors Progress report 1

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1 Quantum noises

Contemporary so-called second-generation gravitational-wave detectors, such as Advanced LIGO [1, 2], Advanced VIRGO [3], and KARGA [4], which are under construction now, will be quantum noise limited over the detection frequency band. At low frequencies, it is dominated by the radiation pressure noise which is due to quantum fluctuation in the amplitude of the optical field [5]; while at high frequencies, the shot noise dominates which arises from the phase fluctuation. There is a trade-off between these two noises that is called Standard Quantum Limit (SQL) [6]. In the linear position meter (the gravitational-wave interferometer is special case of it) the shot noise corresponds to the measurement noise and radiation pressure noise to the back-action noise. The SQL is not an ultimate limit for measurement precision: there are several methods of overcoming it in gravitational-wave detectors. The most well-known examples are Quantum Non-Demolition (QND) measurements and Back-Action Evading (BAE) measurements (see, *e.g.*, [7, 8, 9, 10]). The first method tries to measure conserved dynamical quantity of the system [6, 7, 11, 12]. The second method uses the correlation between the measurement noise and the back-action noise [10, 13, 14, 15, 16, 17].

Though there are many various approaches, applicable for different special cases (*e.g.*, for the certain frequency range), the community looks for other solutions. In this work we investigate multiple optical springs approach.

2 Optical rigidity

The Hamiltonian of a system with a resonator (length L and resonant frequency ω_c) with one movable mirror (resonant frequency ω_0) which motion is being measured by laser (with frequency ω_0):

$$\hat{\mathcal{H}} = \frac{\hbar\omega_m^2}{2}\hat{x}^2 + \frac{\hat{p}^2}{2m} + \hbar\omega_c\hat{e}^\dagger\hat{e} + \hbar G_0\hat{x}\hat{e}^\dagger\hat{e} + i\hbar\sqrt{2\gamma}\big(\hat{a}\hat{e}^\dagger e^{-i\omega_0 t} - \hat{a}^\dagger\hat{e}e^{-i\omega_0 t}\big) \tag{1}$$

The first term corresponds to the mechanical mode ($[\hat{x},\hat{p}] = i\hbar$). The second one describes the cavity mode with annihilation operator $\hat{e}$ and commutator $[\hat{e},\hat{e}^\dagger] = 1$. The third is interaction between the oscillator and light with optomechanical coupling constant $G_0 = \omega_0/L$. The last term describes the pump $\hat{a}$, where γ is half-bandwidth of the resonator.

This Hamiltonian can be linearized if we assume the pump is big so we can replace the operators by the sum of the mean value and small perturbation:

$$\hat{e} \to \bar{e} + \hat{e}, \qquad \hat{e} \ll \bar{e}.$$

Then in the rotating wave approximation we get linearized Hamiltonian:

$$\hat{\mathcal{H}} = \frac{\hbar\omega_m^2}{2}\hat{x}^2 + \frac{\hat{p}^2}{2m} + \hbar\mathit{\Delta}\hat{e}^\dagger\hat{e} + \hbar G_0\hat{x}(\hat{e}^\dagger\bar{e} + \bar{e}^*\hat{e}) + i\hbar\sqrt{2\gamma}\big(\hat{a}\hat{e}^\dagger - \hat{a}^\dagger\hat{e}\big) \tag{2}$$

where $\mathit{\Delta} = \omega_c - \omega_0$ is detuning. We can assume that $\bar{e}$ is real value, so we can simplify the equation by the substitution $g = G_0\bar{e}$.

The Hamilton equation is [18]:

$$\dot{\hat{e}} = -\frac{i}{\hbar}[\hat{e}, \hat{\mathcal{H}}] - \gamma a \tag{3}$$

We find:

$$\dot{\hat{e}} + (\gamma + i\Delta)\hat{e} = -ig\hat{x} + \sqrt{2\gamma}\hat{a} \tag{4}$$

For the output signal we have [18]:

$$\hat{b} = -\hat{a} + \sqrt{2\gamma}\hat{e} \tag{5}$$

The same we can derive for the mechanical mode, and get the system of Langevin equations:

$$\begin{cases} \dot{\hat{e}} + (\gamma + i\Delta)\hat{e} = -ig\hat{x} + \sqrt{2\gamma}\hat{a} \\ \dot{\hat{x}} = \frac{\hat{p}}{m} \\ \dot{\hat{p}} + \gamma_m \hat{p} = -m\omega_m^2 \hat{x} + \hbar g(\hat{e}^\dagger + \hat{e}) + \hat{\zeta}_{th} \end{cases} \tag{6}$$

where $\hat{\zeta}_{th}$ is Brownian thermal force with correlation function

$$\langle \hat{\zeta}_{th}(t)\hat{\zeta}_{th}(t')\rangle = 2m\gamma_m k_B T \Delta(t - t').$$

For the cavity mode in the spectral representation we can get:

$$\hat{e}(\omega) = \frac{g\hat{x}(\omega) + i\sqrt{2\gamma}\hat{a}(\omega)}{\omega - \Delta + i\gamma} \tag{7}$$

$$\hat{e}^\dagger(\omega) = \frac{-g\hat{x}(\omega) + i\sqrt{2\gamma}\hat{a}(-\omega)}{\omega + \Delta + i\gamma} \tag{8}$$

For the mechanical mode from (6) we get:

$$m(\ddot{\hat{x}} + \gamma_m \dot{\hat{x}} + \omega_m^2 \hat{x}) = -\hbar g(\hat{e}^\dagger + \hat{e}) + \hat{\zeta}_{th} = F_{BA} + \hat{\zeta}_{th} \tag{9}$$

Calculate the back-action term:

$$F_{BA} = \hbar g(\hat{e}^\dagger + \hat{e}) = -\hbar g \left\{ g\hat{x}(\omega)\left(\frac{1}{\omega - \Delta + i\gamma} - \frac{1}{\omega + \Delta + i\gamma}\right) + i\sqrt{2\gamma}\left(\frac{\hat{a}^\dagger}{\omega + \Delta + i\gamma} + \frac{\hat{a}}{\omega - \Delta + i\gamma}\right)\right\} =$$
$$= -\frac{2\hbar g^2 \Delta \hat{x}(\omega)}{(\omega - \Delta + i\gamma)(\omega + \Delta + i\gamma)} + 2\hbar g\sqrt{\gamma}\frac{\hat{a}_1(\gamma - i\omega) + \Delta\hat{a}_2}{(\omega - \Delta + i\gamma)(\omega + \Delta + i\gamma)} = -\mathcal{K}(\omega)\hat{x}(\omega) + \hat{F}_n(\omega) \tag{10}$$

where

$$\hat{a}_1 = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \hat{a}_2 = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}},$$

and we also introduce optical rigidity $\mathcal{K}$ and noise term $\hat{F}_n$.

The dynamics of the system can be described by:

$$\hat{x}(\omega) = \chi_{eff}(\omega)[\hat{F}_n + \hat{\zeta_{th}} + G], \tag{11}$$

where G is signal and

$$\chi_{eff}^{-1}(\omega) = -m\omega^2 + \mathcal{K}(\omega) \tag{12}$$

So the optical rigidity effectively modifies the dynamics of the mirror.

3 Spectral density

Now let's descibe the output in simplier way. From the equations Eq.(7),(5) we can derive the two-photon ouput quadratures:

$$\hat{\mathbf{b}}(\omega) = \mathbb{R}\hat{a}(\omega) + 2\sqrt{\gamma}\mathbb{L}\hat{X}(\omega) \tag{13}$$

where

$$\mathbb{L} = \frac{1}{\mathcal{D}(\omega)}\begin{bmatrix} \gamma - i\omega & -\mathit{\Delta} \\ \mathit{\Delta} & \gamma - i\omega \end{bmatrix} \tag{14}$$

$$\mathcal{D}(\omega) = (\gamma - i\omega)^2 + \mathit{\Delta}^2 \tag{15}$$

$$\hat{X}(\omega) = \bar{E}\frac{k_0\hat{x}(\omega)}{\sqrt{\tau}}\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad k_0 = \omega_0/c, \qquad \tau = L/c \tag{16}$$

$$\mathbb{R} = 2\gamma\mathbb{L} - \mathbb{I} \tag{17}$$

and $\bar{E}$ is amplitude of the classical field in the cavity. In the single-mode approximation the field in the cavity is:

$$\hat{\mathbf{e}}(\omega) = \frac{\mathbb{L}(\omega)}{\sqrt{\tau}}\left(\sqrt{\gamma}\hat{a} + \hat{X}(\omega)\right) \tag{18}$$

Thus the back-action force is:

$$F_{BA} = \frac{2\hbar k_0\bar{E}}{\sqrt{\tau}}\hat{\mathbf{e}}(\omega)\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T = F_n - \mathcal{K}(\omega)\hat{x}(\omega) \tag{19}$$

where

$$F_n = \frac{2\hbar k_0\bar{E}\sqrt{\gamma}}{\sqrt{\tau}}\mathbb{L}(\omega)\begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \tag{20}$$

$$\mathcal{K}(\omega) = \frac{mJ\mathit{\Delta}}{\mathcal{D}(\omega)}, \qquad J = \frac{4\omega_0 I_c}{mcL} = \frac{4\hbar k_0^2\bar{E}^2}{m\tau} \tag{21}$$

which is exactly what we get in the equation (10) As we mentioned before the optical spring term $\mathcal{K}$ can be included in mechanical susceptibility term, thus the dynamics of the system is:

$$\hat{x}(\omega) = \chi_{xx}^{eff}(\omega)\left[\hat{F}_n + G(\omega)\right] \tag{22}$$

where G is external classical force and effective susceptibility is:

$$\chi_{xx}^{eff\,-1} = \chi_{xx}^{-1} + \mathcal{K}(\omega) = -m(\omega^2 + i\gamma_m\omega - \omega_m^2) + \mathcal{K}(\omega) \tag{23}$$

In general the system can be described by the system:

$$\begin{cases} \hat{\mathcal{O}}(\omega) = \hat{\mathcal{O}}^{(0)}(\omega) + \chi_{OF}(\omega)\hat{x}(\omega) \\ \hat{F}(\omega) = \hat{F}^{(0)}(\omega) + \chi_{FF}\hat{x}(\omega) \\ \hat{x}(\omega) = \chi_{xx}^{eff}(\hat{F}_n(\omega) + G(\omega)) \end{cases} \tag{24}$$

where $\hat{\mathcal{O}}$ is output from the measurement system and $\hat{F}$ is the back-action force. The measurement result we get by applying measurement operator $\mathbf{H}$ to the output $\hat{b}$. In the case of homodyne detection:

$$\hat{\mathcal{O}}(\omega) = \mathbf{H}^{\mathrm{T}}\hat{\mathbf{b}}(\omega) = \begin{bmatrix}\cos\zeta \\ \sin\zeta\end{bmatrix}^{\mathrm{T}} \begin{bmatrix}\hat{b}_c \\ \hat{b}_s\end{bmatrix} = \hat{b}_c\cos\zeta + \hat{b}_s\sin\zeta \tag{25}$$

In our case these parameters are:

$$\chi_{OF}(\omega) = 2\frac{k_0\sqrt{\gamma}}{\sqrt{\tau}}\mathbf{H}^{\mathrm{T}}\mathbb{L}(\omega)\bar{E}\begin{bmatrix}0\\1\end{bmatrix} \tag{26}$$

$$\hat{\mathcal{O}}^{(0)} = \mathbf{H}^{\mathrm{T}}\mathbb{R}\hat{a} \tag{27}$$

$$\chi_{FF}(\omega) = -\mathcal{K}(\omega) \tag{28}$$

$$\hat{F}^{(0)} = \frac{2\hbar k_0\bar{E}\sqrt{\gamma}}{\sqrt{\tau}}\begin{bmatrix}1\\0\end{bmatrix}^{T}\mathbb{L}(\omega)\hat{a} \tag{29}$$

The system (24) can be resolved:

$$\hat{\mathcal{O}}(\omega) = \hat{\mathcal{O}}^{(0)}(\omega) + \frac{\chi_{xx}^{eff}\chi_{OF}}{1-\chi_{xx}^{eff}\chi_{FF}}\left[G(\omega) + \hat{F}^{(0)}\right] \tag{30}$$

This can be renormalized to the more convenient form. In particular, we can consider the signal as sum of the classical force and some noise:

$$\hat{\mathcal{O}}^{F}(\omega) = \hat{\mathcal{N}}^{F} + G(\omega) = \frac{\hat{\mathcal{X}}}{\chi_{xx}^{eff}(\omega)} + \hat{\mathcal{F}}(\omega) + G(\omega) \tag{31}$$

where

$$\hat{\mathcal{X}}(\omega) = \frac{\hat{\mathcal{O}}^{(0)}(\omega)}{\chi_{OF}(\omega)} = \sqrt{\frac{\hbar}{\gamma m J}}\frac{\mathcal{D}(\omega)}{\mathbf{H}^{\mathrm{T}}\mathbf{D}}\mathbf{H}^{\mathrm{T}}\mathbb{R}\hat{a} \tag{32}$$

$$\mathbf{D}(\omega) = \mathcal{D}(\omega)\mathbb{L}(\omega)\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\Delta\\ \gamma - i\omega\end{bmatrix} \tag{33}$$

$$\hat{\mathcal{F}}(\omega) = \hat{F}_n = \sqrt{mJ\hbar\gamma}\begin{bmatrix}1\\0\end{bmatrix}^{\mathrm{T}}\mathbb{L}(\omega)\hat{a} \tag{34}$$

The spectral density of output can be calculated:

$$S^{F}(\omega) = \frac{S_{\mathcal{X}\mathcal{X}}}{|\chi_{xx}^{eff}|^2} + S_{\mathcal{F}\mathcal{F}} + 2\Re\left\{\frac{S_{\mathcal{X}\mathcal{F}}}{\chi_{xx}^{eff}}\right\} \tag{35}$$

And the corresponding spectral densities are:

$$S_{\mathcal{X}\mathcal{X}} = \frac{\hbar}{4\gamma m J}\frac{|\mathcal{D}(\omega)^2|}{\mathbf{H}^{\mathrm{T}}\mathbf{D}\mathbf{D}^{\dagger}\mathbf{H}}\mathbf{H}^{\mathrm{T}}\mathbb{R}\mathbb{R}^{\dagger}\mathbf{H} = \frac{\hbar}{4\gamma m J}\frac{1}{\left|\mathbf{H}^{\mathrm{T}}\mathbb{L}\begin{bmatrix}1\\0\end{bmatrix}\right|^2} \tag{36}$$

$$S_{\mathcal{F}\mathcal{F}} = \gamma m \hbar J \begin{bmatrix}1\\0\end{bmatrix}^{\mathrm{T}} \mathbb{L}\mathbb{L}^{\dagger}\begin{bmatrix}1\\0\end{bmatrix} \tag{37}$$

$$S_{\mathcal{X}\mathcal{F}} = \frac{\hbar}{2}\frac{\mathcal{D}(\omega)}{\mathbf{H}^{\mathrm{T}}\mathbf{D}}\mathbf{H}^{\mathrm{T}}\mathbb{R}\mathbb{L}^{\dagger}\begin{bmatrix}1\\0\end{bmatrix} = \frac{\hbar}{2}\frac{\mathbf{H}^{\mathrm{T}}\mathbb{L}\begin{bmatrix}1\\0\end{bmatrix}}{\mathbf{H}^{\mathrm{T}}\mathbb{L}\begin{bmatrix}0\\1\end{bmatrix}} \tag{38}$$

It's useful to have these spectral densities in different normalizations. The connection between them is:

$$S^x = S^F|\chi^{eff}_{xx}|^2 \tag{39}$$

$$S^h = S^F\left(\frac{2}{mL\omega^2}\right)^2 \tag{40}$$

4 Multiple Optical Springs

In this approach, we are not trying to use some precise techniques of the noise cancellation (like in back-action evasion), but modifying the dynamics of test mass and therefore enhance the response of the detector to the gravitational-wave signal (amplify this signal). In other words, the fact that the SQL for the force depends on the dynamics of the mirrors allows us to modify this dynamics in the way to overcome the free-mass SQL.

The equation of motion for the position of the test mass x describes the dynamics of the system:

$$x(\Omega) = \chi(\Omega)F(\Omega), \tag{41}$$

where $\chi(\Omega)$ is a mechanical susceptibility and $F(\Omega)$ is an external force. In case of free mass the susceptibility is simply $-1/m\Omega^2$, but in the case of detuned Fabry-Pérot cavity with one movable mirror we have to add another term (optical rigidity $K(\Omega)$) to the susceptibility which describes the interaction:

$$\chi(\Omega) = \left(-m\Omega^2 + K(\Omega)\right)^{-1}. \tag{42}$$

This term is similar to the one describing the spring in the oscillator, that is why this phenomenon is called optical spring [19, 20, 21]. The idea, proposed in the paper [22], is to use the frequency dependence of the optical rigidity to compensate the inertia term, and therefor to decrease the force SQL, that depends on the susceptibility [23]:

$$S_F^{SQL}(\Omega) = 2\hbar|\chi^{-1}(\Omega)|, \tag{43}$$

and thus the larger we can make the susceptibility the better precision we get.

4.1 Two optical springs

While the frequency and cavity bandwidth is much smaller than detuning, we can expand the optical rigidity in Taylor series:

$$K \approx \bar{K} - i\Gamma_{opt}\Omega - m_{opt}\Omega^2 + \mathcal{O}(\Omega^3), \tag{44}$$

where

$$\bar{K} = \frac{mJ\varDelta}{(\varDelta^2+\gamma^2)^2}, \qquad \Gamma_{opt} = -\frac{2mJ\gamma\varDelta}{(\gamma^2+\varDelta^2)^4}, \qquad m_{opt} = -\frac{MJ\varDelta(\varDelta^2-3\gamma^2)}{(\varDelta^2+\gamma^2)^2} \tag{45}$$

As soon as $\bar{K}, \Gamma_{opt}, m_{opt}$ depend only on the parameters of the system: cavity bandwidth, cavity detuning and renormalized optical power, we can choose the m_{opt} in the way to compensate the positive inertia and thus if we combine two carriers we can cancel the constant $\bar{K}_{1,2}$ and inertia terms:

$$\bar{K}_1 + \bar{K}_2 = 0, \qquad m + m_{opt,1} + m_{opt,2} = 0. \tag{46}$$

This canceling significantly reduce the force SQL comparing to the free mass one (in Eq. (42) and (43) we do not take into account the damping term proportional to Ω):

$$\frac{[S_F^{SQL}]_{\text{modified}}}{[S_F^{SQL}]_{\text{free mass}}} = \left|\frac{[\chi(\Omega)_{\text{modified}}]}{m\Omega^2}\right| \approx \left|\frac{|m_{opt}|-m}{m}\right|. \tag{47}$$

By changing the effective mass $|m_{opt}| \to m$ we can achieve the ratio much smaller than one.

This approach works at low frequencies and breaks down at high frequencies since we have to take into account higher terms of Taylor series (44) and that limits us in achieving better sensitivity in broader frequency range.

4.2 Multiple optical springs

The idea that follows directly from this feature - add more optical springs and thus cancel the higher terms of the Taylor expansion. Unfortunately that doesn't work well, because the parameter space becomes very large and it's impossible to solve the equations.

Thus the other approach is to minimize the response in some frequency band. That means we have to minimize functional:

$$G = \int_{f_{min}}^{f_{max}} \ln|\chi_{eff}^{-1}(f)|^2 \, d\ln f, \tag{48}$$

where effective susceptibility with N optical springs is:

$$\chi_{eff}^{-1}(\omega) = -m\omega^2 + \sum_{i=0}^{N} \mathcal{K}_i(\omega) \tag{49}$$

Each $\mathcal{K}_i$ has three parameters to optimize: $J_i, \varDelta_i, \gamma_i$, thus the parameter space has $3N$ dimensions.

5 Optimization of the response function

6 Output spectral density

7 Optimization of the spectral density

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