The Laser Interferometer Gravitational-Wave Observatory

http://www.ligo.caltech.edu

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Introduction To Signal Processing & Data Analysis

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LIGO-**G1200759**

Fourier Transforms

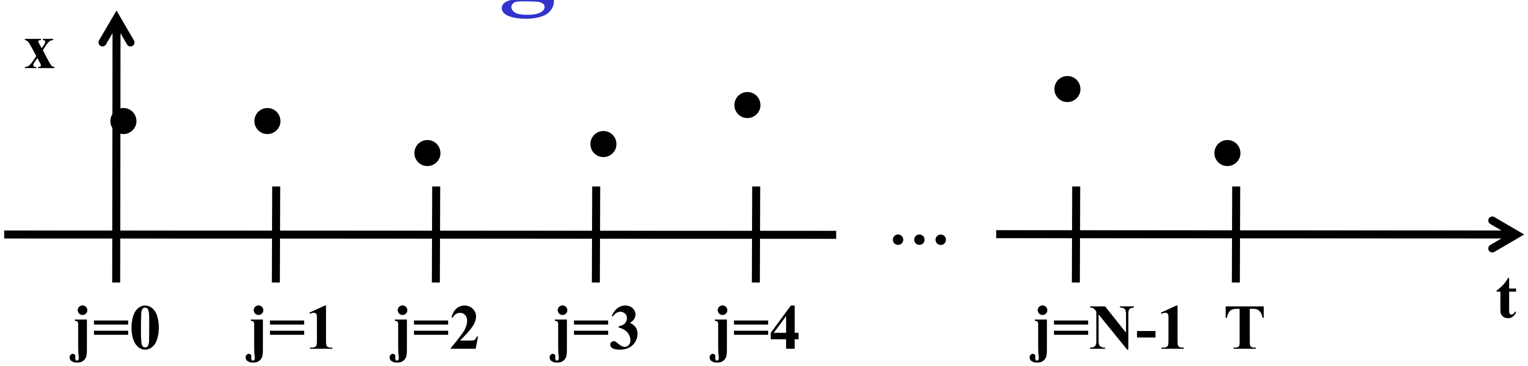
$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(\omega) e^{i\omega t} d\omega$$

$$\omega = 2\pi f$$

$$e^{i\omega t} = \cos \omega t + i \sin \omega t$$

Digital Data



Sample rate, f_s:

$$f_s = 1/(\Delta t)$$

Duration T, number of samples N, Nyquist frequency:

$$f_s T = N; \qquad f_{Nyquist} = f_s / 2$$

Time index j:

$$t_j = j\Delta t; \qquad j = 0,1,2,\ldots N-1;$$

Frequency resolution:

$$\Delta f = 1/T$$

Frequency index k:

$$f_k = k/T; \qquad f_k T = k; \qquad k = 0,1,2,\ldots N-1$$

Discrete Fourier Transforms (DFT)

$$\tilde{x}_k = \sum_{j=0}^{N-1} x_j e^{-2\pi ijk/N}$$

$$x_j = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}_k e^{2\pi ijk/N}$$

The DFT is of order N^2. The Fast Fourier Transform (FFT) is a fast way of doing the DFT, of order $N\log_2 N$.

DFT Aliasing

$$f \rightarrow -f; \qquad \tilde{x}_{-k} = \tilde{x}_k^*$$

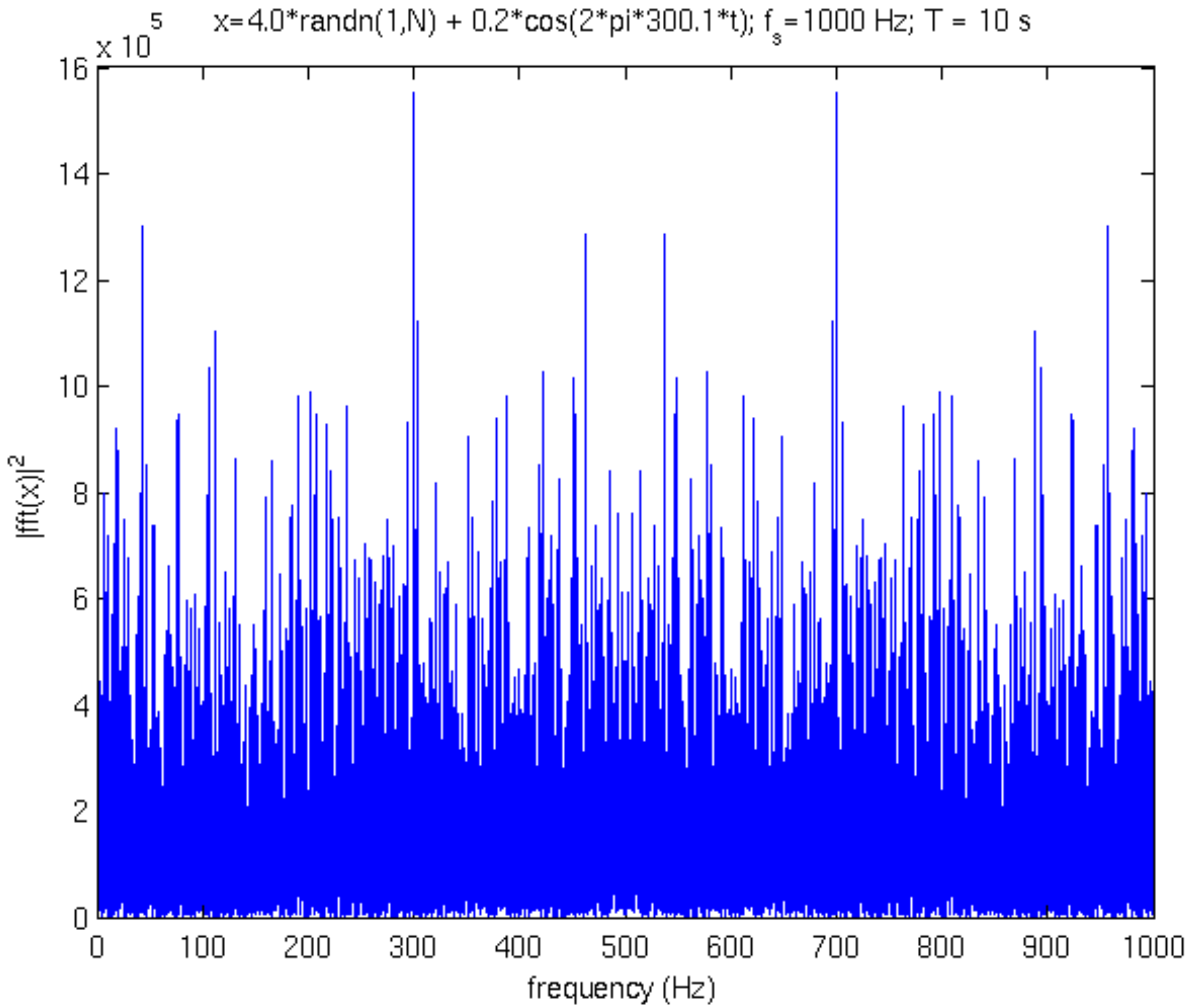
For integer m:

$$f \rightarrow f \pm m f_s; \qquad \tilde{x}_{k \pm mN} = \tilde{x}_k\,; \qquad \tilde{x}_{N-k} = \tilde{x}_k^*$$

$$Useful \quad Band: \qquad [0, f_{Nyquist}] \rightarrow k = [0, N/2]$$

Power outside this band is aliased into this band. Thus need to filter data before digitizing, or when changing f_s, to prevent aliasing of unwanted power into this band.

DFT Aliasing



DFT Orthogonality

$$\sum_{j=0}^{N-1} e^{2\pi ijk'/N} e^{-2\pi ijk/N} = \sum_{j=0}^{N-1} e^{2\pi ij(k'-k)/N} = \sum_{j=0}^{N-1} \left[e^{2\pi i(k'-k)/N} \right]^{j}$$

$$S = \sum_{j=0}^{N-1} r^{j}; rS = \sum_{j=0}^{N-1} r^{j+1}; rS = \sum_{j'=1}^{N} r^{j'=j+1} = \sum_{j'=0}^{N-1} r^{j'} - 1 + r^{N}; S = \frac{1-r^{N}}{1-r}$$

$$\sum_{j=0}^{N-1} \left[e^{2\pi i(k'-k)/N} \right]^{j} = \frac{1-\left[e^{2\pi i(k'-k)/N} \right]^{N}}{1-e^{2\pi i(k'-k)/N}} = \frac{1-e^{2\pi i(k'-k)}}{1-e^{2\pi i(k'-k)/N}} = 0, k \neq k'; N, k = k'$$

$$\boxed{\sum_{j=0}^{N-1} e^{2\pi ijk'/N} e^{-2\pi ijk/N} = N\delta_{kk'}}$$

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Parseval's Theorem

$$\sum_{j=0}^{N-1} \left| x_j \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| \tilde{x}_k \right|^2$$

$$\sum_{j=0}^{N-1} x_j y_j = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}_k \tilde{y}_k^*$$

Correlation Theorem

$$c_{j'} = \sum_{j=0}^{N-1} x_j y_{j+j'}$$

$$\tilde{c}_k = \tilde{x}_k \tilde{y}_k^*$$

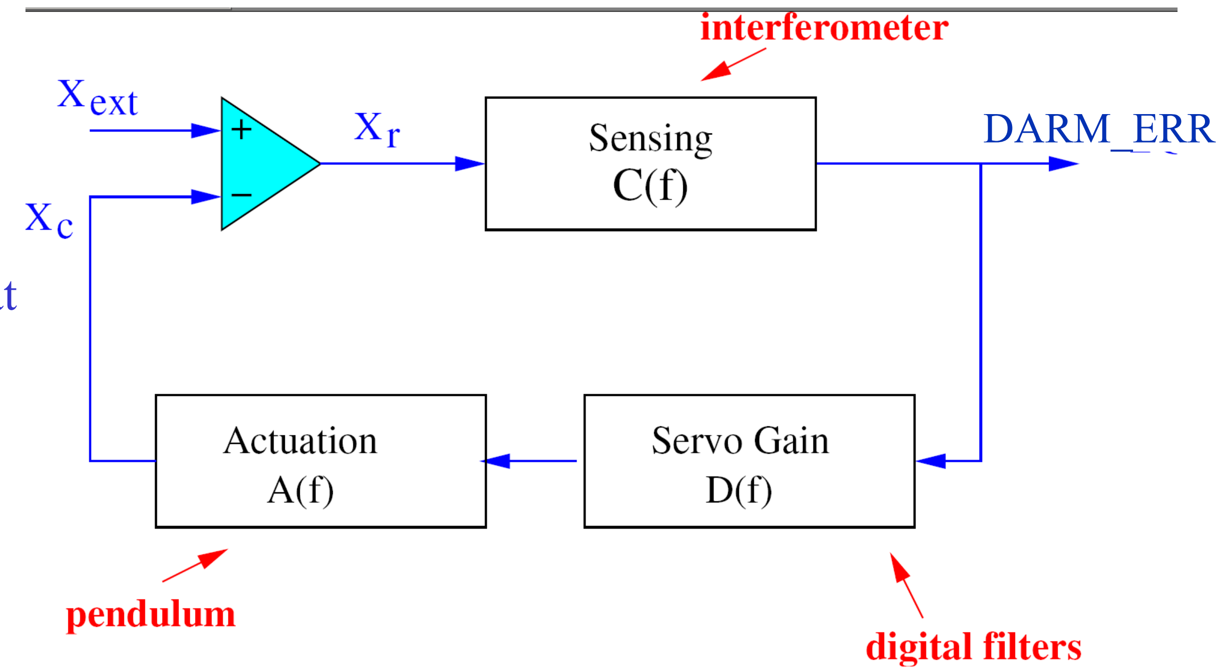
Convolution Theorem

$$C_{j'} = \sum_{j=0}^{N-1} x_j y_{j'-j}$$

$$\tilde{C}_k = \tilde{x}_k \tilde{y}_k$$

Example: Calibration

- Measure open loop gain G, input unity gain to model
- Extract gain of sensing function C=G/AD from model
- Produce response function at time of the calibration, R=(1+G)/C
- Now, to extrapolate for future times, monitor calibration lines in DARM_ERR error signal, plus any changes in gains.
- Can then produce R at any later time t.
- A photon calibrator, using radiation pressure, gives results consistent with the standard calibration.



$$G(f) = C(f)D(f)A(f)$$

$$R(f) = \frac{1+G(f)}{C(f)}$$

$$x_{ext}(f) = R(f)DARM\_ERR(f)$$

One-sided Power Spectral Density (PSD) Estimation

$$P_k = \frac{2\left\langle \left| \tilde{x}_k \right|^2 \right\rangle \Delta t^2}{T}$$

Note that the absolute square of a Fourier Transform gives what we call "power". A one-sided PSD is defined for positive frequencies (the factor of 2 counts the power from negative frequencies). The angle brackets, < >, indicate "average value". Without the angle brackets, the above is called a periodogram. Thus, the PSD estimate is found by averaging periodograms. The other factors normalize the PSD so that the area under the PDS curves gives the RMS^2 of the time domain data.

Gaussian White Noise

$$\langle n_j \rangle = 0; \qquad \left\langle |n_j|^2 \right\rangle = \frac{1}{N}\sum_{j=0}^{N-1} |n_j|^2 = \sigma^2$$

$$\left\langle |\tilde{n}_k|^2 \right\rangle = \frac{1}{N}\sum_{k=0}^{N-1} |\tilde{n}_k|^2 = \sum_{j=0}^{N-1} |n_j|^2 = N\sigma^2$$

Parseval's Theorem is used above.

Power Spectral Density, Gaussian White Noise

$$P_k = \frac{2N\sigma^2\Delta t^2}{T} = 2\sigma^2\Delta t = \frac{2\sigma^2}{f_s}$$

This ratio is constant, independent of f_s.

$$\sqrt{P} = \frac{\sqrt{2}\sigma}{\sqrt{f_s}}$$

The square root of the PSD is the Amplitude Spectral Density. It has the units of sigma per root Hz.

$$\sum_{k=0}^{N/2} P_k \Delta f = \sum_{k=0}^{N/2} \frac{2\sigma^2}{f_s}\frac{1}{T} = \frac{N}{2}\frac{2\sigma^2}{N} = \sigma^2 \rightarrow Area = RMS^2$$

For gaussian white noise, the square root of the area under the PSD gives the RMS of time domain data.

Amplitude and Phase of a spectral line

$$x_j = A\cos(2\pi f t_j + \phi_0) = A\frac{e^{2\pi i f t_j + i\phi_0} + e^{-2\pi i f t_j - i\phi_0}}{2}$$

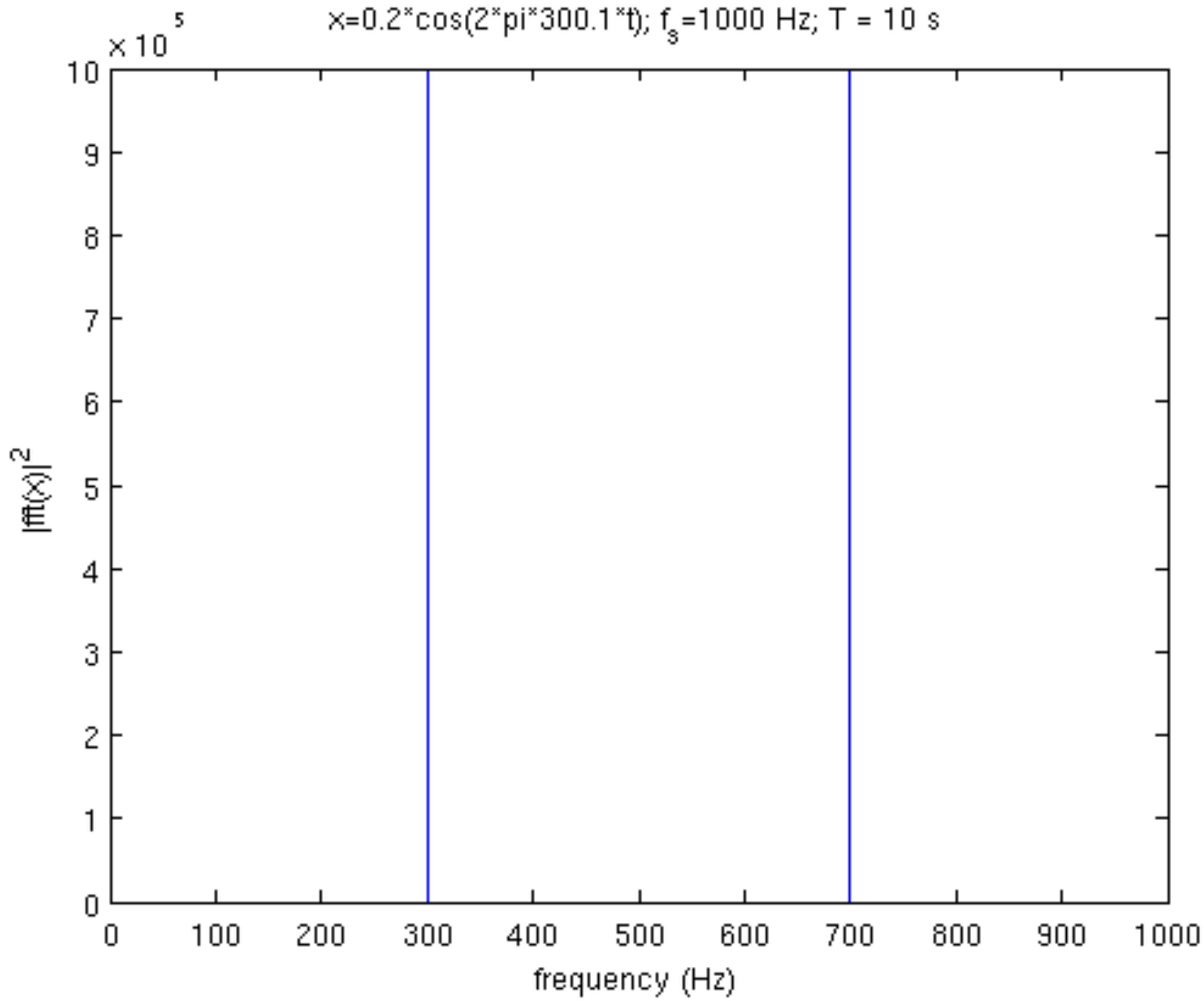
$$fT = k$$

If the product of f and T is an integer k, we call say this frequency is "bin centered".

$$\tilde{x}_k = \frac{AN}{2}e^{i\phi_0}$$

For a sinusoidal signal with a "bin centered" frequency, all the power lies in one bin.

DFT Bin Centered Frequency



DFT Leakage

$$x_j = A\cos(2\pi f t_j + \phi_0) = A\frac{e^{2\pi i f t_j + i\phi_0} + e^{-2\pi i f t_j - i\phi_0}}{2}$$

$$fT = \kappa; \qquad \Delta\kappa = \kappa - k$$

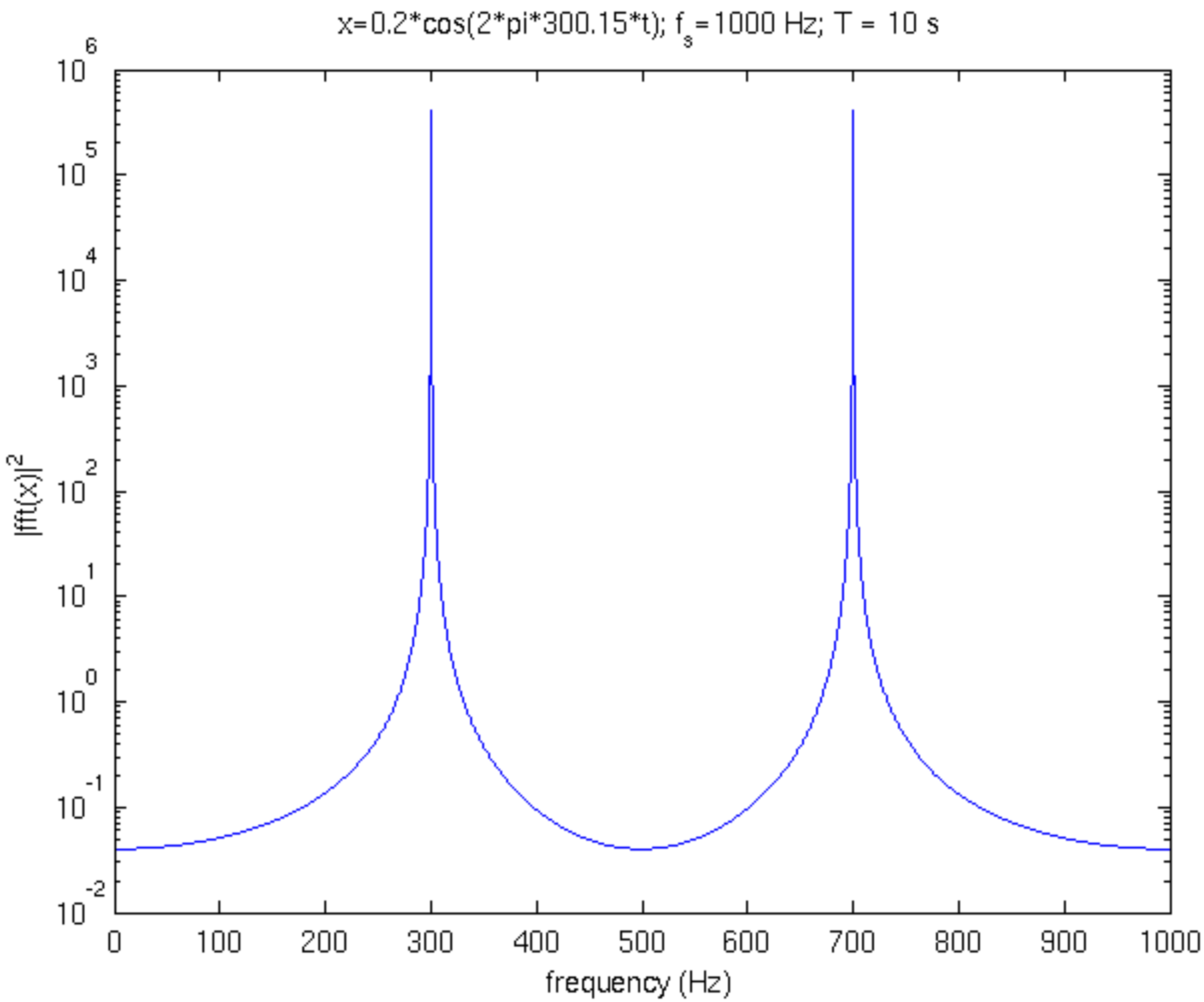
For kappa not an integer get sinc function response:

$$\tilde{x}_k = \frac{AN}{2}e^{i\phi_0}\left[\frac{\sin(2\pi\Delta\kappa)}{2\pi\Delta\kappa} - i\frac{1-\cos(2\pi\Delta\kappa)}{2\pi\Delta\kappa}\right]$$

$$\left|\tilde{x}_k\right|^2 = \frac{A^2N^2}{4}\frac{\sin^2(\pi\Delta\kappa)}{\pi^2\Delta\kappa^2}$$

For a signal (or spectral disturbance) with a non-bin-centered frequency, power leaks out into the neighboring bins.

DFT Leakage



x=0.2*cos(2*pi*300.15*t); f_s=1000 Hz; T = 10 s
|fft(x)|^2
10^6
10^5
10^4
10^3
10^2
10^1
10^0
10^-1
10^-2
0
100
200
300
400
500
600
700
800
900
1000
frequency (Hz)

Windowing reduces leakage

$$Hann \quad Window: \qquad w_j = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi j}{N-1}\right)\right]$$

$$\tilde{w}_k = \frac{N}{2}\delta_{k,0} - \frac{N}{4}\delta_{k,1} - \frac{N}{4}\delta_{k,-1}; \quad for \qquad N \cong N-1$$

$$x_j^w = w_j x_j \rightarrow \tilde{x}_k^w = \frac{1}{N}\sum_{k'=0}^{N} \tilde{x}_{k'}\tilde{w}_{k-k'}$$

$$\tilde{x}_k^w = \frac{1}{2}\tilde{x}_k - \frac{1}{4}\tilde{x}_{k-1} - \frac{1}{4}\tilde{x}_{k+1}$$

Corrections to PSD

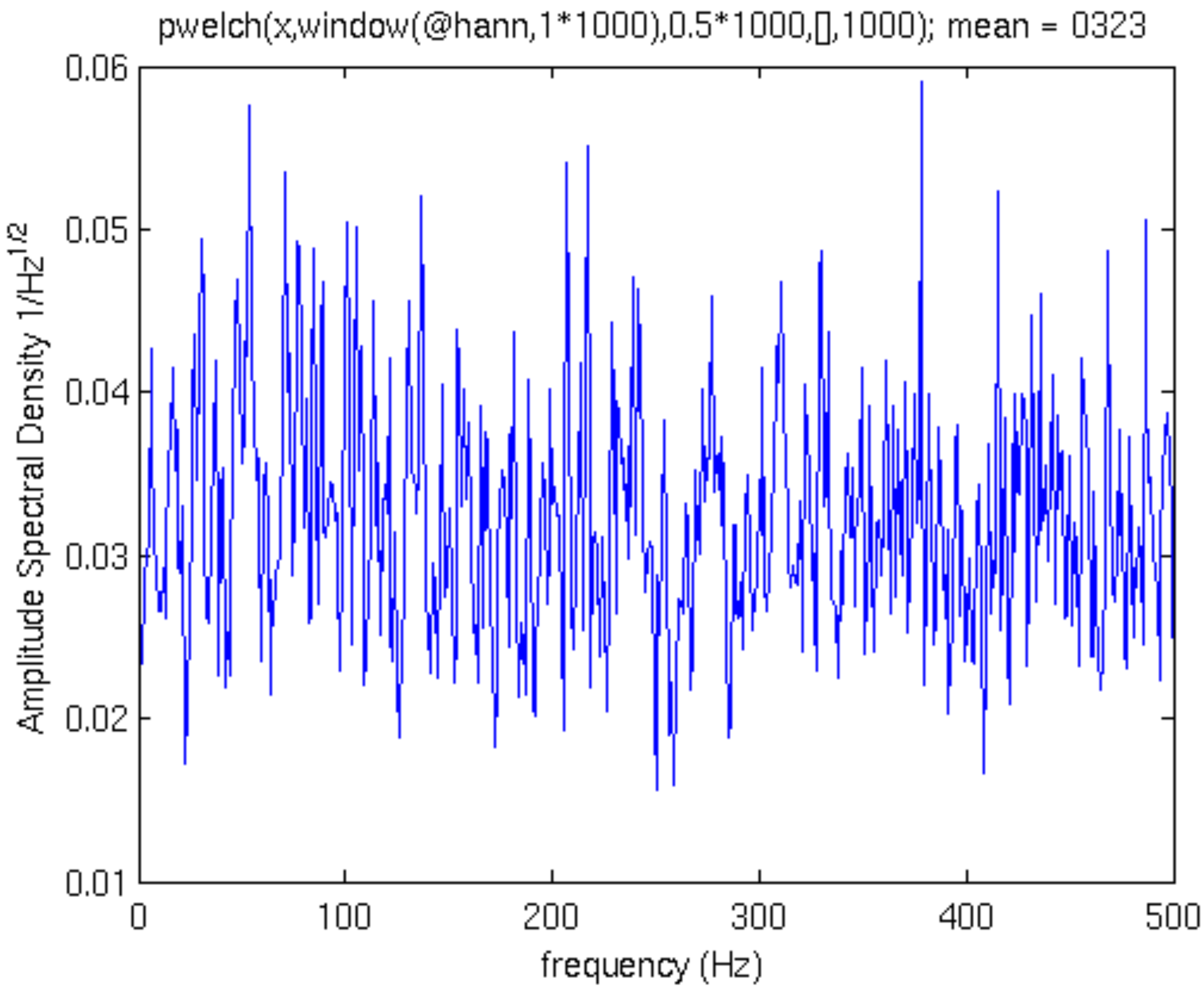
$$Amplitude \quad Correction: \qquad \tilde{x}_k = 2\tilde{x}_k^w$$

$$Energy \quad Correction: \qquad \left|\tilde{n}_k\right|^2 = \frac{8}{3}\left|\tilde{n}_k^w\right|^2$$

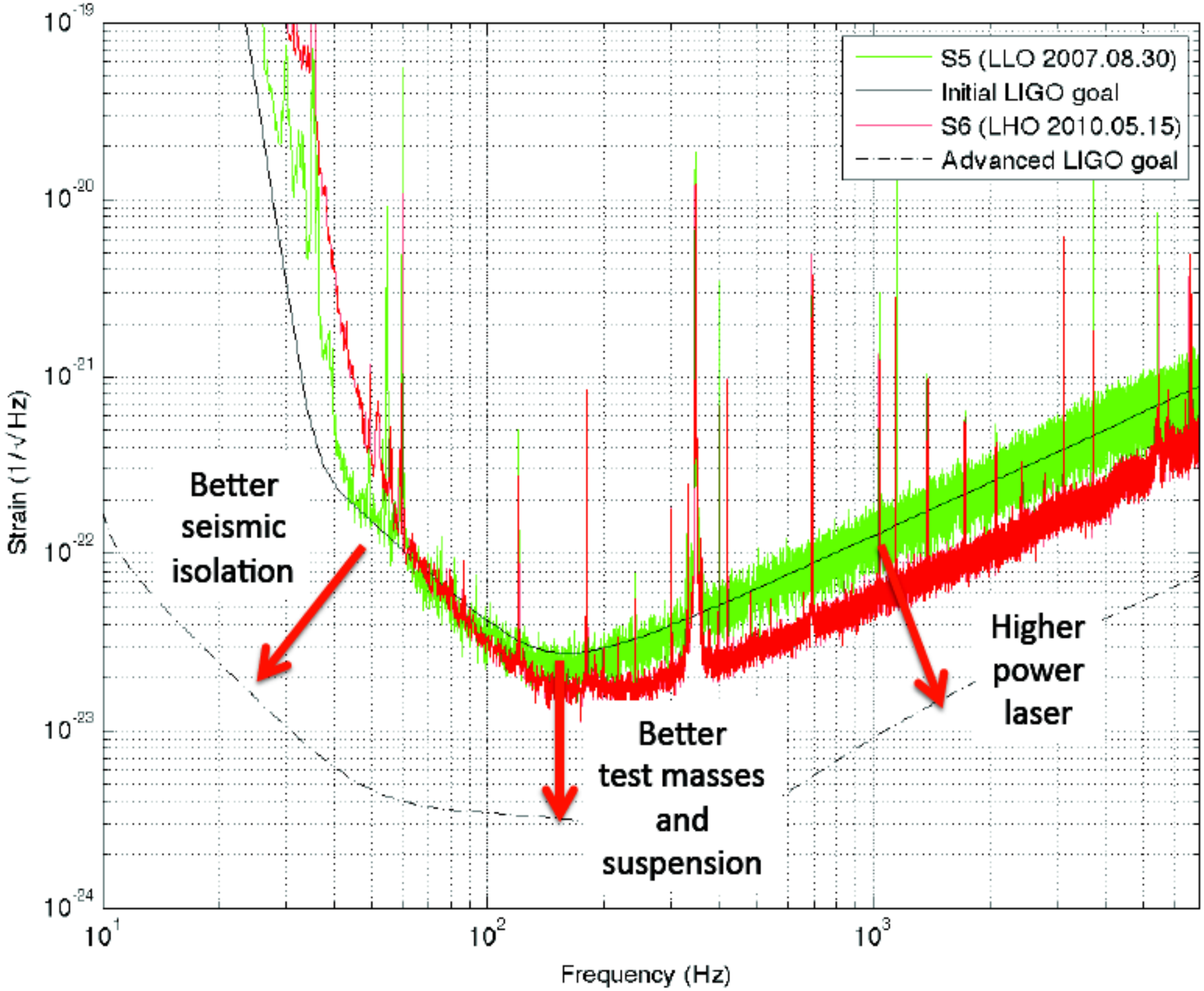
$$SNR^2 = \frac{1}{2}\frac{A^2 T}{S_k} \rightarrow \frac{1}{2}\frac{2^2 (A/2)^2 T/2}{2^2 (3/8) S_k} = \frac{1}{2}\frac{A^2 (2T/3)}{S_k}$$

$$\Delta f \rightarrow \frac{3}{2}\Delta f$$

Example PSD Estimation



For iLIGO and aLIGO



PSD Statistics

$$\tilde{n} = x + iy$$

$$|\tilde{n}|^2 = x^2 + y^2; \qquad if <x>=<y>=0; <x^2>=<y^2>=1$$

$$\mathrm{P}(x,y)dxdy = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dxdy$$

$$r = \sqrt{x^2 + y^2}; \phi = \tan^{-1}(y/x); dxdy \rightarrow rdrd\phi$$

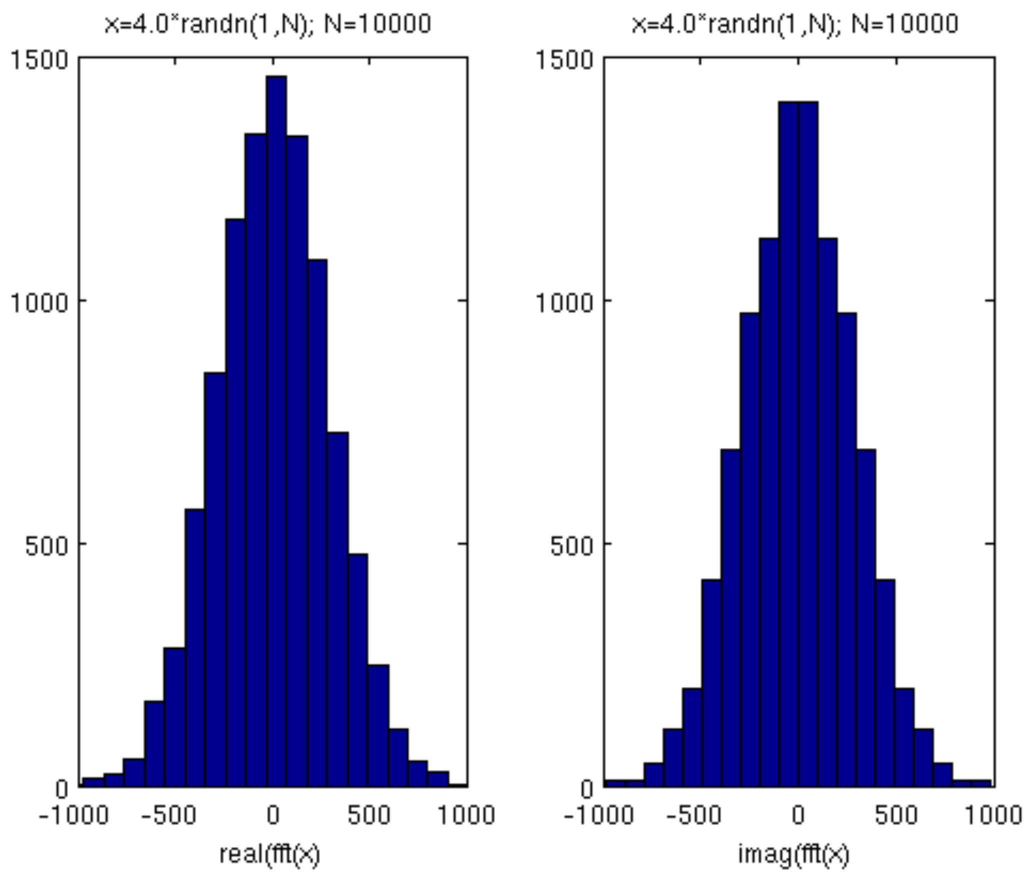
Rayleigh Distribution:

$$\mathrm{P}(r,\phi)drd\phi = \frac{1}{2\pi} re^{-r^2/2} drd\phi; \qquad \mathrm{P}(r)dr = re^{-r^2/2}dr$$

Chi-squared for 2 degrees of freedom:

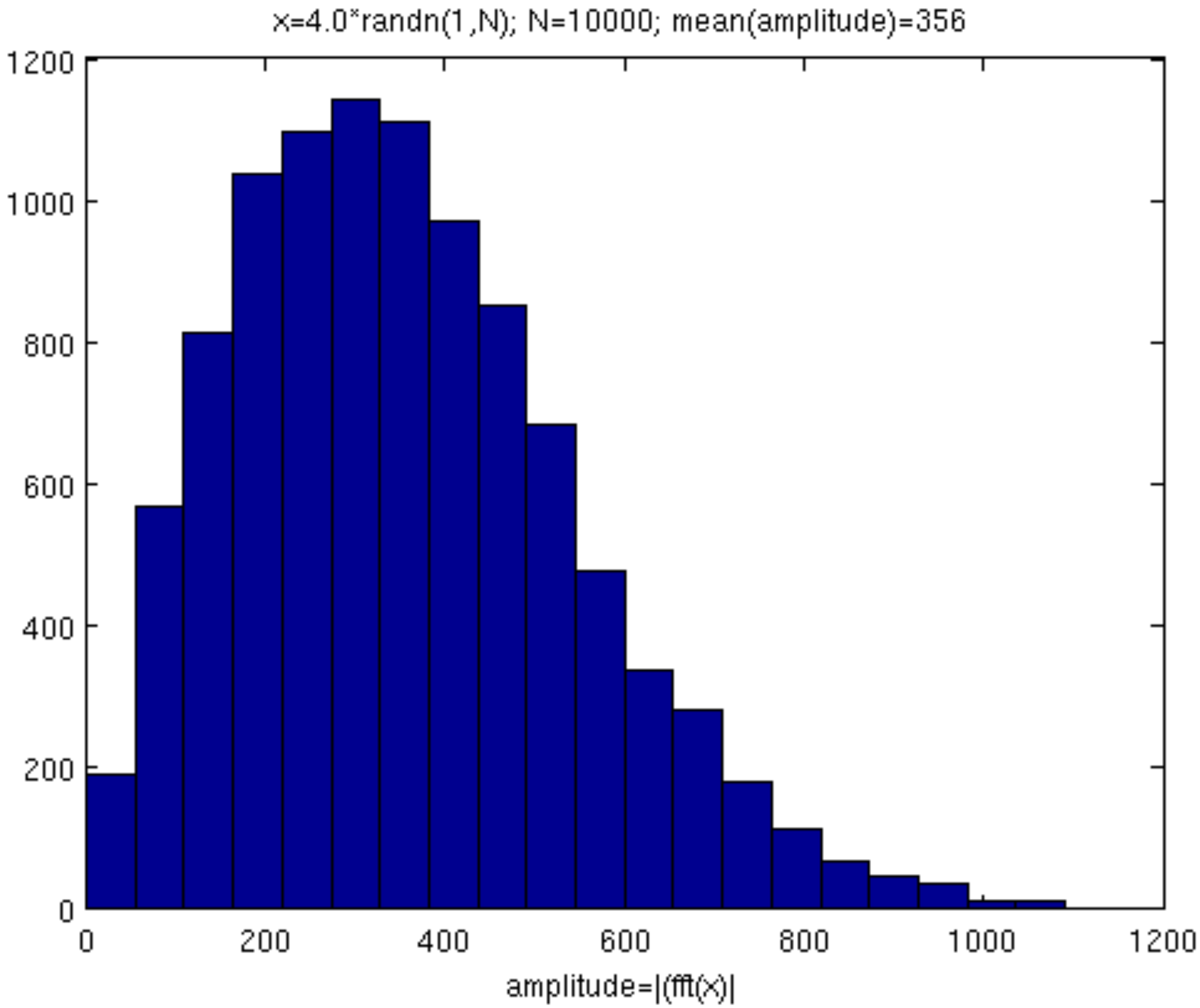
$$\rho = r^2; \frac{1}{2} d\rho = rdr; \qquad \mathrm{P}(\rho)d\rho = \frac{1}{2} e^{-\rho/2} d\rho$$

Histograms, Real and Imaginary Part of DFT



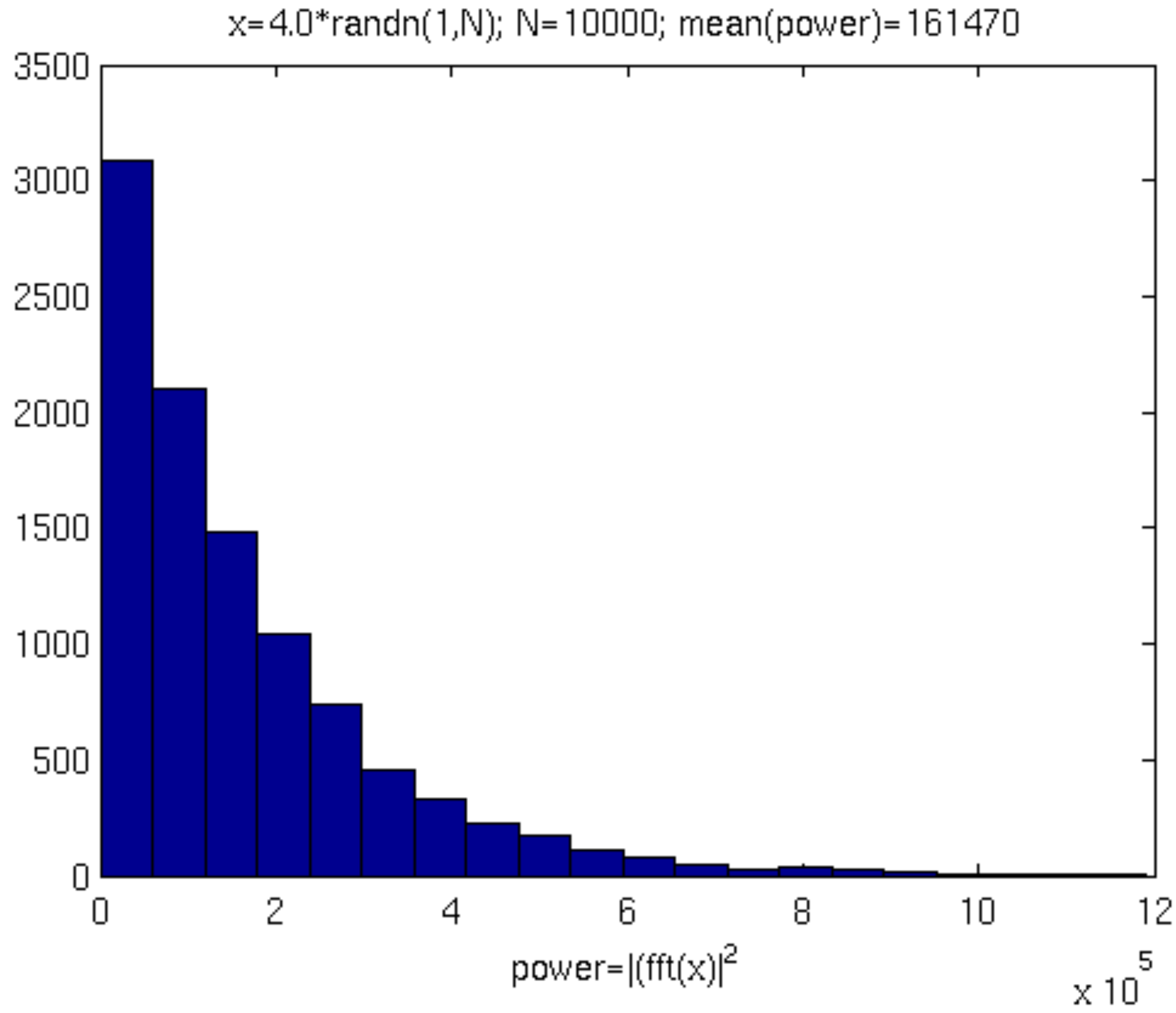
For gaussian noise, the above are also gaussian distributions.

Rayleigh Distribution



Histogram of the square root of the DFT power.

Histogram DFT Power



Result is a chi-squared distribution for 2 degrees of freedom.

Chi-Squared Statistics

A chi-squared variable with ν degrees of freedom is the sum of the squares of ν gaussian distributed variables with zero mean and unit variance.

$$\rho = x^2 + y^2 + z^2 + \ldots; \qquad if < x >=< y >=< z >= \ldots = 0$$

$$r^2 = x^2 + y^2 + z^2 + \ldots; \qquad < x^2 >=< y^2 >=< z^2 >= \ldots = 1$$

$$e.g., dxdydz \rightarrow r^2 \sin\theta dr d\theta d\phi; \qquad \rho = r^2; d\rho = 2rdr$$

$$\mathrm{P}(\rho)d\rho \propto \rho^{1/2} e^{-\rho/2} d\rho$$

$$\mathrm{P}(\rho)d\rho = \frac{1}{2^{\upsilon/2}\Gamma(\upsilon/2)} \rho^{\frac{\upsilon}{2}-1} e^{-\rho/2} d\rho$$

Maximum Likelihood

Likelihood of getting data x for model h for Gaussian Noise:

$$P(x\,|\,h) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x_1-h_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-(x_2-h_2)^2}{2\sigma_2^2}} \frac{1}{\sqrt{2\pi}\sigma_3} e^{\frac{-(x_3-h_3)^2}{2\sigma_3^2}} \ldots$$

$$\chi^2 = \sum_j \frac{(x_j - h_j)^2}{\sigma_j^2} \Rightarrow -2\left(\sum_j \frac{x_j h_j}{\sigma_j^2} - \frac{1}{2} \sum_j \frac{h_j h_j}{\sigma_j^2} \right)$$ **Chi-squared**

$$e.g., h_j = f(h_0, \cos\iota, \Phi_0, \psi)$$

$$\frac{\partial \chi^2}{\partial h_0} = 0, \qquad \frac{\partial \chi^2}{\partial \cos\iota} = 0, \qquad \frac{\partial \chi^2}{\partial \Phi_0} = 0, \qquad \frac{\partial \chi^2}{\partial \psi} = 0$$

Minimize Chi-squared = Maximize the Likelihood

Matched Filtering

$$\ln\Lambda = \left(\sum_j \frac{x_j h_j}{\sigma_j^2} - \frac{1}{2}\sum_j \frac{h_j h_j}{\sigma_j^2}\right) = \left(\sum_k \frac{\tilde{x}_k^* \tilde{h}_k}{S_k} - \frac{1}{2}\sum_k \frac{\tilde{h}_k^* \tilde{h}_k}{S_k}\right)$$

$$\ln\Lambda = (x,h) - \frac{1}{2}(h,h)$$

This is called the log likelihood; x is the data, h is a model of the signal.

$$e.g., h \rightarrow Ah; \qquad \ln\Lambda = A(x,h) - \frac{A^2}{2}(h,h)$$

$$\partial \ln\Lambda / \partial A = (x,h) - A(h,h) = 0$$

Maximizing over a single amplitude gives:

$$A = \frac{(x,h)}{(h,h)}; \qquad \ln\Lambda_{\max} = \frac{1}{2}\frac{(x,h)^2}{(h,h)}; \qquad SNR \propto \sum_k \frac{\tilde{x}_k^* \tilde{h}_k}{S_k} \Big/ \sqrt{\sum_k \frac{\tilde{h}_k^* \tilde{h}_k}{S_k}}$$

This can be generalized to maximize over more parameters, giving the formulas for matched-filtering in terms of dimensionless templates.

Frequentist Confidence Region

Matlab simulation for signal: $s(t) = A\cos 2\pi ft + B\sin 2\pi ft$

- Estimate parameters by maximizing likelihood or minimizing chi-squared
- Injected signal with estimated parameters into many synthetic sets of noise.
- Re-estimate the parameters from each injection

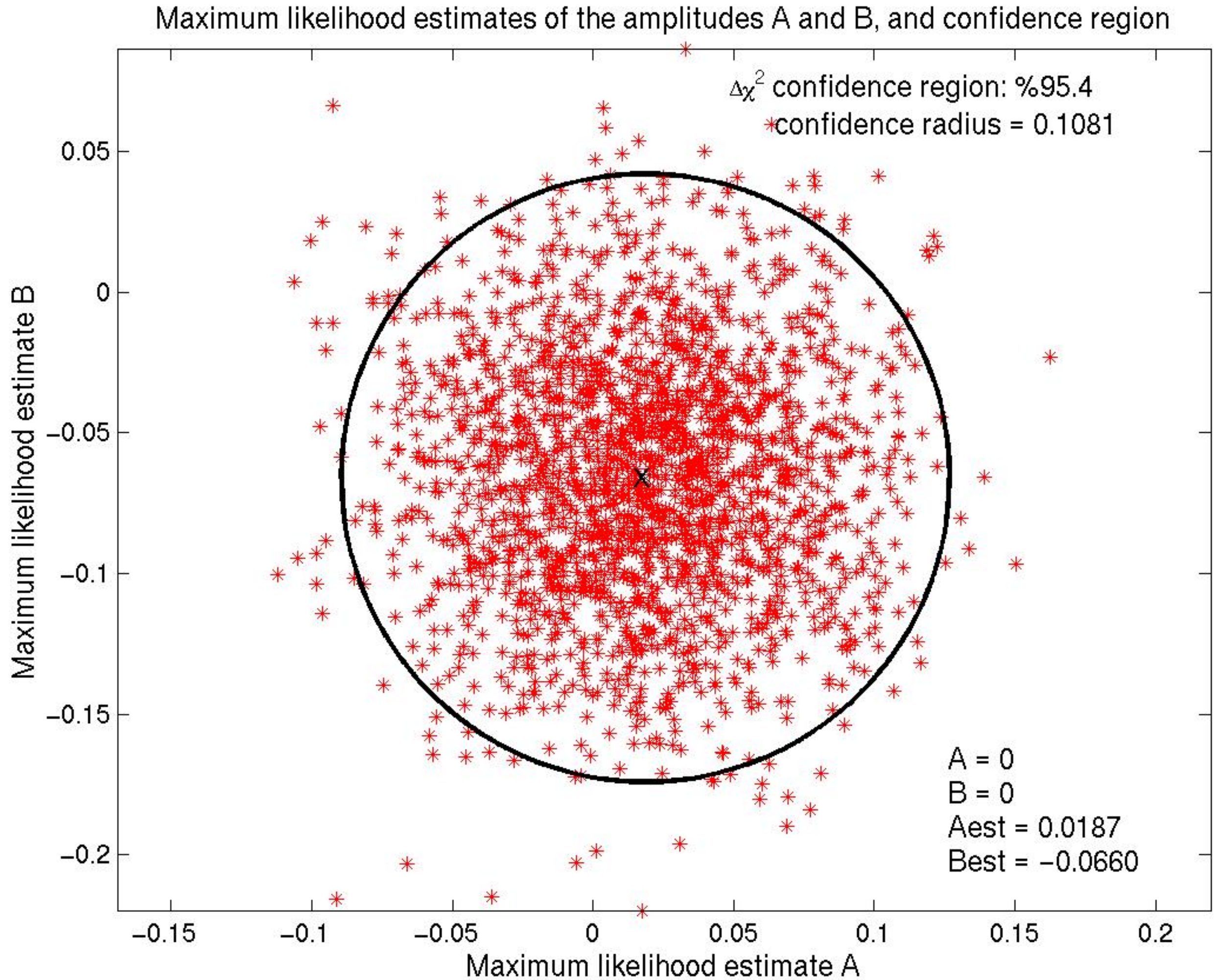


Figure courtesy Anah Mourant, Caltech SURF student, 2003.

Bayesian Analysis

Bayes' Theorem:

$$P(a)P(b \mid a) = P(b)P(a \mid b)$$

Likelihood:

$$P(x \mid h) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x_1 - h_1)^2}{2\sigma_1^2}} \frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-(x_2 - h_2)^2}{2\sigma_2^2}} \ldots$$

Posterior Probability:

$$P(h \mid x) = \frac{P(h)}{P(x)} P(x \mid h)$$

Confidence Interval:

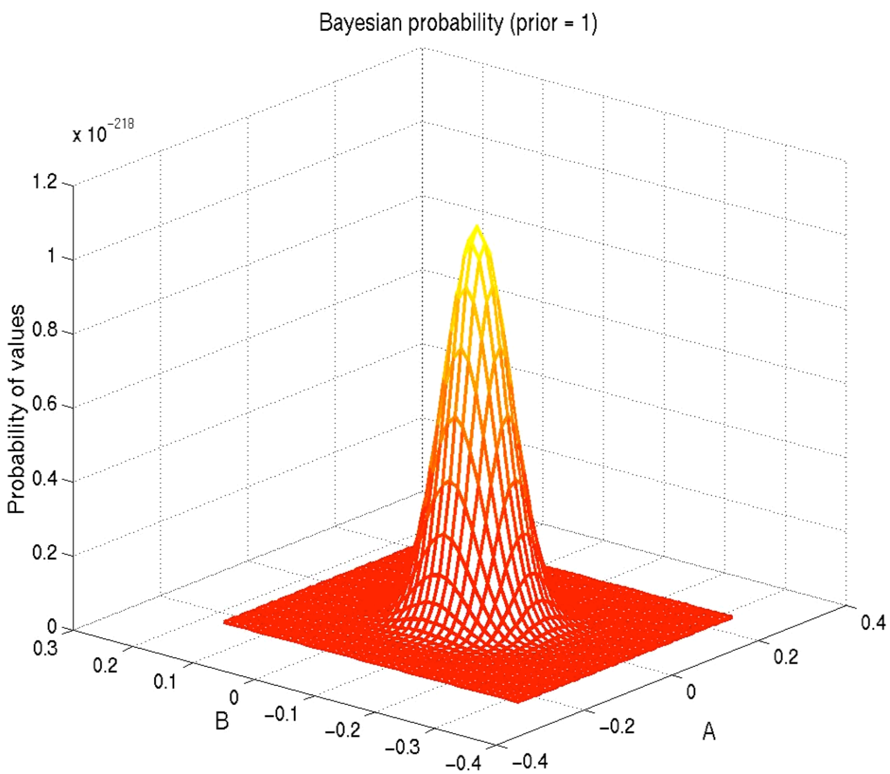
$$C = \int_0^{h_C} P(h \mid x) dx$$

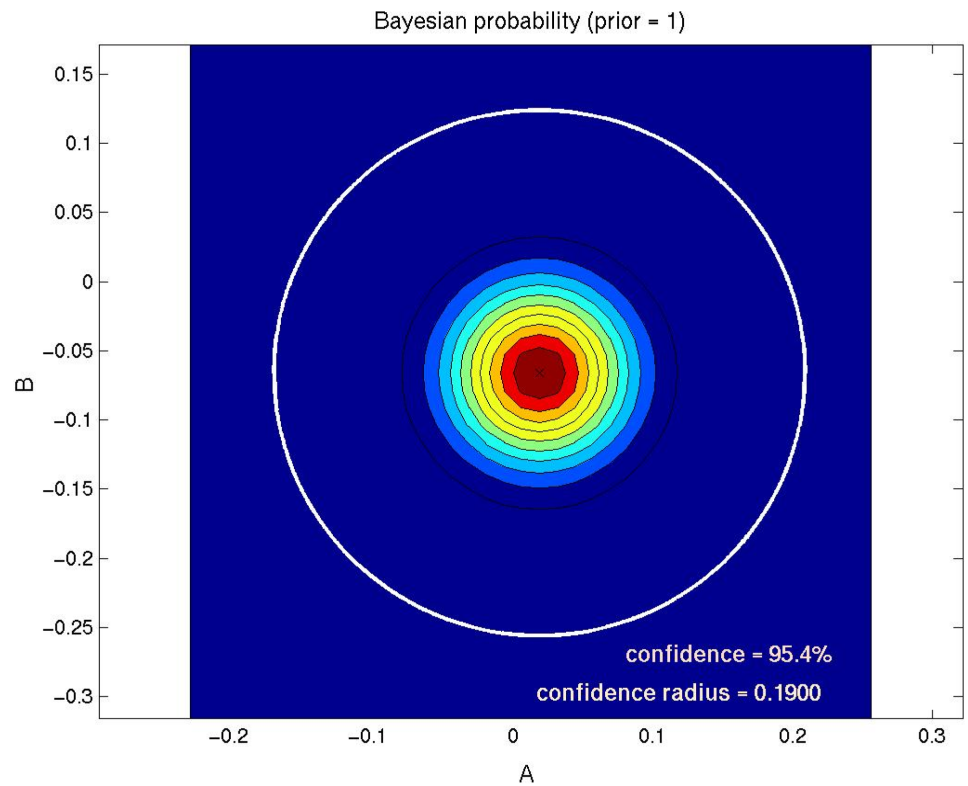
Prior

Bayesian Confidence Region

Matlab simulation for signal: $s(t) = A\cos 2\pi ft + B\sin 2\pi ft$

Uniform Prior: $P(A,B;x) \propto \exp(-\chi^2/2)$





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