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Analytic Marginalisation of Phase Parameter

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1 Introduction

Does anyone really care about the phase? The overall phase parameter ϕ of the gravitational wave marks the phase of the wave at some arbitrary reference time. It is useful in that it is a necessary parameter to describe the signal, and maintaining a coherent model of the overall phase between detectors is important. But this parameter reveals nothing about the astrophysics of the source, and it is difficult to imagine a model that predicts that one phase should be preferred over any other. We can safely say the prior on phase is uniform in the range $[0, 2\pi)$, and the posterior is uninteresting, so let's not measure it at all!

As the parameter is a necessary component of the signal, in a sense we cannot get by without using it somehow. Typically the phase is maximised, either separately for each IFO as in the standard CBC pipeline, or coherently over the network combination as in the coherent F-statistic. But the likelihood of the data is a known function of ϕ, and knowing it at one value, say $\phi = 0$, we can calculate it analytically for all other ϕs. Marginalising this resulting function will allow us to ignore the ϕ parameter in our likelihood itself, without losing any other information by maximisation.

Here we derive the partially marginalised analytic likelihood function independent of ϕ, apply it to parameter estimation for compact binaries. We find excellent agreement with the standard results in terms of parameter estimation and evidence calculation, and a large speed increase from dropping the dimensionality of the parameter space.

2 Likelihood

The model gravitational wave at some reference phase is defined as a complex vector $\mathbf{h_0}$, and the data $\mathbf{d}$ with noise standard deviation $\boldsymbol{\sigma}$, which could be in the time or frequency domain. The number of data points, the dimensionality of the data, is D. We shall define the inner product as $\mathbf{a}.\mathbf{b} = \sum_j a_j b_j / \sigma_j^2$ in the usual way. The phase parameter acts on the waveform such that $h = h_0 e^{i\phi}$.

Starting explicitly with the joint posterior on ϕ and $\mathbf{d}$ using the standard Gaussian likelihood,

$$p(\mathbf{d}|I)p(\phi|\mathbf{d}, I) = p(\phi|I)p(\mathbf{d}|\phi, I) \tag{1}$$

$$= \frac{1}{2\pi}\left(\prod_k (2\pi)^{\frac{1}{2}}\sigma_k\right)^{-1} \exp\left[-\frac{1}{2}(\mathbf{d} - \mathbf{h_0}e^{i\phi}).(\overline{\mathbf{d}} - \overline{\mathbf{h_0}}e^{-i\phi})\right] \tag{2}$$

$$= \frac{1}{2\pi}\left(\prod_k (2\pi)^{\frac{1}{2}}\sigma_k\right)^{-1} \exp\left[-\frac{1}{2}\mathbf{d}.\overline{\mathbf{d}}\right] \exp\left[-\frac{1}{2}\mathbf{h_0}e^{i\phi}.\overline{\mathbf{h_0}}e^{-i\phi}\right] \exp\left[\frac{1}{2}(\mathbf{d}.\overline{\mathbf{h_0}}e^{-i\phi} + \overline{\mathbf{d}}.\mathbf{h_0}e^{i\phi})\right] \tag{3}$$

where we have assumed the prior distribution $p(\phi|I) = \frac{1}{2\pi}$. The last term in the exponential is related to the standard matched filter output, and is the only term remaining in ϕ. We want to marginalise the joint posterior over ϕ. This can be found in the standard books of integrals and is noted in e.g.[1, 2], but we will derive it anyway. The final result is given in eq. 11. We will find the solution by changing variables to

$z = \exp(-i\phi)$ and doing a contour integral on the unit circle in the complex plane $|z| = 1$.

$$p(\mathbf{d}|I)\int_0^{2\pi} d\phi p(\phi|\mathbf{d}, I) = \frac{1}{2\pi}\left(\prod_k (2\pi)^{\frac{1}{2}}\sigma_k\right)^{-1} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right] \exp\left[-\frac{1}{2}\|\mathbf{h_0}\|^2\right] \int_0^{2\pi} d\phi \exp\frac{1}{2}\left[\mathbf{d}.\overline{\mathbf{h_0}}e^{-i\phi} + \overline{\mathbf{d}}.\mathbf{h_0}e^{i\phi}\right] \tag{4}$$

$$p(\mathbf{d}|I) = \frac{1}{2\pi}\left(\prod_k (2\pi)^{\frac{1}{2}}\sigma_k\right)^{-1} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right] \exp\left[-\frac{1}{2}\|\mathbf{h_0}\|^2\right] \oint \frac{-idz}{z} \exp\frac{1}{2}\left[\mathbf{d}.\overline{\mathbf{h_0}}\overline{z} + \overline{\mathbf{d}}.\mathbf{h_0}z\right] \tag{5}$$

Now let's expand the exponential and collect terms in z, which we did by Cauchy product, using $\overline{z} = z^{-1}$

$$\begin{aligned}
\exp(A(z+\overline{z})) &= \exp\frac{1}{2}\left[\mathbf{d}.\overline{\mathbf{h_0}}z^{-1} + \overline{\mathbf{d}}.\mathbf{h_0}z\right] && (6)\\
&= \left(1 + \sum_{n=1}^{\infty}\frac{A^n z^n}{n!}\right)\left(1 + \sum_{m=1}^{\infty}\frac{\overline{A}^m}{z^m m!}\right) && (7)\\
&= 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{A^m z^m}{m!}\frac{\overline{A}^{n-m}}{z^{n-m}(n-m)!} && (8)\\
&= 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{A^m\overline{A}^{n-m}z^{2m-n}}{m!(n-m)!} && (9)
\end{aligned}$$

We know that $\oint z^{-n}dz = 0$ when $n \neq 1$, so find the terms in z^0 because of the z^{-1} prefactor in eq. 5, and we want to use the residue theorem $\oint \frac{f(a)}{z-a}dz = 2\pi i \sum_k \mathrm{Res}(f, a_k)$, for the singularity at $z = 0$. The sum was re-expanded in terms of z^{2m-n}, so we only keep the terms with $n = 2m$.

$$2\pi i\left(1 + \sum_{m=1}^{\infty}\frac{A^m\overline{A}^m}{(m!)^2}\right) = 2\pi i \mathrm{I}_0(2\sqrt{A\overline{A}}) \tag{10}$$

where $\mathrm{I}_0(x)$ is the modified Bessel function of the first kind. Substituting back in $A = \frac{1}{2}\mathbf{d}.\overline{\mathbf{h_0}}$, recalling eq. 5, and cancelling the factors $2\pi i\frac{-i}{2\pi} = 1$ we get

$$p(\mathbf{d}|I) = \left(\prod_k (2\pi)^{\frac{1}{2}}\sigma_k\right)^{-1} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right] \exp\left[-\frac{1}{2}\|\mathbf{h_0}\|^2\right] \mathrm{I}_0\left(|\mathbf{d}.\overline{\mathbf{h_0}}|\right) \tag{11}$$

3 LALInference Implementation

We have implemented eq. 11 in LALInference by adjusting the standard likelihood function, creating a new `LALInferenceMarginalisedPhaseLogLikelihood` function, which is enabled with the `--margphi` command line option. We used the GSL implementation of the modified bessel function `gsl_sf_bessel_I0_scaled_e` [3].

3.1 Comparison

Figure 1 shows a comparison of the posterior PDFs with and without the use of the marginalisation. The distributions are identical within the expected statistical flutctuations given the number of samples.

We also compared the evidences output by the nested sampling algorithm in both cases. We found on the same injection as figure 1, with identical noise realisations, the old and new log Bayes factors were 368.60 and 368.62, well within the expected uncertainty for a single run at this resolution.

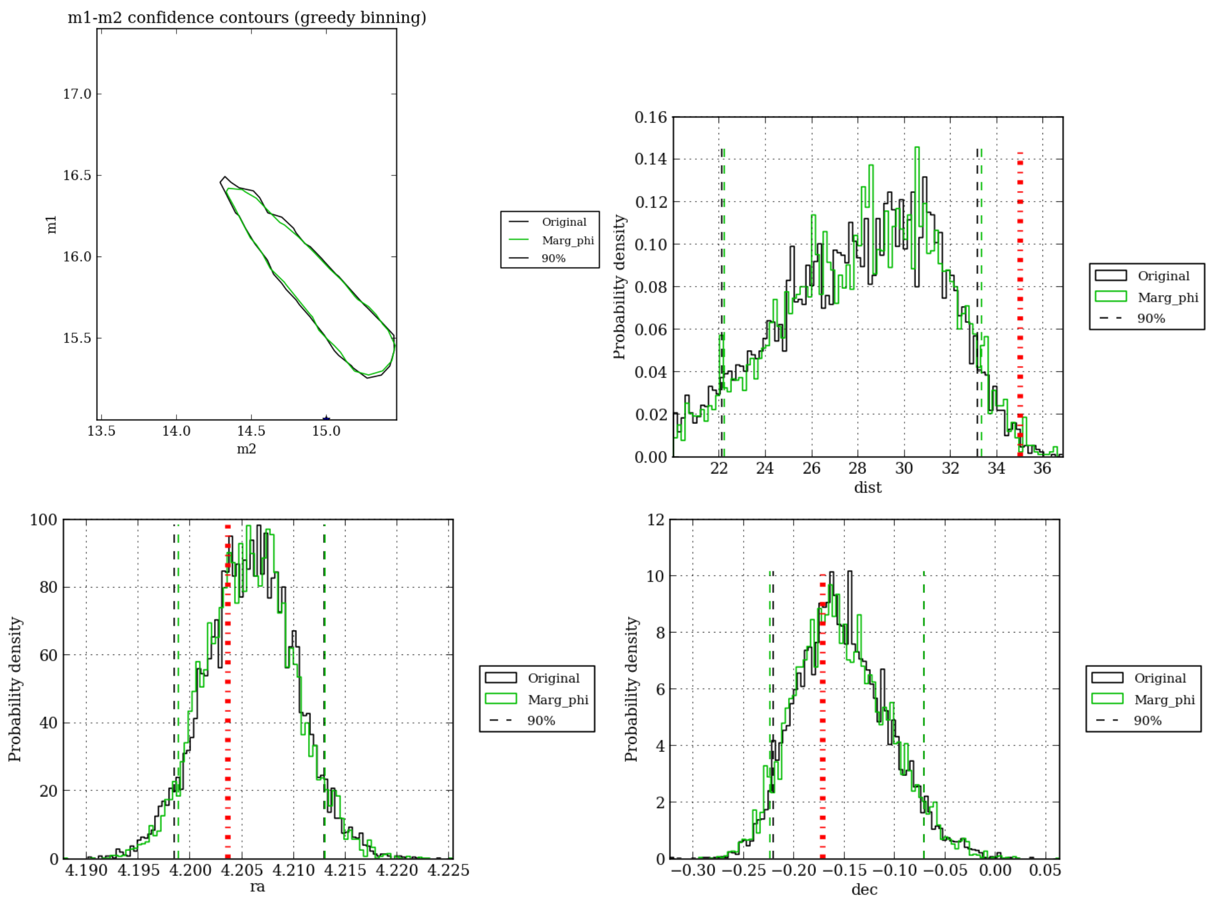

Figure 1: Comparison of posterior PDFs for some parameters with and without the marginalised phase likelihood. The distributions are statistically the same, all passing the K-S test.

3.2 Run-time

When performing the comparison, we noticed that the overall run time of the nested sampling code was reduced from 266 min to 66 min ($4\times$ faster) in this test example, even with a unoptimised implementation of the new likelihood. The reason for this seems primarily to be that the marginalisation over ϕ has decreased the autocorrelation length of the MCMC sub-chains used in the nested sampling, especially toward the end of the run. This can be understood heuristically because when the code has locked onto the posterior mode with ϕ unmarginalised the maximum likelihood points are still relatively close to the minimum likelihood points that are obtained by inverting the phase of the template $\phi \to \phi + \pi$. Therefore it is to be expected that many jumps will result in a bad choice of ϕ, especially given the correlation between ϕ and the polarisation angle ψ in PDFs. Further runs are ongoing to determine the typical size of this effect over different signal types.

References

[1] Duncan Brown. *Searching for gravitational radiation from binary black hole MACHOs in the galatic halo*. The University of Wisconsin-Milwaukee, 2004.

[2] James Clark. *An evidence based search for transient gravitational waves from neutron stars*. The University of Glasgow, 2008.

[3] *GSL reference manual*. URL: `https://www.gnu.org/software/gsl/manual/html_node/Regular-Modified-Cylindrical-Bessel-Functions.html`.