Frequency Domain LQR Estimation

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1 Introduction

This note applies the frequency domain LQR method to an estimator for modal damping of a suspension. Preliminary results suggest the noise performance above 10 Hz can be improved by an order of magnitude while simultaneously improving the damping of the modes.

The Linear Quadratic Regulator (LQR) method, while known for its idealized guaranteed stability and relative robustness, does not on its own permit the application of frequency dependent cost functions. However, frequency weighted variables can be worked into the LQR cost function by augmenting the system plant model.

Much of this work is based off the frequency domain LQR work done by Gupta in [2, 1, 3].

2 Frequency Domain LQR

This section develops the theory behind the frequency domain LQR method.

The system plant is specified in Eq. (1) and Eq. (2) with the usual LTI state space notation. The state vector is x, the actuator input is u, and the output is y.

$$\dot{x} = Ax + Bu \tag{1}$$
$$y = Cx + Du \tag{2}$$

Typically the infinite horizon LQR cost function is expressed as

$$J = \int_0^\infty [x^T Q x + u^T R u] dt \tag{3}$$

This cost function can be converted to the frequency domain by invoking Parseval's Theorem.

$$J = \frac{1}{2} \int_{-\infty}^\infty [x(-jw)^T Q x(jw) + u(-jw)^T R u(jw)] dw \tag{4}$$

In this form, Q and R can also be generalized as frequency dependent matrices. However, since we really want to define the cost in the frequency domain, we shall specify the cost function initially in the frequency domain and then use Parseval's theorem to convert it into a time domain form that can be minimized by solving the Riccati equation given by the standard LQR method.

$$J = \frac{1}{2} \int_{-\infty}^\infty [\bar{x}^T(-jw)^T \bar{x}(jw) + \bar{u}(-jw)^T \bar{u}(jw)] dw \tag{5}$$

$\bar{x}$ is a filtered version of x and $\bar{u}$ is a filtered version of u. Weighting matrices do not need to be specified at this point because all relative weights are incorporated into the filtering functions defined in Eqs. (6) and (7) below. χ and μ represent intermediate states.

$$\dot{\chi} = F_x\chi + G_x x \qquad \bar{x} = H_x\chi + S_x x \tag{6}$$

$$\dot{\mu} = F_u\mu + G_u u \qquad \bar{u} = H_u\mu + S_u u \tag{7}$$

Invoking Parseval's theorem to return to the time domain and plugging in the second column of Eqs. (6) and (7), the cost function becomes

$$J = \int_0^\infty [x^T S_x^T S_x x + 2x^T S_x^T H_x\chi + \chi^T H_x^T H_x\chi + \mu^T H_u^T H_u\mu + 2\mu^T H_u^T S_u u + u^T S_u^T S_u u]dt \tag{8}$$

Simplifying the notation Eq. (8) can be rewritten in a more familiar form as

$$J = \int_0^\infty [z^T Q z + u^T R u + 2z^T N u]dt \tag{9}$$

The augmented state z and the weighting matrices Q, R, and N are

$$z = \begin{bmatrix} x \\ \chi \\ \mu \end{bmatrix} \tag{10}$$

$$Q = \begin{bmatrix} S_x^T S_x & S_x^T H_x & 0 \\ H_x^T S_x & H_x^T H_x & 0 \\ 0 & 0 & H_u^T H_u \end{bmatrix} \tag{11}$$

$$R = S_u^T S_u \tag{12}$$

$$N = \begin{bmatrix} 0 \\ 0 \\ H_u^T S_u \end{bmatrix} \tag{13}$$

The system to plug into the LQR algorithm is augmented with the filtering states

$$\dot{z} = \bar{A}z + \bar{B}u \tag{14}$$

where

$$\bar{A} = \begin{bmatrix} A & 0 & 0 \\ G_x & F_x & 0 \\ 0 & 0 & F_u \end{bmatrix} \tag{15}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \\ G_u \end{bmatrix} \tag{16}$$

The control law is then found by solving the algebraic Riccati equation

$$\bar{A}^T P + P\bar{A} - (P\bar{B} + N)R^{-1}(P\bar{B} + N)^T + Q = 0 \tag{17}$$

and finally

$$u = R^{-1}(P\bar{B} + N)^T z \tag{18}$$

where the feedback gain is

$$K = R^{-1}(P\bar{B} + N)^T \tag{19}$$

The usual restrictions on the weighting matrices apply where $Q \geq 0$ and $R > 0$. These requirements have implications for the filtering functions chosen to apply the frequency weighting. The filtering function on the state x must have at least as many poles as zeros, meaning the transfer function is proper or strictly proper.

The filtering function for the control input is less flexible. Note that $R = S_u^T S_u$ in Eq. (12). In general, the only way to have a nonzero S_u matrix is if the filtering function has the same number of zeros as poles. Thus, it appears we can only use proper transfer functions, not even strictly proper transfer functions. This restriction can be relaxed if the vector $\bar{u}$ contains the control effort in addition to the filtered control effort. In other words, the S_u matrix is forced to have nonzero elements even when the number of zeros is less than the number of poles. More or less the end result is the similar, because in either case the cost function contains terms relating to both μ and u by the requirement of $R > 0$. This means no matter what we do there is always a broadband weight and a filtered weight on the control effort. However, by allowing for strictly proper weighting functions on the control effort we get more freedom in the design of the frequency dependent part of the cost function (which also tends to help prevent 'ill-conditioning' warnings in Matlab's LQR command), and we can directly choose how much we want to weight the broadband control effort term.

3 Application to an Estimator for Modal Damping a Suspension

This section applies the frequency domain LQR theory to an estimator used in modal damping. The cost function for the estimating modes appropriately defind in the time domain. However, the cost on the noise sensor is more appropriately defined in the frequency domain. The noisy sensor can thus be given low cost at the mode frequencies and high cost above 10 Hz where gravitational waves are to be measured.

The typical modal system and estimator state space equations are

$$\dot{q} = A_m q + B_m u \tag{20}$$

$$x = \phi q \tag{21}$$

$$y = Cx \tag{22}$$

$$\dot{\hat{q}} = A_m \hat{q} + B_m u - L_m(C\hat{x} - y) \tag{23}$$

The modal estimator error is

$$\tilde{q} = \hat{q} - q \tag{24}$$

$$\dot{\tilde{q}} = A_m \tilde{x} - L_m C_m \tilde{q} \tag{25}$$

To design the estimator feedback with the LQR method it is well known that the state space matrices of the error equation (25) are first transposed to make a fictitious dual system with the same dynamics.

$$\dot{\xi} = A_m^T \xi + C_m^T u \tag{26}$$

u represents the sensor input to the estimator which we want to filter to weight the cost of the appropriate frequencies of this signal. The following filtering function is chosen.

$$\dot{\mu} = F\mu + G^T u \tag{27}$$
$$\bar{u} = H\mu + Su \tag{28}$$

$\bar{u}$ is the filtered version if the sensor signal which will be weighted in the LQR algorithm. The augmented state space to which LQR is directly applied is

$$\dot{z} = \bar{A}z + \bar{B}u \tag{29}$$

where

$$z = \begin{bmatrix} \xi \\ \mu \end{bmatrix} \tag{30}$$

$$\bar{A} = \begin{bmatrix} A_m^T & 0 \\ 0 & F \end{bmatrix} \tag{31}$$

$$\bar{B} = \begin{bmatrix} C_m^T \\ G \end{bmatrix} \tag{32}$$

The augmented feedback gain is found by

$$\bar{L} = \begin{bmatrix} L_q^T & L_u^T \end{bmatrix} = \mathrm{lqr}(\bar{A},\bar{B},Q,R,N) \tag{33}$$

where

$$\dot{z} = \bar{A}z - \bar{B}Lz \tag{34}$$

$$Q = \begin{bmatrix} Q_m & 0 \\ 0 & H^T H \end{bmatrix} \tag{35}$$

$$R = S^T S \tag{36}$$

$$N = \begin{bmatrix} 0 \\ H_u^T S_u \end{bmatrix} \tag{37}$$

Q_m is the cost on reconstructing the modes. This is the same Q that would be chosen for the typical time domain LQR method. A simple way to specify this matrix is to set equal cost on the modal displacements and zero cost on the modal velocities. For example, a double suspension could have

$$Q_m = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{38}$$

We are not quite done yet because this solution was found by converting the estimator dynamics into a fictitious dual system. To apply these results we must convert the system back by transposing all the state space matrices.

$$\begin{bmatrix} \dot{\tilde{q}} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_m & 0 \\ 0 & F^T \end{bmatrix} \begin{bmatrix} \tilde{q} \\ \sigma \end{bmatrix} - \begin{bmatrix} L_q C_m & L_q G^T \\ L_u C_m & L_u G^T \end{bmatrix} \begin{bmatrix} \tilde{q} \\ \sigma \end{bmatrix} \tag{39}$$

However, the estimator state equations must be defined in terms of the estimated state vector $\hat{q}$, not the unknown estimation error vector $\tilde{q}$. To achieve this the modal plant dynamics from (20) are added to the top row. Rearranging some terms then yields finally the estimator state equations that are applied to the system.

$$\begin{bmatrix} \dot{\hat{q}} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_m - L_q C_m & -L_q G^T \\ -L_u C_m & F^T - L_u G^T \end{bmatrix} \begin{bmatrix} \hat{q} \\ \sigma \end{bmatrix} + \begin{bmatrix} B_m & L_q \\ 0 & L_u \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \tag{40}$$

$$\hat{q}_{out} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \hat{q} \\ \sigma \end{bmatrix} \tag{41}$$

3.1 Results with a Double Pendulum Simulation

This double pendulum model was created from a triple pendulum with a locked test mass. This is the same model used to generate modal damping on the LASTI triple pendulum.

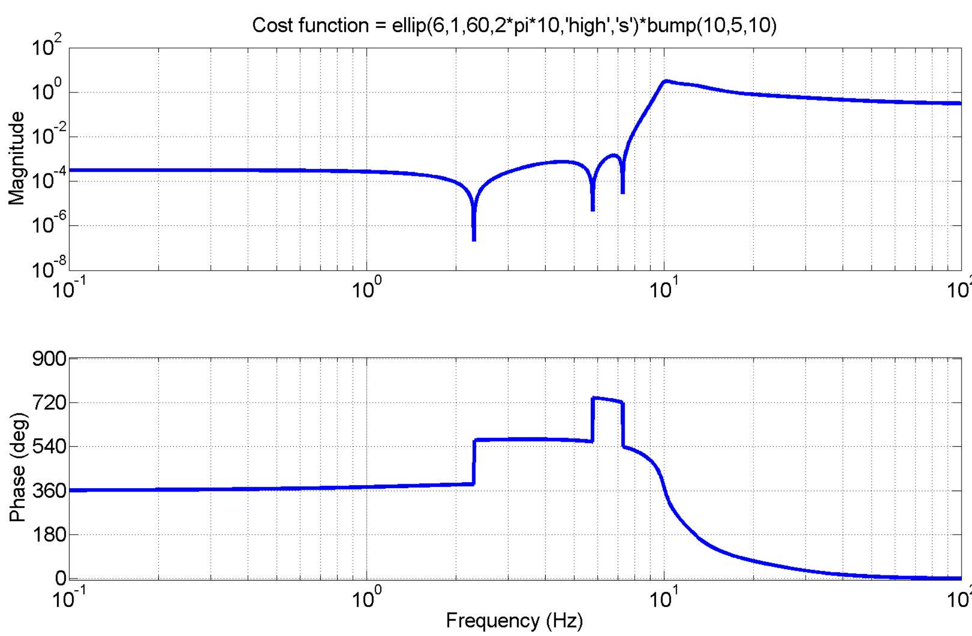

Figure 1: The sensor noise cost function for the frequency domain modal estimator.

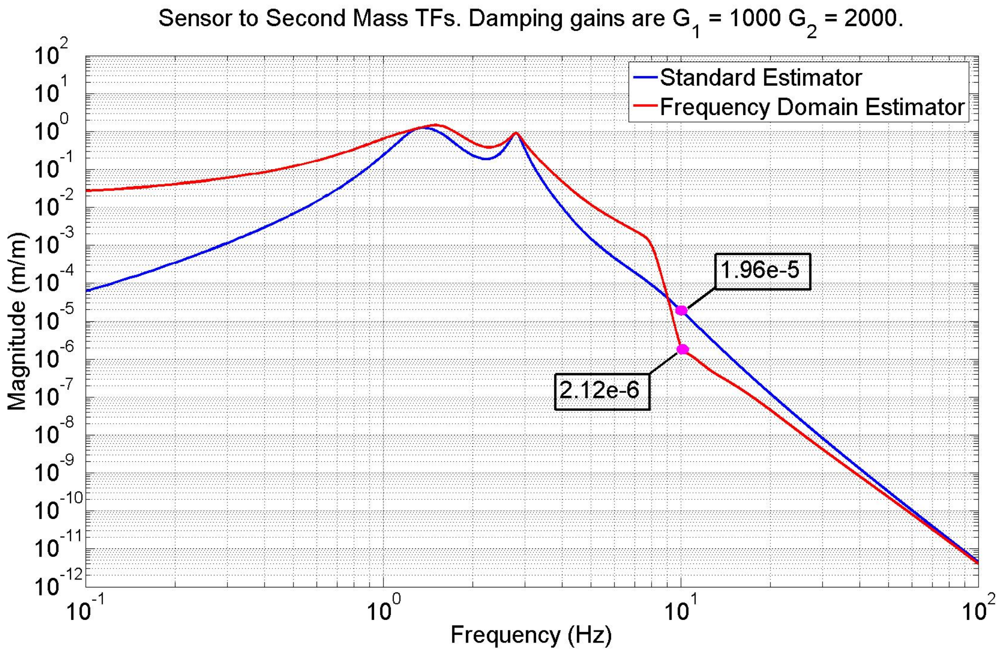

Figure 2: The sensor noise to the second mass TFs for closed loop system with either estimator.

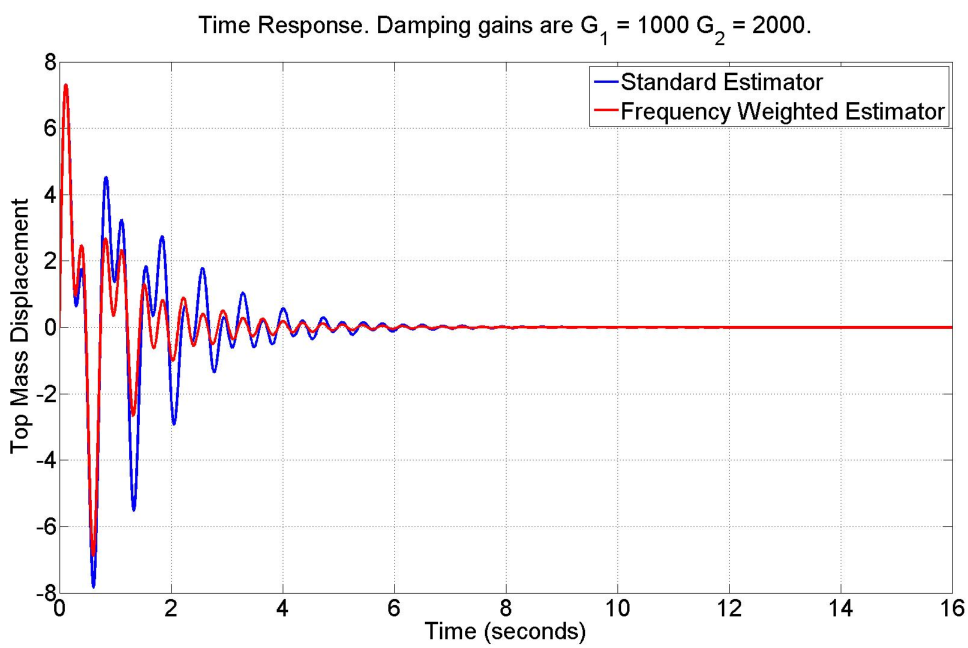

Figure 3: The damping response of the closed loop system with either estimator using the same damping gains.

References

[1] N. Gupta and R. Du Val. A New Approach for Actrive Control of Rotocraft Vibration. *Journal of Guidance, Control, and Dynamics*, 5(2):143–150, 1982.

[2] Narendra Gupta. Frequency-Shaped Cost Functionals: Extension of Linear-Quadratic-Guassian Design Methods. *Journal of Guidance, Control, and Dynamics*, 3(6):529–535, 1980.

[3] Narendra Gupta. Robust Control/Estimator Design By Frequency-Shaped Cost Functionals. *IEEE*, 1981.