Estimating sensitivity of the Einstein@Home search *S5R5*

R. Prix, K. Wette
(Dated: Thu May 31 11:42:49 2012 +0200)
(commitID: bd800542426b91ece4a0dbb05456c02e2ef0160d-CLEAN)
(DCC: LIGO-T1200272-v1)

Notes on semi-analytically estimating the Hough-on-$\mathcal{F}$-stat sensitivity of the E@H run *S5R5*.

I. ESTIMATING SENSITIVITY OF HOUGH-ON-$\mathcal{F}$ STATISTIC

We derive an estimate for the sensitivity of the Hough-on-$\mathcal{F}$ statistic [1] used in the E@H seach *S5R5*. This builds on the sensitivity-estimation methods developed in [2] and [3], adapted to the **non-**χ^2 distributed Hough-on-$\mathcal{F}$ statistic.

The Hough-on-$\mathcal{F}$ statistic is defined as the number $n \le N$ out of a total of N segments where the $\mathcal{F}$-statistic value in a template λ crossed a predefined threshold $\mathcal{F}_{\mathrm{th}}$. The phase parameters λ of the templates include the sky-position $\vec{n}$, frequency f, and higher-order frequency derivatives $\dot{f}, \ddot{f}, \ldots$.

The statistic $2\mathcal{F}_k$ in segment k for signals in Gaussian noise follows a χ^2-distribution with 4 dof and non-centrality parameter ρ_k^2, i.e.

$$P\left(2\mathcal{F}|\rho_k\right) = \chi^2_4(2\mathcal{F};\rho_k^2)\,, \tag{1}$$

where $\rho_k = \rho_k(\lambda_{\mathrm{s}},\lambda)$ is the expected SNR in segment k of a signal with phase parameters λ_{s} when the closest (coarse-grid) template is in λ. This can be expressed in terms of the perfectly-matched SNR $\rho_{\mathrm{opt},k} \equiv \rho_k(\lambda_{\mathrm{s}},\lambda_{\mathrm{s}})$ by introducing the per-segment mismatch $\mu_k(\lambda_{\mathrm{s}},\lambda)$ as

$$\rho_k^2(\lambda_{\mathrm{s}},\lambda) = \left[1-\mu_k(\lambda_{\mathrm{s}},\lambda)\right]\,\rho_{\mathrm{opt},k}^2\,. \tag{2}$$

The probability of a threshold-crossing in segment k is therefore

$$P\left(\mathcal{F}_k > \mathcal{F}_{\mathrm{th}}|\rho_k\right) = \int_{2\mathcal{F}_{\mathrm{th}}}^{\infty} \chi^2_4(2\mathcal{F};\rho_k^2)\,d(2\mathcal{F}) = 1 - \mathrm{cdf}\,\chi^2_4(2\mathcal{F}_{\mathrm{th}};\rho_k^2)\,. \tag{3}$$

Note that the per-segment SNR ρ_k will not be exactly constant across segments, for several reasons:

(i) the closest template λ to λ_{s} will generally be different in every segment, and be subject to a different metric mismatch function μ_k.

(ii) The optimal SNR $\rho_{\mathrm{opt},k}$ in general varies over segments as a function of their start-time (except if the start-times or segment-lengths are multiples of a sidereal day) due to the time-varying antenna-pattern.

(iii) in case of non-stationary noise, $\rho_{\mathrm{opt},k}$ further varies over segments as a function of the noise-PSD for that segment S_k.

(iv) $\rho_{\mathrm{opt},k}$ varies as a function of the amount of data $\Delta T_{\mathrm{data},k}$ used per segment k.

where the antenna-pattern variation (ii) should be very small for the E@H run *S5R5* as $\Delta T = 25\,$hours, and we also don't expect large noise-floor variations (iii) in S_k over different segments.

However, in order to be able to continue we have to approximate the per-segment SNR as constant, i.e. $\rho_k \approx \bar{\rho}$, which we write as

$$\bar{\rho}^2 = \left[1-\bar{\mu}\right]\,\rho_{\mathrm{opt}}^2\,. \tag{4}$$

We define the per-template per-segment threshold-crossing probability $p_{\bar{\rho}}$ in the presence of a signal with constant per-segment SNR $\bar{\rho}$ as

$$p_{\bar{\rho}}(\mathcal{F}_{\mathrm{th}};\bar{\rho}) \equiv P\left(\mathcal{F} > \mathcal{F}_{\mathrm{th}}|\bar{\rho}\right) = 1 - \mathrm{cdf}\,\chi^2_4(2\mathcal{F}_{\mathrm{th}};\bar{\rho}^2)\,, \tag{5}$$

and consequently the per-template per-segment false-alarm probability is $p_0(\mathcal{F}_{\mathrm{th}}) \equiv P\left(\mathcal{F} > \mathcal{F}_{\mathrm{th}}|\bar{\rho}=0\right)$.

Given $p_{\bar{\rho}}$, we can express the (discrete) probability distribution for the Hough number count n as

$$P\left(n|N,\,p_{\bar{\rho}}\right) = \binom{N}{n}\,p_{\bar{\rho}}^n\,\left(1-p_{\bar{\rho}}\right)^{N-n}\,. \tag{6}$$

The overall Hough false-alarm and false-dismissal probabilities are therefore

$$p^H_{f_{\rm A}}(n_{\rm th}, \mathcal{F}_{\rm th}) = P\left(n \geq n_{\rm th} | \mathcal{F}_{\rm th}, \bar{\rho} = 0\right) = \sum_{n=n_{\rm th}}^{N} P\left(n|N, p_0\right) \,, \tag{7}$$

$$p^H_{f_{\rm D}}(n_{\rm th}, \mathcal{F}_{\rm th}, \bar{\rho}) = P\left(n < n_{\rm th} | \mathcal{F}_{\rm th}, \bar{\rho}\right) = \sum_{n=0}^{n_{\rm th}-1} P\left(n|N, p_{\bar{\rho}}\right) \,. \tag{8}$$

Estimating the sensitivity of a search typically consists of injecting signals drawn from a population Π_{h_0} with fixed h_0, and determining the overall false-dismissal probability $p^H_{f_{\rm D}}$ for this population for a fixed threshold $n^*_{\rm th}$, corresponding to a certain false-alarm probability $p^{H\,*}_{f_{\rm A}}$. The amplitude parameter h_0 is varied until a desired confidence level $1 - p^{H\,*}_{f_{\rm D}}$ is obtained, and the corresponding signal amplitude $h^*_0 = h_0(p^{H\,*}_{f_{\rm A}}, p^{H\,*}_{f_{\rm D}})$ is considered a measure for the sensitivity of the search. This critical amplitude is the solution to the equation

$$p^{H\,*}_{f_{\rm D}} = P\left(n < n^*_{\rm th} | \mathcal{F}_{\rm th}, h^*_0, \Pi_{h_0}\right) \,, \tag{9}$$

for given threshold $\mathcal{F}_{\rm th}$ and fixed-h_0 signal population Π_{h_0}.

Following the notation of [2, 4, 5], we can write the optimal (per-segment) SNR $\rho_{\rm opt}$ of a perfectly-matched template ($\lambda = \lambda_{\rm s}$) as

$$\rho_{\rm opt} = \hat{\rho}(h_0)\, \mathcal{R}(\theta) \,, \tag{10}$$

where $\theta \equiv \{\cos\iota, \psi, \vec{n}\}$, and $\mathcal{R}(\theta)$ denotes the geometric antenna-pattern response of the detector network to a GW from direction $\vec{n}$ with amplitude parameters $\{\cos\iota, \psi\}$. Using the notation of Eq. (95) in [5], we can write the response function explicitly as

$$\mathcal{R}^2(\theta) = \frac{25}{4}\, \left(\alpha_1\, A + \alpha_2\, B + 2\alpha_3\, C\right) \,, \tag{11}$$

where $\alpha_i = \alpha_i(\cos\iota, \psi)$ and A, B, C are functions of sky-position $\vec{n}$ (and, generally, data segment k).

For an all-sky search, the signal population Π_{h_0} consists of an isotropic distribution over the sky $\vec{n}$, and uniform distributions over $\cos\iota \in [-1, 1]$ and $\psi \in [-\pi/4, \pi/4]$, and one can show in general [5] that $\langle \mathcal{R}^2 \rangle_\theta = 1$. Following [2] we introduce the optimal per-segment "rms SNR" $\hat{\rho}$ of the signal population Π_{h_0}, namely

$$\hat{\rho} \equiv \sqrt{\langle \rho^2_{\rm opt} \rangle_\theta} = \frac{2}{5} h_0 \sqrt{\frac{T_{\rm data}/N}{\mathcal{S}}} \,, \tag{12}$$

we $\mathcal{S}$ is the overall noise floor, defined as the harmonic mean

$$\mathcal{S}^{-1} \equiv \frac{1}{N_{\rm SFT}} \sum_{X\alpha}^{N_{\rm SFT}} S^{-1}_{X\alpha} \,, \tag{13}$$

over the total number $N_{\rm SFT}$ of SFTs (over all segments), $S_{X\alpha}$ is the per-SFT noise PSD for detector X and time-index α, while

$$T_{\rm data} \equiv N_{\rm SFT}\, T_{\rm SFT} \,, \tag{14}$$

is the total amount of data used (over all segments)[1].

The unknown signal location $\lambda_{\rm s}$ gives rise to a (template-bank dependent) probability distribution for the mismatch $\bar{\mu}$, which affects the measured (average) per-segment SNR $\bar{\rho}$ as seen in (4). We can absorb this effect by introducing an 'effective' response $\mathcal{R}_{\rm eff}(\theta; \bar{\mu})$ which includes the (unknown) mismatch $\bar{\mu}$:

$$\mathcal{R}_{\rm eff}(\theta; \bar{\mu}) \equiv \left[1 - \bar{\mu}(\lambda_{\rm s})\right]\, \mathcal{R}(\theta) \,, \tag{15}$$

---

[1] Under "ideal" data conditions of $N_{\rm det}$ detectors with identical stationary noise-floor $S_{X\alpha} = S_{\rm n}$ without gaps, we have $\mathcal{S} = S_{\rm n}$, and $T_{\rm data} = N_{\rm det}\, N\, \Delta T$.

which allows us to express the (per-segment) SNR $\bar{\rho}$ using Eqs. (10) and (4) as

$$\bar{\rho} = \hat{\rho}(h_0)\,\mathcal{R}_{\rm eff}(\theta;\bar{\mu})\,. \tag{16}$$

The sensitivity equation (9) can be written more explicitly as [2]:

$$p_{f_{\rm D}}^{H\,*} = \int P\left(n < n^*_{\rm th}|\mathcal{F}_{\rm th},\ \bar{\rho} = \hat{\rho}^*\,\mathcal{R}_{\rm eff}\right)\,P\left(\mathcal{R}_{\rm eff}|\Pi_{h_0}\right)\,d\mathcal{R}_{\rm eff}\,, \tag{17}$$

where the threshold on number-count $n^*_{\rm th} = n_{\rm th}(p_{f_{\rm A}}^{H\,*}, N)$ is obtained by inverting Eq. (7).

This equation is to be solved for the minimal "rms-SNR" $\hat{\rho}^*$, which we can translate into a minimum signal amplitude h_0^* using Eq. (12):

$$h_0^* = \frac{5}{2}\,\hat{\rho}^*\,\sqrt{\frac{\mathcal{S}}{\Delta T_{\rm data}}}\,, \tag{18}$$

where $\Delta T_{\rm data} \equiv T_{\rm data}/N$.

A. Possible conventions for expressing sensitivity statements

- Karl has proposed [2] to use $\hat{\rho}^*$ directly to characterize the sensitivity of a search.
- Map has used a functional form inspired by the Hough paper (Eq.6.41) in [1]):

$$h_0^* = \frac{F}{N^{1/4}}\,\sqrt{\frac{\mathcal{S}}{\Delta T_{\rm data}}}\,, \tag{19}$$

  namely $F \equiv 5/2\hat{\rho}^*\,N^{1/4}$.

- We propose to standardize sensitivity statements with respect to only "global" properties of the search, namely $\mathcal{S}$ and $T_{\rm data}$, and absorb all "internal" search properties (e.g. $N, \Delta T, \bar{\mu}, \ldots$) into the sensitivity pre-factor. This could be done either as

$$h_0^* = H\,\sqrt{\frac{\mathcal{S}}{T_{\rm data}}}\,, \tag{20}$$

  where $H \equiv 5/2\,\hat{\rho}^*\sqrt{N}$.

  This is in analogy to what was done for fully-coherent searches (e.g. $H = 11.4$ for a targeted F-statistic search with $p^*_{f_{\rm A}} = 0.01$ and $p^*_{f_{\rm D}} = 0.1$ [6]).

- One could even absorb *all* search-specific parameters (including $T_{\rm data}$) into a single "sensitivity factor" σ_h (with dimensions of $\sqrt{S_{\rm n}}$, i.e. $\sqrt{\rm Hz}^{-1}$), and express

$$h_0^* = \frac{\sqrt{\mathcal{S}}}{\sigma_h}\,, \tag{21}$$

  where $\sigma_h^{-1} \equiv 5/2\hat{\rho}^*\,\Delta T_{\rm data}^{-1/2}$. This latter definition has the advantage of quantifying the overall "intrinsic" sensitivity of the search, independently of the detector noise level $\sqrt{\mathcal{S}}$.

B. Biased sensitivity approximation

Note that we can equivalently express Eq. (17) in the form of an average, namely

$$p_{f_{\rm D}}^{H\,*} = \langle P\left(n < n^*_{\rm th}|\mathcal{F}_{\rm th},\ \bar{\rho} = \hat{\rho}^*\,\mathcal{R}_{\rm eff}\right)\rangle_{\Pi_{h_0}}\,. \tag{22}$$

A simpler (but biased) sensitivity approximation (used in [1]) proceeds by solving instead

$$p_{f_{\rm D}}^{H\,*} = P\left(n < n^*_{\rm th}|\mathcal{F}_{\rm th},\ \bar{\rho}^2 = \tilde{\hat{\rho}}^{*2}\,\langle\mathcal{R}^2_{\rm eff}\rangle_{\Pi_{h_0}}\right)\,, \tag{23}$$

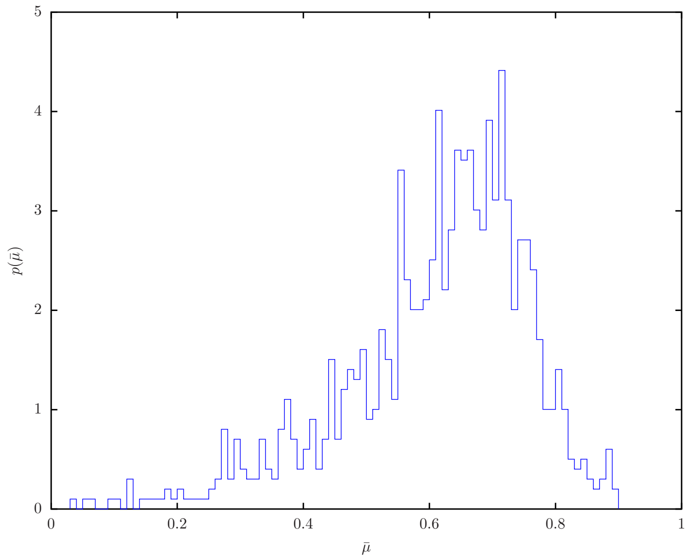

FIG. 1: Effective $\mathcal{F}$-statistic mismatch distribution in SNR, $P(\bar{\mu}|I)$.

where the averaging refers to the assumed signal population Π_{h_0}. The difference is simply whether to compute the false-dismissal probability for the average SNR^2, or the average of the false-dismissal probability as a function of SNR^2. If $P(n < n_{\rm th}|\bar{\rho})$ (which is always monotonically decreasing with $\bar{\rho}$) is a *convex* function of $\bar{\rho}$ (which won't always be the case), then Jensen's inequality states that

$$P\left(n < n_{\rm th}|\hat{\rho}^2 \langle \mathcal{R}^2_{\rm eff}\rangle\right) \leq \langle P\left(n < n_{\rm th}|\hat{\rho}^2\, \mathcal{R}^2_{\rm eff}\right)\rangle\,, \tag{24}$$

and so one would expect $\tilde{\hat{\rho}}^* \leq \hat{\rho}^*$ for the solutions of (22) and (23), respectively. This corresponds to the biased sensitivity approximation *under-estimating* h_0^* and therefore over-estimating the sensitivity of the search.

Further approximations used [1] to solve Eqs. (7) and (8) for $\bar{\rho}^*$ are: (i) approximate the χ^2-distributions with Gaussians, and (ii) Taylor-expand in small per-segment SNR $\bar{\rho}$. In [3] we refer to this as the "weak-signal Gaussian" limit. With these approximations, and using $\langle \mathcal{R}^2_{\rm eff}\rangle = 1 - \langle\bar{\mu}\rangle$, one can solve Eq. (8) for $\tilde{\hat{\rho}}^*\sqrt{1-\langle\bar{\mu}\rangle}$, and then use Eq. (18) to obtain the minimal signal amplitude h_0^*.

Apart from the bias introduced by (23), the "weak-signal Gaussian" approximation was found [3] to be rather unreliable for small false-alarm probabilities $p^H_{f_{\rm A}}$ and segment numbers in the range $N \lesssim 10^3$. We therefore continue with a more exact approach pioneered in [2].

C. Unbiased semi-analytical sensitivity estimation

We can numerically generate the probability distribution $P(\mathcal{R}_{\rm eff}|\Pi_{h_0})$ for the effective response via Monte-Carlo simulation of the assumed signal- and mismatch distributions. Note that $P(\mathcal{R}|\Pi_{h_0})$ is fully specified in terms of the antenna-pattern response of the detectors for the given signal population. However, we need to prescribe an *ad-hoc* mismatch-distribution $P(\bar{\mu}|I)$ for the Hough-on-$\mathcal{F}$ stat search grids. Drawing values for $\mathcal{R}_{\rm eff}$ is achieved by drawing $\mathcal{R}$ and $\bar{\mu}$ independently, and computing $\mathcal{R}_{\rm eff} = (1-\bar{\mu})\,\mathcal{R}$.

Given a probability distribution $P(\mathcal{R}_{\rm eff}|\Pi_{h_0})$ we can numerically solve the integral in Eq. (17) for $\hat{\rho}^*$. Karl has coded this up in octapps, and using this framework we arrive at the following sensitivity estimate.

II. ESTIMATING *S5R5* SENSITIVITY

In order to estimate the sensitivity, we use the *S5R5* parameters: $N = 121$, $\mathcal{F}_{\rm th} = 2.6$. The quoted sensitivity refers to a 90% confidence level, corresponding to ${p^H_{f_{\rm D}}}^* = 0.1$.

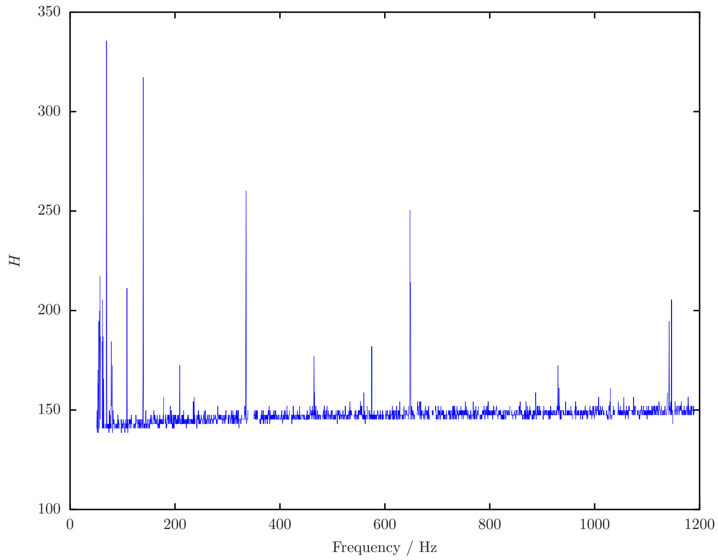

FIG. 2: Predicted sensitivity factor H as a function of frequency.

The two input parameters that need to be specified are the false-alarm probability $p^{H\ *}_{f_A}$, and the effective $\mathcal{F}$-statistic mismatch distribution in SNR, $P(\bar{\mu}|I)$:

- The false-alarm probability can equivalently be expressed as a number-count threshold $n_{\rm th}$, for which we use the values of the loudest candidates found in the *S5R5* post-processing, e.g. see Fig. 3 in the E@H *S5R5* paper. We use $n_{\rm th} \in [70, 76]$ in the table below, which roughly covers the $\pm 3\sigma$ range in $n_{\rm th}$.
- We use the mismatch distribution obtained by Miroslav from his follow-up pipeline. The distribution has mean $\langle\bar{\mu}\rangle \approx 0.61$ and standard deviation $\sigma_{\bar{\mu}} \approx 0.15$.

Running `S5R5Sensitivity.m`[2] which uses octapps[3], we obtain the following result:

Number-count thresholds	$n_{\rm th}$	70	71	72	73	74	75	76
False-alarm probabilities	$p^{H\ *}_{f_A}$	6.0e-13	1.6e-13	3.9e-14	9.4e-15	2.2e-15	5.0e-16	1.1e-16
[Biased (naive) estimate]	$\hat{\rho}^*_0$	2.6	2.7	2.7	2.7	2.7	2.8	2.8
Mean SNR	$\hat{\rho}^*$	5.1	5.2	5.3	5.4	5.4	5.5	5.6
Map's F-factor	F	42.5	43.1	43.8	44.5	45.2	45.8	46.5
Sensitivity factor	H	140.8	143.1	145.3	147.5	149.8	152.0	154.2
Sensitivity scale $[s^{1/2}]$	σ_h	31.0	30.5	30.0	29.6	29.1	28.7	28.3

A. Predicting the *S5R5* upper limits

To predict the h_0 upper limits obtained by the *S5R5* search, we additionally need:

- The number count threshold $n_{\rm th}$ in every 0.5 Hz frequency band for which upper limits are quoted. These were supplied by Paola as significances/critical ratios CR, which were converted to number counts using $n_{\rm th} = \sigma\text{CR} + \langle n_{\rm th}\rangle$, with $\sigma = 4.8$ and $\langle n_{\rm th}\rangle = N(1 + \mathcal{F}_{\rm th})e^{-\mathcal{F}_{\rm th}}$. This gave 31 distinct values of $n_{\rm th}$, ranging from 69 to 120.

---

[2] git-version a84e432abebdff6e800861caff59380cec0fde0d-CLEAN
[3] git-version 8c592f0ea8084e77f6fbf9e0ffc54ff3d19a9133-CLEAN

- The noise floor $\mathcal{S}$, harmonically averaged over SFTs and detectors; this was supplied by Map and Paola.

For each 0.5 Hz frequency band, we calculate the sensitivity factor H using the appropriate n_{th}. Then using the mean noise floor $\mathcal{S}$ in the 0.5 Hz band, and Eq. (20), we calculate a prediction for h_0. The predicted sensitivity factors H are plotted as a function of frequency in Fig. 2.

[1] B. Krishnan, A. M. Sintes, M. A. Papa, B. F. Schutz, S. Frasca, and C. Palomba, Phys. Rev. D. **70**, 082001 (2004), arXiv:gr-qc/0407001.

[2] K. Wette, Phys. Rev. D. **85**, 042003 (2012), DCC LIGO-P1100151-v2, URL `https://dcc.ligo.org/cgi-bin/private/DocDB/ShowDocument?docid=75488`.

[3] R. Prix and M. Shaltev, Phys. Rev. D. **85**, 084010 (2012), arXiv:1201.4321.

[4] R. Prix, Phys. Rev. D. **75**, 023004 (2007), gr-qc/0606088.

[5] R. Prix, Tech. Rep. (2010), (LIGO-T0900149-v2), URL `https://dcc.ligo.org/cgi-bin/DocDB/ShowDocument?docid=1665`.

[6] B. Abbott et al. (LIGO Scientific Collaboration), Phys. Rev. D. **69**, 082004 (2004).