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CALIFORNIA INSTITUTE OF TECHNOLOGY
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**Technical Note** **LIGO-T1300472–v1** 2013/11/15

# Design of a coating-less reference cavity with total internal reflection

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**Abstract**

Thermal noise is one of the fundamental noise sources that limits frequency stabilization of lasers using an external cavity. This thermal noise comes from the mirror substrate and multi-layered reflection coatings. The use of reflective coatings, and hence noise from them, can be eliminated by the technique of Total Internal Reflection (TIR). This can be used in the building stabler optical cavities, a necessity in the field of gravitational wave detection. The project aims to search for an optimal design and substrate material for the cavity using TIR and reduction in thermal noise.

# Part I

# Motivation

Laser can be used as a precise timing clock source and also to measure lengths precisely as is the case with Gravitational Wave(GW) detectors. Therefore, any fluctuation in laser frequency amounts to reduction in precision. In general, a laser is stabilized using frequency reference like an atomic line or using modes of an optical cavity. In the case of a cavity, the length is not constant but fluctuates due to long-term drifts in temperature and pressure and also due to short-term fluctuations like acoustic vibrations that alter the cavity length. Fluctuations may also arise from the fact that changes in pressure and temperature change the refractive index of the medium which in turn changes the optical length of the cavity [5]. Sophisticated measurements like gravitational wave detection require laser noise to be strongly reduced. To achieve this, active stabilization of the laser cavity is needed. The reason behind the study of thermal noise in resonator cavities is that it sets a limit on the frequency stability of optical cavities, and thus on their performance too. In case of Fabry Perot resonators, the noise comes from the mirror coating and the substrate. Although the use of multilayer coating allows one to obtain high reflectivities $\simeq 1 - 10^{-6}$, the Brownian Thermal noise due to the large mechanical loss becomes a hindrance [3]. Therefore, the use of coating-less resonator cavities, based on the principle of Total Internal Reflection (TIR), can be used to eliminate the noise from the coating completely.

Previous attempts of using coating-less TIR cavities have been proposed by V. B. Braginsky and S. P. Vyatchanin [4] where corner reflecting prisms were used. However, there were losses from the discontinuity in the prism shape i.e. the corners and also from the anti-reflection coating used on the front surfaces of the prism. Another technique called Frustrated Total internal reflection (FTIR) has been used by Schiller et. al [2] as shown in Fig.(1). Design and characterization of a highly reflective mirror without coatings using the principle of TIR has also been shown in [18].

The project aims to focus on developing methods of calculating Thermal noise in the TIR optical cavity, work towards achieving frequency stabilization and ultimately search for an optimal design and material for the TIR reference cavity for the Thermal noise reduction using simulations constructed using COMSOL and MATLAB software packages.

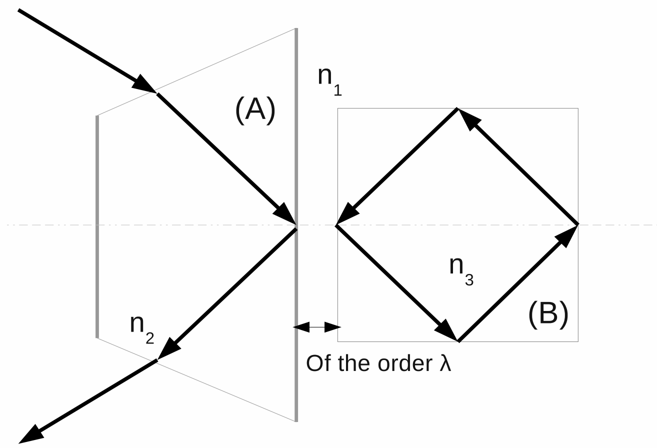


Figure 1: Rough diagram of a Schiller TIR cavity. The laser is injected into cavity (B) using prism (A). The beam is forced to stay within the cavity due to TIR.

## TIR Cavity

The coating noise from the mirrors of laser resonator cavities can be eliminated if the coating itself is not present. The idea is implemented using TIR which gives no loss during reflection unlike any coated mirror which will have a finite transmission coefficient. However, to be used a resonator, light must allowed to enter and leave the cavity. For this purpose, evanescent coupling is used. Every TIR has an associated evanescent wave that travels out of the cavity and decays in amplitude exponentially. The half-value of decay is of the order of the wavelength $\lambda$ (see Chapter 7 of [17]). Hence, if another transparent surface with higher optical density is brought close enough to the half-value, the wave will leak into this material. This is called Frustrated Total Internal Reflection (FTIR). The construction of the cavity is such that, the leaked beam is trapped inside the cavity and a resonator is formed as shown in Fig.(1). The refractive index $n_1$ is lower than $n_2$ or $n_3$ to allow TIR and the separation between the two materials is of the order of the wavelength $\lambda$ as shown in the diagram. This project aims to simulate a TIR cavity similar to that proposed by Schiller *et. al* [2] and optimize the cavity configuration in order to minimize the other sources thermal noise.

# Part II

# Thermal Noise and FDT

The study of noise becomes important in the study of gravitational waves because of the magnitude of the signal due to a gravitational wave is small enough to be of the same order as the noise due to several factors: mechanical disturbances, thermal fluctuations in shape, change in physical properties like refractive index and so on. Once the acoustic or seismic disturbances are taken care of by the use of mechanical-filters, the other noises in a microscopic scale are to be addressed. In case of LIGO (a snap of a reference cavity is

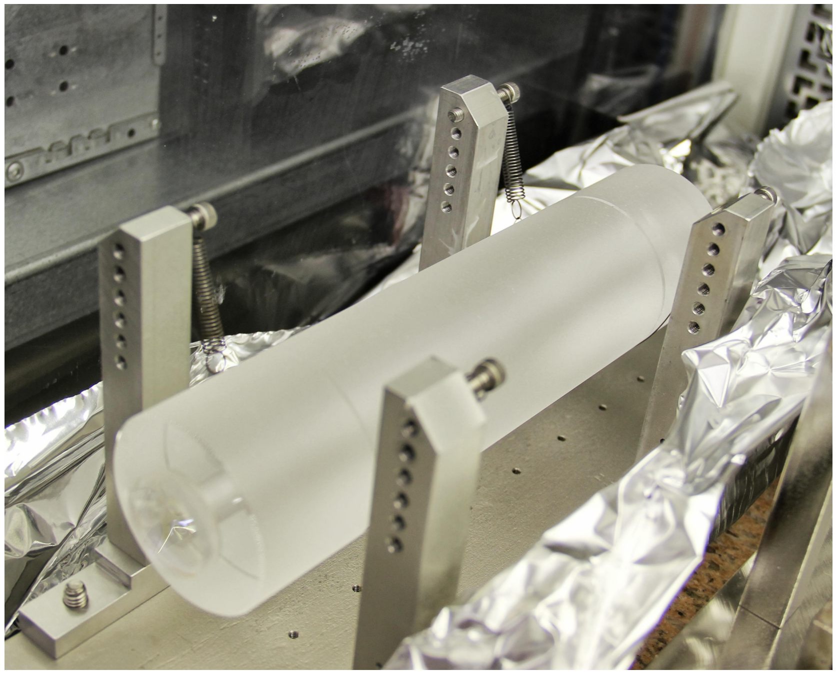

Figure 2: Photograph of a reference cavity used by LIGO for frequency stabilization. Courtesy: Erica Chan

shown in Fig.(2)) these microscopic noises come from the mirror substrate and coating of the resonator cavity which can be calculated with the aid of the Fluctuation Dissipation Theorem (FDT).

## Fluctuation Dissipation Theorem

A system in equilibrium is not absolutely at rest. There are fluctuations that occur in the quantities about the equilibrium value. Considering the case of a Brownian particle (like a pollen grain in water): the incessant motion of the surrounding molecules colliding with the Brownian particle result in fluctuation of the position of the particle [1]. In his theory of Brownian motion, Einstein showed that this fluctuation is related to the diffusivity of the medium which in turn is related to the viscosity [1]. Hence, the fluctuations in the quantity at equilibrium is related to the dissipative mechanism. This is the a special case of the Fluctuation Dissipation Theorem (FDT) which is a more generalized relationship between the random force and the frictional constant. It is a way to relate the dissipative mechanism to the fluctuations in a thermodynamic quantity. It presents us a theoretical approach to quantify "noise" in a system by looking at the dissipative mechanism associated with the quantity of interest.

A particular case of electrical noise was identified by Nyquist [6] prior to the work of Callen and Welton and the noise spectrum was found to follow:

$$E^2 d\nu = 4k_B T R d\nu \tag{1}$$

where $E$ is the random voltage developed across the end of an electrical conductor, $R$ is the

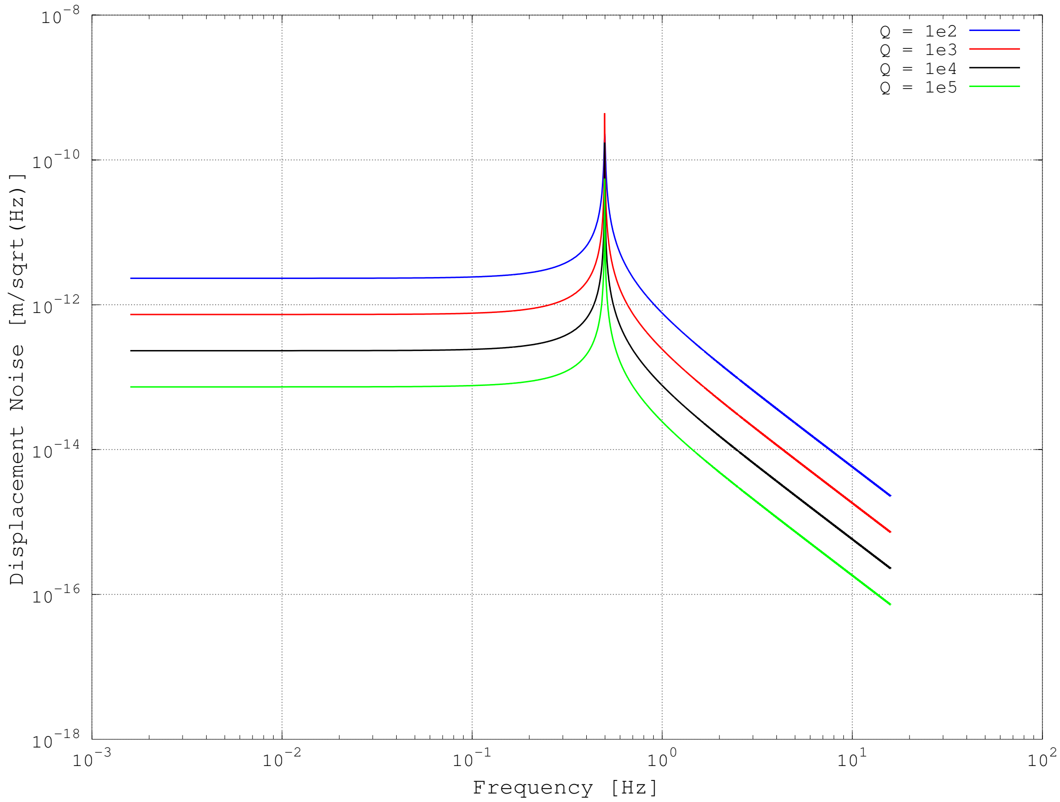


Figure 3: Spectral density in case of velocity damping

resistance, $T$ is the temperature and $k_B$ the Boltzmann constant.

In the paper by Callen and Welton [7], a relationship is formulated between the general impedance of a system and the fluctuations caused due to the forces. According to them, if a system has densely populated energy levels, it can absorb energy when acted upon by a periodic force. The fluctuations have been considered as sinusoidal perturbation to the Hamiltonian of the unperturbed system following which time-dependent perturbation theory is used to obtain the expression for the power loss as a function of the angular frequency of the perturbations $\omega$. The generalization of the Nyquist relation as obtained by Callen and Welton goes as:

$$\langle V \rangle^2 \backsimeq \frac{2}{\pi} \int_{\omega=0}^{\infty} R(\omega)\, E(\omega, T)\, d\omega \tag{2}$$

where $E(\omega, T)$ is the energy at a particular frequency $\omega$ and $R(\omega)$ is the resistance. At high temperatures, $E(\omega, T) \backsimeq k_B T$ which gives the result obtained by Nyquist. The generalized resistance that appears in the expression arises out of dissipation of the system.

## An example of Mechanical noise

Peter Saulson, in his work [11], has shown the use of FDT to evaluate the thermal noise in case of dissipative simple harmonic motion. He has shown two separate cases of damping - that due to velocity damping i.e. by a dissipative force of the form $F_{diss} = -fv$ and that due to internal damping which is modelled by a complex spring constant after the anelastic

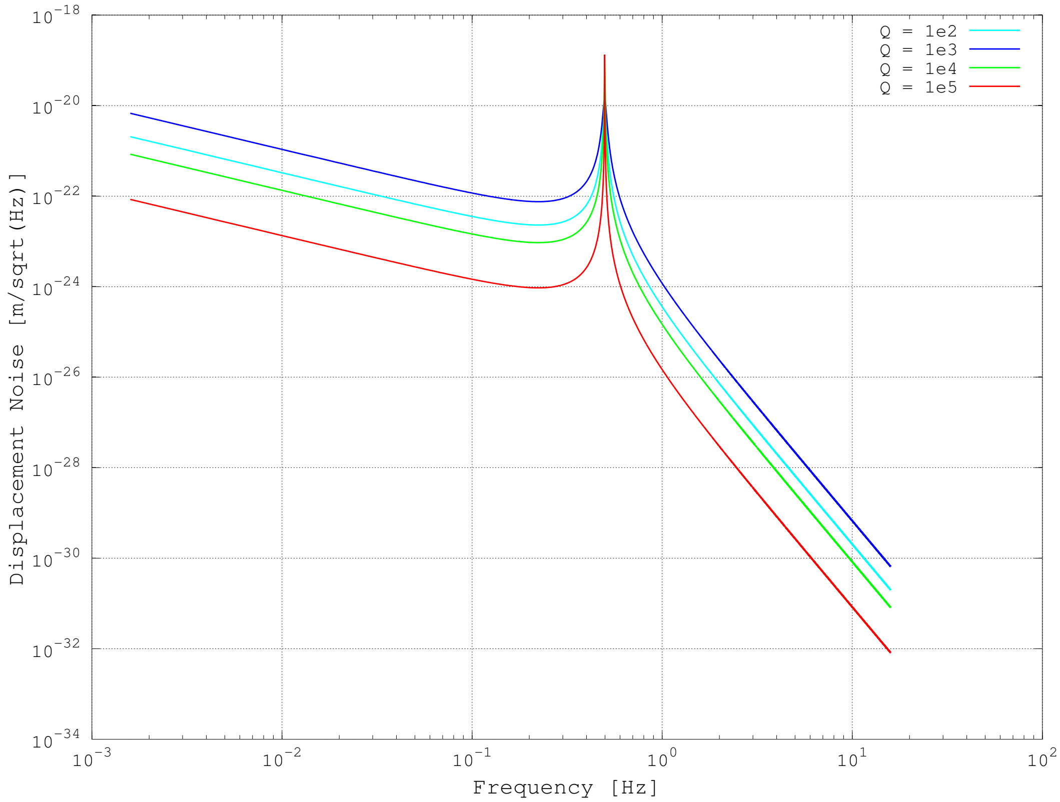


Figure 4: Spectral density in case of internal damping. The results are for constant loss factor $\phi$

model of Zener [13]. In the case of velocity damping, the equation of motion is given as:

$$m\ddot{x} + f\dot{x} + kx = F_{th} \tag{3}$$

where $x$ is the displacement, $f$ is the damping coefficient, $k$ is the stiffness constant and $F_{th} = 4k_BTf$. Going over to the frequency domain, it follows from Eq.(3) that:

$$\mid x(\omega) \mid^2 = \frac{4k_BTf}{(k - m\omega^2)^2 + (f\omega)^2} \tag{4}$$

Fig.(3) shows the variation of the power spectral density for a pendulum of length 1m and mass 1kg for different $Q$ factors. The $Q$ factor is a measure of the rate of energy loss from an oscillator due to damping. It is the ratio of the natural frequency to the half-power bandwidth. Under heavy damping, the $Q$ factor is low and vice versa. One can also think of the $Q$ factor as measure of how sharply peaked is the frequency response of the oscillator near the natural frequency. With high values of the $Q$ factor, the system shows strong response at resonance with a small bandwidth and vice versa. In the case of internal damping, the dissipation is considered through a complex spring constant. The Hooke's law in this case is:

$$F = -k\{1 + i\phi(\omega)\}x \tag{5}$$

where $\phi(\omega)$ is the lag between the response of the system $x$ to a force oscillating a frequency $\omega$. The equation of motion in this case is:

$$m\ddot{x} + k\left\{1 + i\phi(\omega)\right\}(x - x_g) = F_{th} \tag{6}$$

Here, $\frac{x}{x_g}$ is the vibration transfer function. The damping coefficient gets replaced by the quantity $\frac{k\phi}{\omega}$ and the amplitude associated with $\omega$ is:

$$\mid x(\omega) \mid^2 = \frac{4k_B T k\phi}{\omega\left[\left(k - m\omega^2\right)^2 + (k\phi)^2\right]} \tag{7}$$

The model of calculating the quantity $\phi$ is taken from the anelastic model by Zener [1] where:

$$\phi = \Delta \frac{\omega\tau}{1 + (\omega\tau)^2} \tag{8}$$

there $\tau$ is a frequency at which damping shows a maximum. The quantity $\Delta$ is the relaxation strength which in the case of a two spring and a dashpot model is the ratio $\frac{k_2}{k_1}$ where $k_2$ is the spring constant of the spring attached to the dashpot. This ratio is usually $\ll 1$. Fig.(4) shows the variation of power spectral density with the frequency for a pendulum of length 1m and of mass 1kg for constant loss factor $\phi$ (i.e. $\phi$ is independent of $\omega$) for each of $Q$ factors.

The technique of applying the FDT directly to the noise calculation in the LIGO test masses was put forward by Levin [8]. The Levin approach uses the FDT directly to evaluate the noises where the only term to be calculated computationally is the power loss due to dissipation.

# Levin's Approach and Brownian Thermal Noise

In his approach towards calculating the Brownian Thermal noise in the LIGO test masses, Levin [8] considered fluctuations in LIGO's readout variable $x(t)$ to be the phase shift in the light reflected from the surface. He considered the quantity mathematically as:

$$x(t) = \int f(\mathbf{r})\, y(\mathbf{r}, t)\, d^2 r \tag{9}$$

where $f$ is the form factor proportional to the intensity profile of the laser beam, normalized such that $\int f(\mathbf{r})\, d^2 r = 1$ and $y(\mathbf{r}, t)$ is the displacement of the mirror face in the direction of the beam. In case of gaussian beam, $f$ is a gaussian that mimics the laser's intensity. An oscillating pressure $P(\mathbf{r}, t)$, scaled according to the form factor, acts on the test mass surface which perturbs the system from equilibrium.

$$P(\mathbf{r}, t) = F(t)\, f(\mathbf{r}) = F_0 \cos(\omega t)\, f(\mathbf{r}) \tag{10}$$

Here $F_0$ is the amplitude of the applied force. The Hamiltonian that would result in this force term, therefore, becomes $H_{int} = -F(t)\, x$. Using Eq.(9), the Hamiltonian takes the form:

$$H_{int} = -\int P(\mathbf{r}, t)\, y(\mathbf{r}, t)\, d^2 r \tag{11}$$

This Hamiltonian is the generator of the time evolution of $x(t)$. Using the expression complex impedance which goes as:

$$Z(\nu) = \frac{2\pi i \nu x(\nu)}{F(\nu)} \tag{12}$$

where $x(\nu)$ and $F(\nu)$ are the Fourier transforms of the readout and the force $F(t)$, the real part of Eq.(12) gives the relation with the power dissipated $W_{diss}$:

$$|\ \Re\{Z(f)\}\ | = \frac{2W_{diss}}{F_0^2} \tag{13}$$

Using this relation in the expression relating the impedance to the dissipation due to Callen and Welton, the final expression of the spectral density, as obtained by Levin [8], is:

$$S_x(\nu) = \frac{2k_B T W_{diss}}{\pi^2 \nu^2 F_0^2} \tag{14}$$

where $F_0$ is the amplitude of the oscillating force applied, $\nu$ is the frequency and $W_{diss}$ is the time averaged power dissipation due to internal stresses developed due to the applied pressure. For other sources of thermal noise such as the Thermoelastic and Thermorefractive noise, the calculation of $W_{diss}$ in Eq.(14) turns out to be different depending on the mechanism of the dissipation.

In the case of the Brownian noise, the dissipative mechanism is assumed to be the mechanical loss of the material of the test mass. The idea is the fact that the molecules of the substrate material vibrate about the mean position which causes an overall change in shape of the substrate. The idea is similar to the dissipation of energy of a Brownian particle. The difference being the fact that due to a mechanical loss factor, the material will get compressed and extended in time. This results in elastic potential energy being developed inside which can be calculated from elastic theory [8]. The expression for $W_{diss}$ follows:

$$W_{diss} = 2\pi\nu U_{max}\phi(\nu) \tag{15}$$

# Thermoelastic (TE) noise

The TE noise is a result of the fluctuations in the face of the mirror on which the light beam is incident. Although Brownian noise results in the same behaviour, the mechanism which causes the face to fluctuate is different for the two cases. In case of the TE noise, it is the random thermal expansion of the material while in the case of Brownian Thermal noise, it is the stochastic Brownian motion of the particles of the material that results in the fluctuation. Details about the TE noise for the case of the laser beam getting reflected off the mirror face has been evaluated by Liu and Thorne [14]. In order to calculate the spectral density of these fluctuations, the approach taken is that of Levin [8]. One applies a time-varying sinusoidal pressure at the frequency of interest to the face of the substrate where the beam is being reflected, scaled according to gaussian intensity profile of the light beam:

$$P = F_0 \frac{e^{-\frac{r^2}{r_0^2}}}{\pi r_0^2} \cos(\omega t) \tag{16}$$

where $F_0$ is the force amplitude, $r_0$ is the beam radius and $\omega$ is the frequency of interest. This time varying pressure will in turn generate stresses, which are solved by means of the equation of stress balance [15], in the test mass which in turn result in the generation of a temperature gradient. The heat flow results in dissipation which is time averaged to give the quantity $W_{diss}$ (as calculated in [15]),

$$W_{diss} = \left\langle \frac{TdS}{dt} \right\rangle = \left\langle \int_V \frac{\kappa}{T} (\nabla T)^2 \, dV \right\rangle \tag{17}$$

which in turn is related to the spectral density according to the Fluctuation Dissipation Theorem [7, 8] according to Eq.(18). In the above expression $T$, $S$ and $\kappa$ correspond to the temperature, entropy and the thermal conductivity respectively.

$$S(\omega) = \frac{8k_B T W_{diss}}{F_0^2 \omega^2} \tag{18}$$

It should be noted that according to the approach considered by Liu and Thorne, the TE noise is generated only at the face of the mirror where there is phase shift due to the fluctuation of the position of the face where light is reflected. The following is the expression of the spectral density in the case of an infinite test mass [14]:

$$S_q^{ITM}(\omega) = \frac{8(1+\sigma)^2 \kappa \alpha_l^2 k_B T^2}{\sqrt{2\pi} C_V^2 \rho^2 r_0^3 \omega^2} \tag{19}$$

where $\alpha_l$ is the linear expansion coefficient, $\sigma$ is the Poisson ratio, $C_V$ is the specific heat at constant volume and $\rho$ is the density of the material.

# Thermorefractive (TR) Noise

While the reason for TE noise and the Brownian noise is the fluctuation in the position of the mirror face, the reason for TR noise is the random fluctuation in the index of refraction, $n$ with the temperature. The optical path that light travels in the substrate is going to change if $n$ changes. This adds an extra random phase to the light as it travels through the material. Heinert's work [10] on TR noise presents a comprehensive calculation of the TR noise in finite sized test masses. The approach taken is once again that due to Levin [8] and is similar to one followed by Liu and Thorne [14]. In this case, one uses an oscillating heat source at the desired frequency directly unlike in TE noise calculation where one indirectly develops temperature gradient through an oscillating pressure. This is because in the case of the TR noise, the extra random phase is added due to the change in the refractive index which occurs throughout the entire beam path inside the material. The change in the refractive index is caused due to the fluctuating temperature. Thus, to calculate the fluctuation in the temperature, the heat source is applied. Note that the heat source given by Eq.(20) is independent of $z$, the direction of propagation, which imples that we consider the heat source to be present all through the beam path. This is however an approximation. In general, the thermal gradient along the $z$ direction should also be considered. This is because in reality the beam radius changes with $z$ and therefore the virtual heat source should also have a $z$ dependence.

$$q(\vec{r}, T) = \frac{\beta}{\pi r_0^2} T_0 F_0 e^{-\frac{r^2}{r_0^2}} \cos(\omega t) \tag{20}$$

The parameter $\beta = \frac{\partial n}{\partial T}$ is the thermorefractive parameter. The heat injection, or in other words entropy injection, results in a temperature field that evolves in time following the inhomogenous Heat equation. The dissipated work is calculated as given in Eq.(17) and the spectral density follows from Eq.(18) and is given by:

$$S_z(\omega) = \frac{16}{\pi} k_B T_0^2 \frac{H R^2 \kappa \beta^2}{r_0^4 C_P^2} \times \sum_{n=1}^{\infty} \frac{k_n^2}{\left[J_0(a_n)\right]^2} \frac{K_n^2}{\omega^2 + \frac{\kappa^2}{C_P^2} k_n^4} \tag{21}$$

where $K_n = \int_0^1 J_0(a_n \rho) e^{-((R/r_0)\rho)^2} \rho d\rho$, $J_0(x)$ is the zeroth order Bessel function, $a_n$ is the $n$th root of $J_1(x)$, the first order Bessel function and $k_n = \frac{a_n}{R}$. The details of the calculation along with the expression for $S_z(\omega)$ is provided in [10].

# Part III

# Finite Element Analysis(FEA)

At this point, it is worthwhile to recall that the aim of the project is to look into the noise in the TIR cavity as shown in Fig.(1). Owing to the geometry of the TIR cavity, analytic calculation of the noise maybe difficult and hence the FEA approach. To realize the above aim, a Finite Element Model to compute the various noise sources using COMSOL with MATLAB was used. However, the first step towards building a successful model is a sanity check with the already known results for Thermoelastic and Thermorefractive noise calculation for the relatively simpler case of cylindrical test masses [16, 10, 14]. In certain cases of the analytic calculations, the cylindrical test mass was assumed to be much larger in dimensions compared to the beam spot size and was approximated by an infinite half space. In our case, we always model finite sized test masses and check for an asymptotic match with the analytic results when the dimensions of the cylinder is made larger.

## Using COMSOL with MATLAB

Here we have calculated the Thermoelastic and Thermorefractive noise profiles for cylindrical test masses only. The TE and TR noises are calculated separately in two different COMSOL models. The algorithm for building up the model followed from the works of Heinert [10] and Liu and Thorne [14]. To summarize, the components of the model were as follows:

- A cylinder with insulated walls (see Appendix for an explanation) is built out of the material desired - typically Silicon, Sapphire, Fused Silica etc.

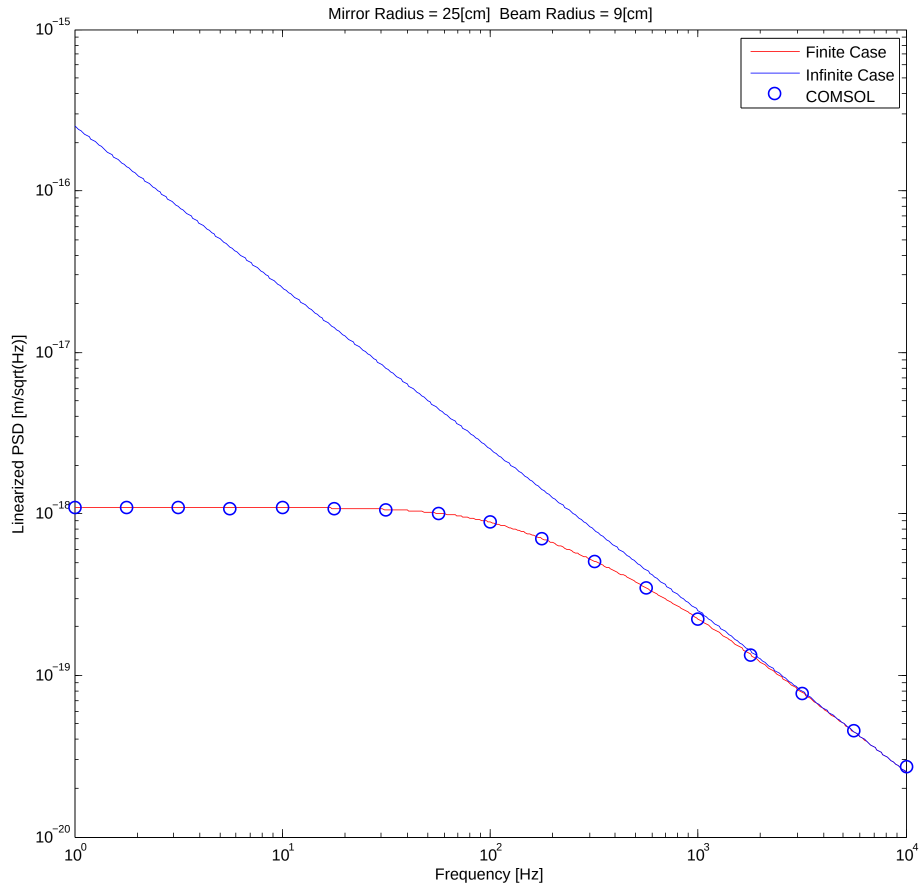


Figure 5: TR noise spectrum for Sapphire at 10K. The solid lines represent the analytic solutions for the finite(red) and infinite(blue) cases. The blue circles is the result of the FE model.

- The *Solid Mechanics* and *Heat Transfer in Solids* modules of COMSOL were used for the calculation of the TE noise since we look into the thermal gradients generated due to application of a pressure on one of the faces of the cylinder (see [14]).
- The *Heat Transfer in Solids* module of COMSOL was used for the calculation of the TR noise spectrum since the thermal gradients were to be studied due to the presence of a virtual heat source (see [10]).
- A *Time Dependent Study* was used and Fourier analysis on the signal was performed in MATLAB to extract the response due to only the perturbing term amongst the transient solution and numerical error made by COMSOL.

# Results

As mentioned earlier, the geometry chosen was cylindrical. The dimensions of the cylinder considered here is taken from the paper by Heinert *et. al.* [10]: the mirror radius $25cm$ the beam radius $9cm$ and the mirror height $46cm$. The TR noise spectrum as calculated by the model for the case of Sapphire at 10K is shown in Fig.(5). A comparison is made with the analytic calculations by Heinert [10]. The relative error between the FEA and the

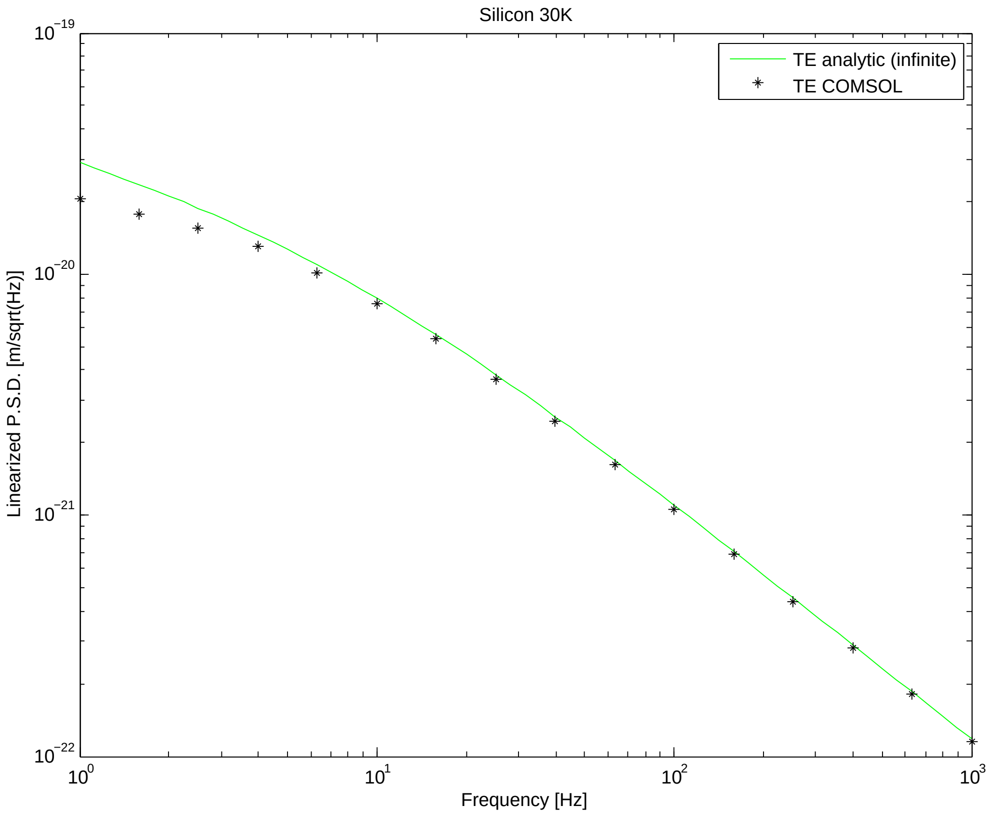


Figure 6: TE noise spectrum for Silicon at 30K. Solid green line is the analytic solution while black asterisk represent the result of FEA.

analytic results is less than 5% in this case. One interesting feature to note about this plot is the deviation of the TR noise from the infinite case and becoming a flat line for lower frequencies. The reason being the themal diffusivity (i.e. the quantity given by $\frac{\kappa}{C_p}$; $\kappa$ is the thermal conductivity in W/mK and $C_p$ is the specific heat per unit mass at constant pressure in J/kgK) for Sapphire at 30K has a high value ($\sim 10^5$ kg/ms [10]). The diffusivity detemines how quickly heat is carried away in a material. As a result at low frequencies, the temperature gradients developed are not steep since heat diffuses away quickly within the time of the pressure cycle. Therefore, the dissipation, which depends on the temperature gradient as given in Eq.(17), is low. For materials with lower diffusivity, this is not the case and the plot is almost a straight line like the infinite test mass case shown in Fig.(5). Interestingly, the same material Sapphire shows a much lower diffusivity at 300K [10]. In which case the plot is different as mentioned above.

The TE noise spectrum as calculated for Silicon at 30K is shown in Fig.(6). The analytic calculation is taken from the paper by Cerdonio *et. al.* [16]. The calculation is however performed for a cylindrical test mass with an infinite radius. The difference between the simulated and analytic results become apparent at the low frequency regions where the finiteness of the geometry becomes important. One may also notice the plot deviates from that of a straight line in the low frequencies. The reason is expected to be the same as cited for the TR case.

Once the simulated results matched fairly with the analytic results, the two mechanisms were put together to check for any cancellation effects. The idea behind thinking of cancellation

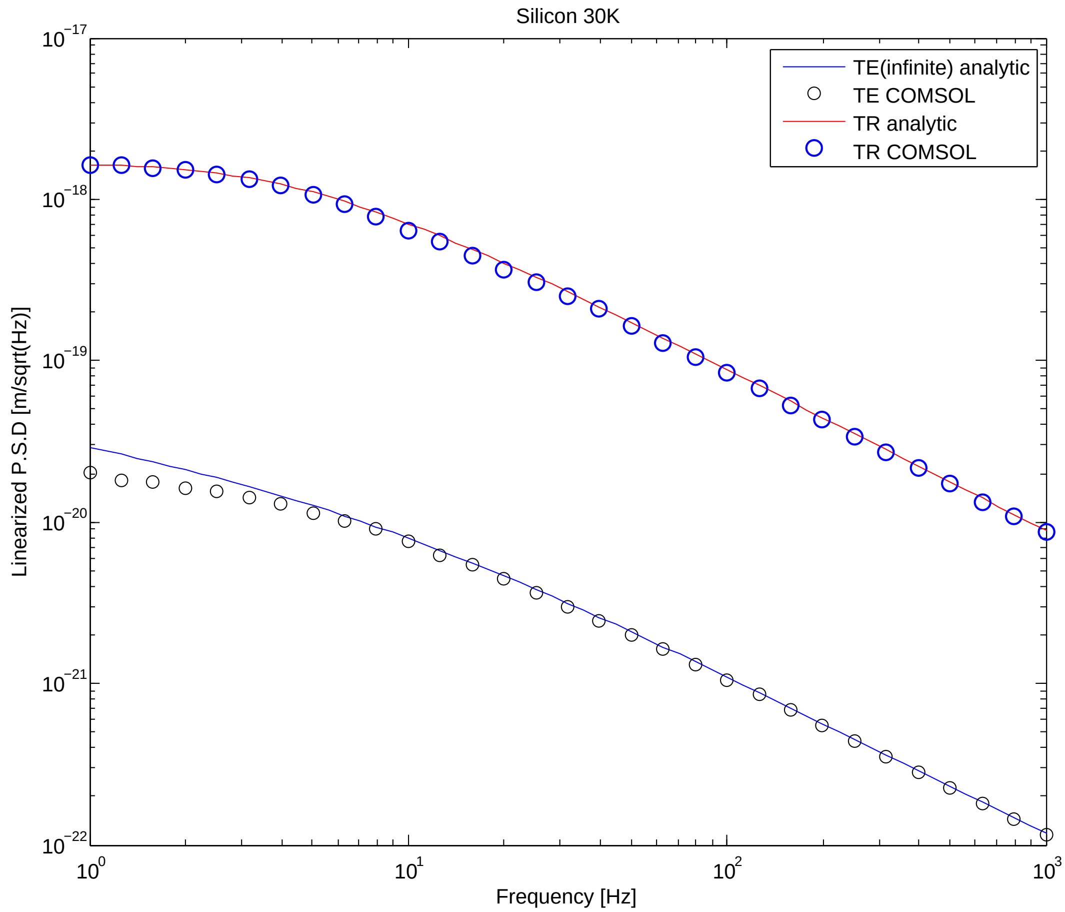


Figure 7: TE and TR noise spectrum for Silicon at 30K. The solid lines represent the analytic results for the TE(blue) and TR(red) noise. The circles correspond to the results of the TE(black) and TR(blue) from Finite Element model.

effects is that both the TE and TR noise arise from thermal fluctuations and are therefore coherent. The TE noise depends on the coefficient of thermal expansion $\alpha$ of the substrate material while the TR noise depends on the thermo-optic coefficient, $\beta$. It was suspected that using a material with an $\alpha$ and $\beta$ with opposite signs might lead to an overall reduction in the total noise spectrum. Using Silicon at temperatures around 30K is an example of such a material where $\alpha$ is negative while $\beta$ is positive. This reduction would, however, be apparent only for those materials for which the TE and TR noise amplitudes are similar. It is to be emphasized at this point that the noise cancellation effects in substrate material has not been previously explored.

Shown in Fig.(7) is a plot of the spectrum of the TE and TR noise for Silicon at 30K. As is clearly evident from the plot, this material doesn't look favourable for compensating TE with TR noise since the TR noise is almost 2 orders of magnitude greater than the TE noise. The cancellation, if it does occur, will not be apparent in this case.

It was noted that the TE and TR noise in the case of Sapphire at 300K was fairly close to each other (see Fig.(8)). We tried a hypothetical material which has all the physical properties of Sapphire at 300K apart from the thermal expansion coefficient which was taken to be the negative of the true value i.e. $-5.4 \times 10^{-6} K^{-1}$ instead of $5.4 \times 10^{-6} K^{-1}$. When the TE

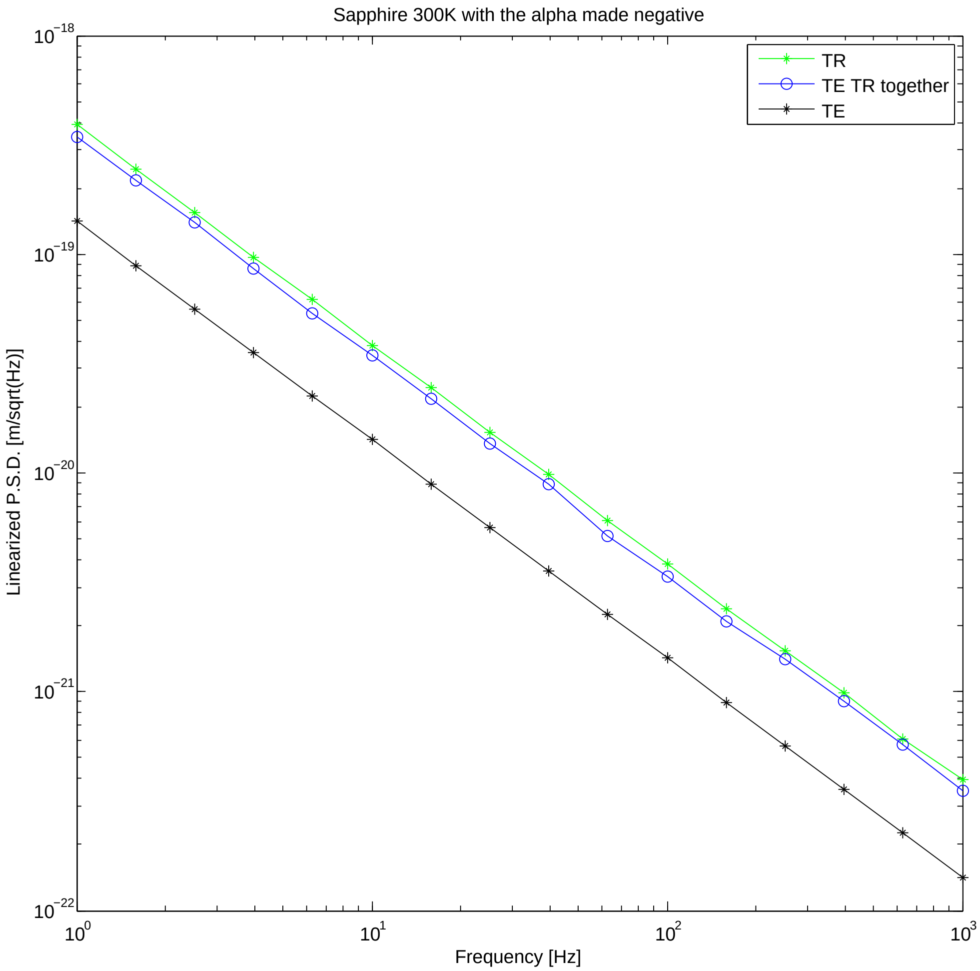


Figure 8: Suspected noise compensation for hypothetical Sapphire at 300K with the linear thermal expansion coefficient, $\alpha_l$ made negative from its usual value. The green solid line with crosses represents the TR noise calculated separately. The black solid line with crosses represents the TE noise calculated separately. The green solid line with circles is the total noise when the two mechanism where put together.

and TR mechanism are considered together in this model, we do notice a certain amount of reduction in the total noise. The results are shown in Fig.(8). It can be seen that the total noise which is plotted in blue is lower than the TR noise calculated individually which is plotted in green. Due to time constraints, further quantitative analysis on the relative reduction and correlation between the noises could not be done during the term of the project.

We can, however, draw certain qualitative conclusions from Fig.(8). Whether the noises are correlated can be determined from the correlated sum which is the algebraic sum of the quantities and the uncorrelated sum which is the square root of the sum of the squares of the two noises. The uncorrelated sum is always greater than each individual summand. The correlated sum can, however, be lower than some of the summands, since the addition is performed taking the relative signs into account. The fact that in this case the total noise, when the two mechanism are put together, is less than the greater of the noise sources suggests that some compensating mechanism exists: the noises are somewhat correlated.

# Part IV

# Conclusion and Future Work

## Remarks on the results

Based on the fact that the results given by our COMSOL model agrees with the analytic results for the special case of the cylindrical test mass, we may conclude that for other complicated geometries like the TIR cavity, this model should suffice. The result shown in Fig.(8) suggests there may be some compensating mechanism for the TE and TR noise. Although the results are for a hypothetical material, a parametric study of the physical constants required for the problem might lead us to such values for which a material exists. It is worth mentioning once again that Silicon is a promising material due to its negative expansion coeffecient at low temperatures.

## Future Work

The entire exercise of noise calculation hinges on the correct usage of the Fluctuation-Dissipation Theorem (FDT). According to the FDT, we can calculate the fluctuation of a variable by perturbing the system from equilibrium by applying the conjugate 'generalized' force. To be applied to the case of the TIR cavity, the perturbation that is to be added to the Hamiltonian is still under consideration. We expect that the present FEA model will give us the correct calculation of the dissipation from the response of the system once this perturbation is applied.

Another task related to the cancellation of the TE and TR noise would be to search for a realistic material for which the TE and the TR noise are close and the material also has a negative thermal expansion coefficient $\alpha$ and a positive thermorefractive coefficient $\beta$. Due to the absence of extensive literature on cancellation, it is also worthwhile to develop further theoretical models.

In the TIR cavities, it is suggested that significant loss occurs due to scattering from the TIR faces [18]. In fact, this is the main source of power loss in the design suggested by Schiller *et. al.* [2] as mentioned in [18]. However, it is also suggested that using super polished mirrors might give high reflectivity $\simeq 99.9999\%$ [18].

# Part V

# Appendix

# Discussion related to the dissipation

The Fluctuation Dissipation Theorem, as the name suggests, relates the fluctuations of a system from equilibrium and the dissipation mechanism. The expression that contains the content of the above statement is Eq.(18) given as:

$$S(\omega) = \frac{8k_B T}{F_0^2 \omega^2} W_{diss}$$

The manner in which $W_{diss}$ is calculated speaks of the dissipative mechanisms present in the system. The expression of the same as used in [14, 10] is given by Eq.(17):

$$W_{diss} = \left\langle \frac{TdS}{dt} \right\rangle = \left\langle \int \frac{\kappa}{T} (\nabla T)^2 d^3 r \right\rangle$$

This equation is derivable from the Heat equation itself. Consider the inhomogenous Heat equation with a heat source pumping heat energy:

$$C_V \frac{\partial T(\vec{r},t)}{\partial t} - \kappa \nabla^2 T(\vec{r},t) = \dot{q}(\vec{r},t) \tag{22}$$

where $T(\vec{r,t})$ is the temperature field. If one multiplies the above equation with the temperture field $T$, one gets:

$$C_V T \frac{\partial T}{\partial t} - \kappa T \nabla^2 T = T\dot{q} \tag{23}$$

Consider the first identity due to Green which says if one has two scalar field $\Phi$ and $\Psi$ then the following holds:

$$\int_V \left(\Phi \nabla^2 \Psi + \nabla\Phi . \nabla\Psi\right) d^3 r = \int_S \Phi \nabla \Psi . d\vec{S}$$

where $V$ denotes the volume integral over any volume while $S$ is the surface that closes that volume. Now consider both $\Phi = \Psi = T$ with the assumption that $\nabla T$ at the surface vanishes. This is the case that is considered in most references (adiabatic boundary conditions). This implies:

$$\int_V \left(T \nabla^2 T + \nabla T . \nabla T\right) d^3 r = 0$$

$$\Rightarrow \int_V T \nabla^2 T \, d^3 r = - \int_V (\nabla T)^2 d^3 r \tag{24}$$

Now consider the time average of the Eq.(23). The system being in equilibrium, the change in $T$ with time i.e. $\frac{\partial T}{\partial t}$ will be small and will also average to zero since the temperature fluctuations are random. Thus $\langle T \frac{\partial T}{\partial t} \rangle$ goes to zero. By applying Eq.(24) and integrating over the volume, Eq.(23) changes as:

$$\left\langle \int_V \kappa (\nabla T)^2 d^3 r \right\rangle = T \left\langle \frac{dQ}{dt} \right\rangle$$

where $Q = \int q \, d^3r$. Since $dQ = TdS$ from thermodynamics, the above equation becomes:

$$\left\langle \int_V \frac{\kappa}{T} (\nabla T)^2 \, d^3r \right\rangle = \left\langle \frac{TdS}{dt} \right\rangle = W_{diss}$$

where $T$ has been moved inside the integral since it hardly changes and is regarded a constant. This is Eq.(17) as used in the references.

A few points of discussion are as follows

- Liu and Thorne never considered the Heat equation but have used Eq.(17) for the dissipation, which is derived entirely by considering the Heat equation. This is because the stresses generated due to the external pressure applied on the test mass resulted in heat generation which was to be calculated.
- Important to note is the fact that the above formula would not be true if $\nabla T \neq 0$ at the surface. This is what Heinert explicitly mentiones thorugh the adiabatic boundary conditions.
- It is evident that a heat source is to be used if Eq.(17) is to be used. Liu and Thorne had to see the surface effects - the way noise is added when beam reflects off the surface. So they applied pressure on the face so that the situation is correctly described and also one avoids injecting heat directly. In their paper they relate the stresses to the temperature gradient which allows them to use the above expression.

# Basics on Waves and Signals

A wave is a function of the parameter $(x - vt)$ in one dimension, $x$ being the displacement, $v$ the velocity and $t$ the time. A simple case considers the harmonic waves which are periodic in $t$. The travelling waves are those that are periodic in both $x$ and $t$. Examples include electromagnetic waves like light and all radiation that fill the EM spectrum. Any periodic time signal, and in general any periodic function, can be written as a sum over the values of its conjugate varible, in this case frequency. This is the Fourier series and is mathematically expressed as:

$$f(t) = \sum_n c_n e^{i\omega_n t} \tag{25}$$

where the set $\{\omega_n = n\omega_0\}$ forms the set of frequencies present in the signal. The value of $\omega_0$ is the fundamental frequency and is given by $\frac{2\pi}{T}$, with $T$ being the period of the function. The coefficients $c_n$, called the Fourier coefficients, give a weight factor to each frequency that is present and is known as the amplitude associated with that frequency. Usually, the coefficients decrease in magnitude with the increase in the frequency simply because of the fact that a smooth signal cannot have a component that vibrates very rapidly. In case there is just one frequency in the entire summation, the signal is said to be monochromatic.

When averaging over time signals that are periodic, one gets the result as zero unless the averaging time is faster than the time period $\frac{2\pi}{\omega}$ of the wave. This is expected since each component in the sum of Eq.(25) would contribute equally in both positive and negative halves of the cycle. Thus, for periodic signals the Root Mean Squared ($rms$) value is considered in place of the average value. The expression follows the name itself: square the signal, take the mean and then take the square root.

$$f_{rms} = \sqrt{\frac{1}{T}\int_{T} \{f(t)\}^2 \, dt} \tag{26}$$

In case of real-valued monochromatic signals, the rms amplitude is $\frac{1}{\sqrt{2}}$ times the amplitude of the signal. The rms quantity is of practical interest because it is a non-zero quantity that stays constant in time and can be measured. The peak to peak definition follows as being twice the amplitude or $2\sqrt{2}$ times the rms - from the lower peak to the higher peak. The intensity or the power per unit area goes as the square of the signal. Hence, when integrated over a time period, it will be proportional to the rms value of the signal.

To do away with the fact that the time signal is periodic, the Fourier transform is used. This tool gives a continuous function of the conjugate variable and the value of the function gives the contribution of that component. In case of a time signal $x(t)$, the fourier transform is given by $\tilde{x}(\omega)$ as follows:

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t} dt \tag{27}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{i\omega t} d\omega \tag{28}$$

The power carried by the wave is propotional to $\mid x(t) \mid^2$. Thus by integrating $\mid x(t) \mid^2$ over a time period, one get the power provided by the signal. Alternatively, one can obtain the power also by integrating the fourier transform over the all the frequency range. This is known as the Parseval's theorem. So that for any small enough interval $\omega$ to $\omega + \Delta\omega$, the quantity $\mid \tilde{x}(\omega) \mid^2 \Delta\omega$ is proportional to the power due to the frequency $\omega$. This way power can be also calculated from the frequency spectrum of the signal. Hence$\mid \tilde{x}(\omega) \mid^2$ is taken to be the power spectral density.

In case of stochastic processes, signals are not described by a single function but by a family of functions called its sample functions [1]. The sample functions are not generally periodic nor do they vanish as $t \to \pm\infty$. Therefore, to handle them, one defines "clipped functions" as follows:

$$x_T = \begin{cases} x(t) & \frac{-T}{2} \le t \le \frac{T}{2} \\ 0 & otherwise \end{cases} \tag{29}$$

The techniques of Fourier series can be applied to this function. This way, one gets the definition of power spectral density as:

$$\lim_{T\to\infty} \frac{\mid A_T(\omega) \mid^2}{T} = G(\omega) \tag{30}$$

where,

$$A_T(\omega) = \int_{-T/2}^{T/2} x(t)\, e^{-i\omega t} dt \tag{31}$$

which are the usual fourier coefficients of the clipped function $x_T(t)$.

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