Optical simulations within and beyond the paraxial limit

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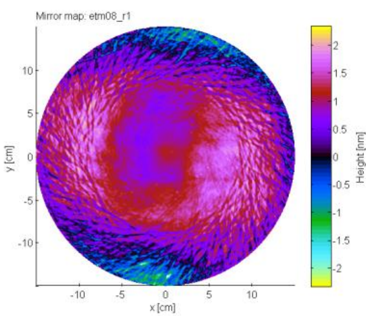
Simulating realistic optics

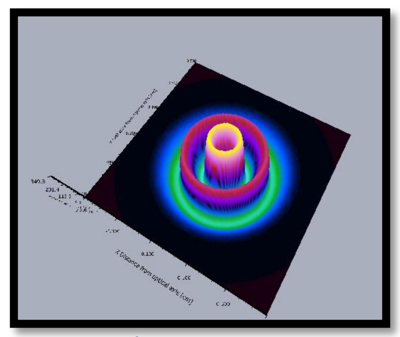
- We need to know how to accurately calculate how distortions of optical elements effect the beam



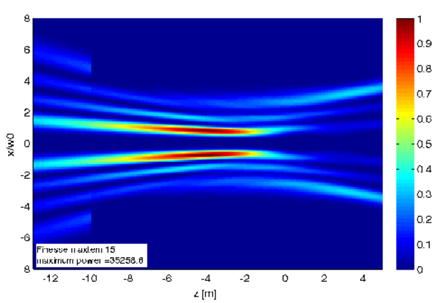
Surface and bulk distortions

- Thermal effects
- Manufacturing errors
- Mirror maps

Ideal beam



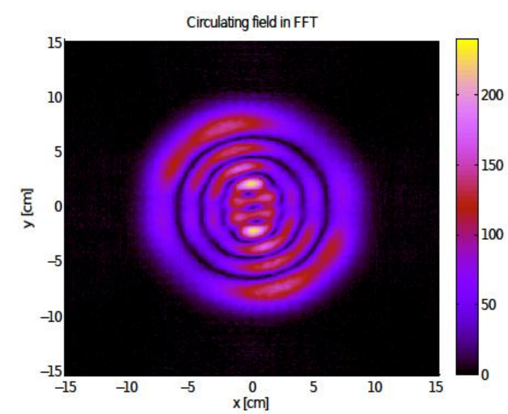
- Gaussian beam
- Higher order modes like Laguerre-Gaussian (LG) beams



Finite element sizes

- Beam clipping

Final distorted beam

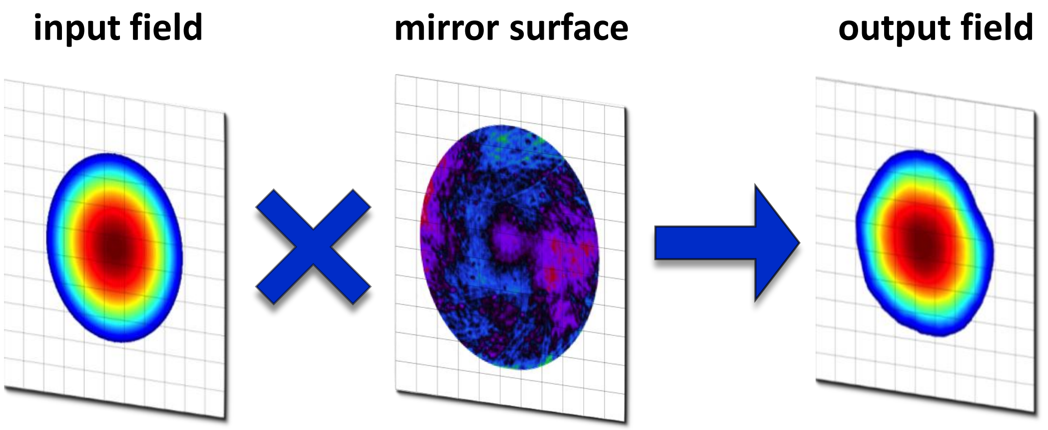


Methods to simulate light

- Fast-Fourier Transform (FFT) methods
  - Solving scalar wave diffraction integrals with FFTs
- Modal method
  - Represent beam in some basis set, usually eigenfunctions of the system
- Rigorous Simulations
  - What to use when the above breakdown?
  - Resort to solving Maxwell equations properly

FFT Methods

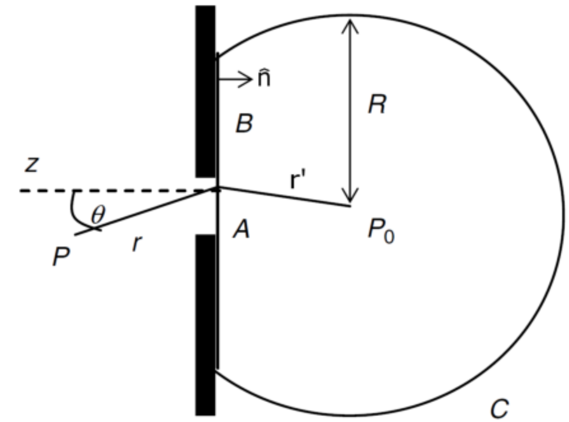
The effect of a mirror surface is computed by multiplying a grid of complex numbers describing the input field by a grid describing a function of the mirror surface.



- An FFT method commonly refers to solving the scalar diffraction integrals using Fast Fourier Transforms
- Can quickly propagate beam through complicated distortions, useful when studying non-eigenmode problems
- Quasi-time domain, cavity simulations can require computing multiple round trips to find steady state

Scalar Diffraction

- Diffraction mathematically based on Greens theorem, then making many approximations to make it solvable



Number of approximations

Helmholtz-Kirchhoff integral equation

$$U(P_0) = \frac{1}{4\pi} \iint_S \frac{e^{ik|\vec{r}|}}{|\vec{r}|} \left[U(\vec{r}) \left(ik - \frac{1}{|\vec{r}|} \right) - \frac{\partial U(\vec{r})}{\partial \vec{n}} \right] ds$$

Solve with Numerical Integration

Rayleigh-Sommerfeld (RS) integral equation

$$U(P_0) = \frac{-i}{\lambda} \iint_A E(x,y) e^{-ikz} \left(\frac{1}{r} - ik \right) \frac{e^{ik|\vec{r}'|}}{|\vec{r}'|} cos(\hat{n},\hat{r}) dxdy$$

Difference is in approximation of $|r|$ and $\cos(\hat{n},\hat{r})$, which leads to limitations in accuracy at wider angles and proximity to aperture

Fresnel Diffraction

$$U(P_0) \quad = \quad -\frac{ik}{\lambda z_0} e^{ik(z_0 + (x_0^2 + y_0^2)/2z_0)} \left(E_a(x,y,0) \otimes e^{ik\frac{x^2+y^2}{2z_0}} \right)$$

Fraunhofer Diffraction

$$U(P_0) \quad = \quad -\frac{ik}{\lambda z_0} e^{ik(z_0 + (x_0^2 + y_0^2)/2z_0)} \mathcal{F}(E_a(x,y,0))$$

Solve with FFTs [1][2][3]

Paraxial Approximations

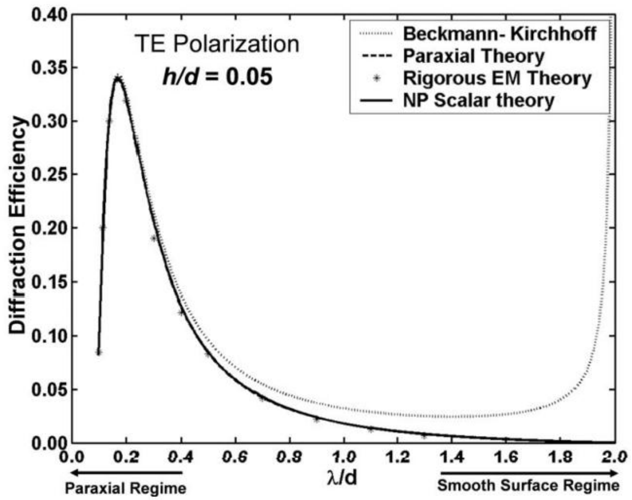
[1] Introduction to Fourier Optics, Goodman
[2] Shen 2006, Fast-Fourier-transform based numerical integration method for the Rayleigh–Sommerfeld diffraction formula
[3] Nascov 2009, Fast computation algorithm for the Rayleigh–Sommerfeld diffraction formula using a type of scaled convolution

Scalar Diffraction

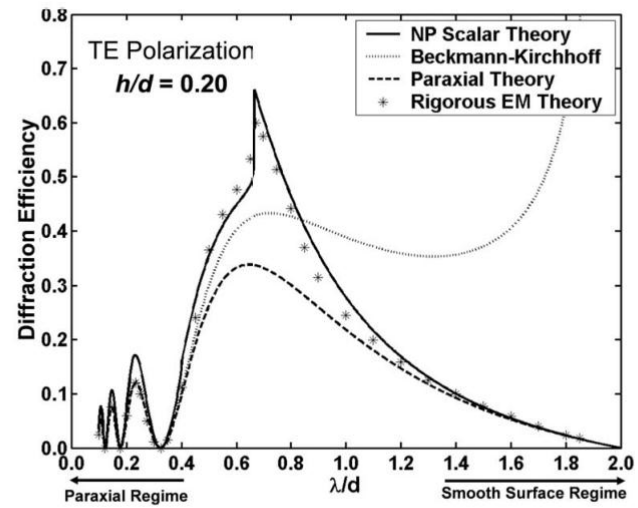
- Active research topic [1] looking at sinusoidal gratings and FFT methods, in particular non-paraxial methods

Plots showing the 1st order diffraction efficiency of a sinusoidal gratings [1]

$$\frac{h}{d} = \frac{\text{Grating height}}{\text{Grating period}}$$



Smooth surface, small h/d



Rough surface, large h/d

- h/d ~ 10^{-9} for LIGO mirrors, at first looks as if scalar theories should predict accurate results

- But how do results appear when looking at ppm differences?

[1] Harvey 2006, Non-paraxial scalar treatment of sinusoidal phase gratings

Fresnel and Fraunhofer conditions

Conditions on distance from aperture [1]

Fraunhofer Diffraction: $$N_{fr} = \frac{R^2}{\lambda z_0} < 0.5$$

Fresnel Diffraction: $$N_{fr} = \frac{R^2}{\lambda z_0} \geq 0.5$$

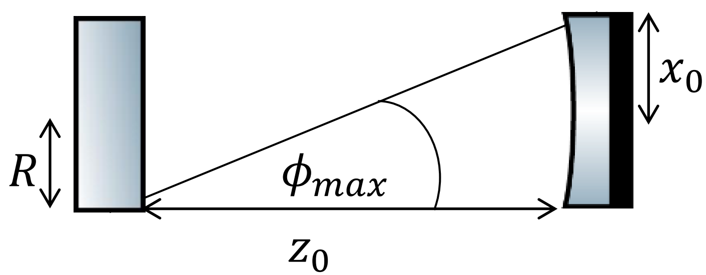
Conditions on distance from beam axis

$$\mathrm{N}_{rs1} = \frac{(x_o + R)^2}{4 z_o} \ll 1$$

In practice $< 10^{-3}$

$$N_{rs2} = \sqrt{\frac{z_0^2}{z_0^2 + (x_o + R)^2}} \cong 1$$

N_{fr}, Fresnel number
N_{rs1}, Rayleigh number 1
N_{rs2}, Rayleigh number *2*
R, Aperture/mirror radius
z_0, Distance to plane
x_0, largest distance from beam axis



Using N_{rs1} with the upper 10^{-3} limit, we can find the max angle for a given distance to a plane

$$|\theta_{max}| = \arctan\left(\sqrt{\frac{4 \times 10^{-3}}{z_o}}\right)$$

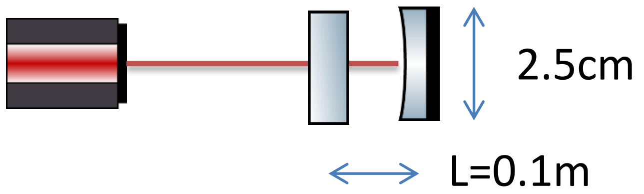
If you are too close to the aperture or looking at a point too far from the beam axis, you should be using Rayleigh-Sommerfeld Diffraction!

[1] Angular criterion to distinguish between Fraunhofer and Fresnel diffraction, Medina 2004

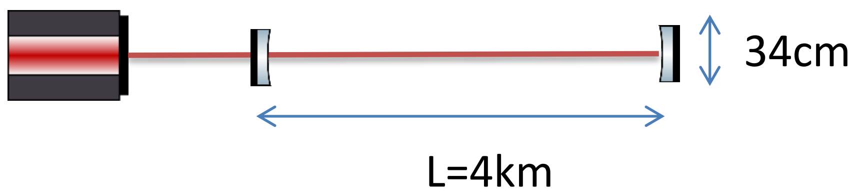
Examples

- Testing difference between Rayleigh-Sommerfeld and Fresnel diffraction

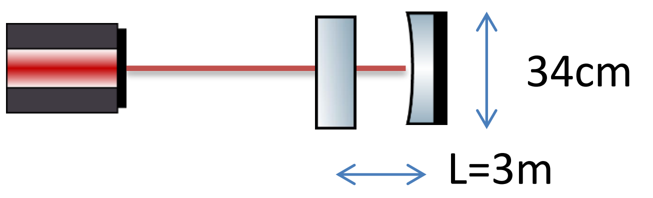
- Small cavity
  - Small mirrors and short
  - Larger angle involved



- LIGO arm cavity
  - Big mirrors and long
  - Small angles involved



- Khalili like Cavity
  - Big mirrors and short
  - Larger angles involved

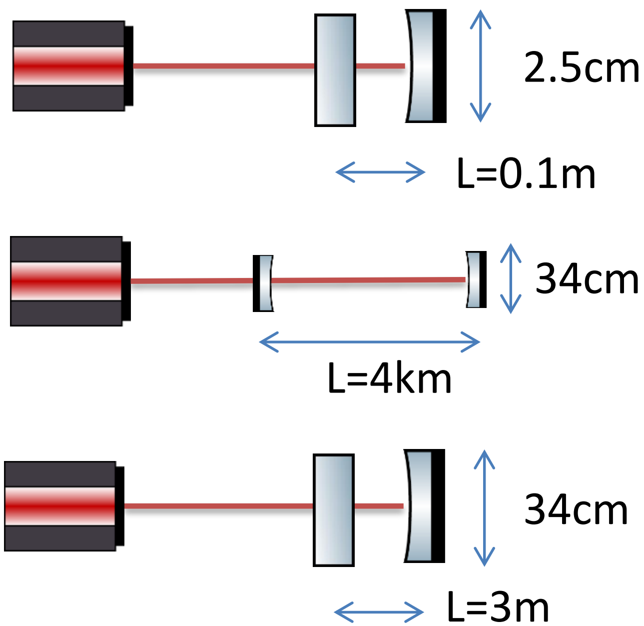


Examples

- How do the conditions look for each example

For Fresnel diffraction to be (safely) valid :

$$N_{rs1} < 10^{-3} \text{ and } 1 - N_{rs2} \cong 0$$

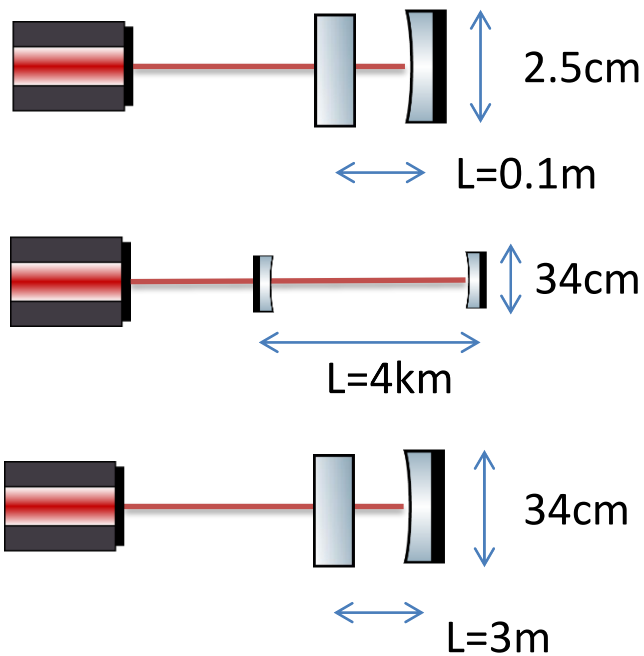


Fresnel Limit, θ_{max}	Max angle ϕ_{max}	N_{fr}	N_{rs1}	$1 - N_{rs2}$
$11°$	$\sim 14°$	$\sim 10^3$	$\sim 10^{-5}$	$\sim 10^{-5}$
$0.05°$	$\sim 0.005°$	$\sim 10^{-4}$	$\sim 10^{-6}$	$\sim 10^{-9}$
$2°$	$\sim 9.6°$	~ 0.96	$\sim 10^{-2}$	$\sim 10^{-3}$

Examples

- Round trip beam power difference between Rayleigh-Sommerfeld and Fresnel FFT

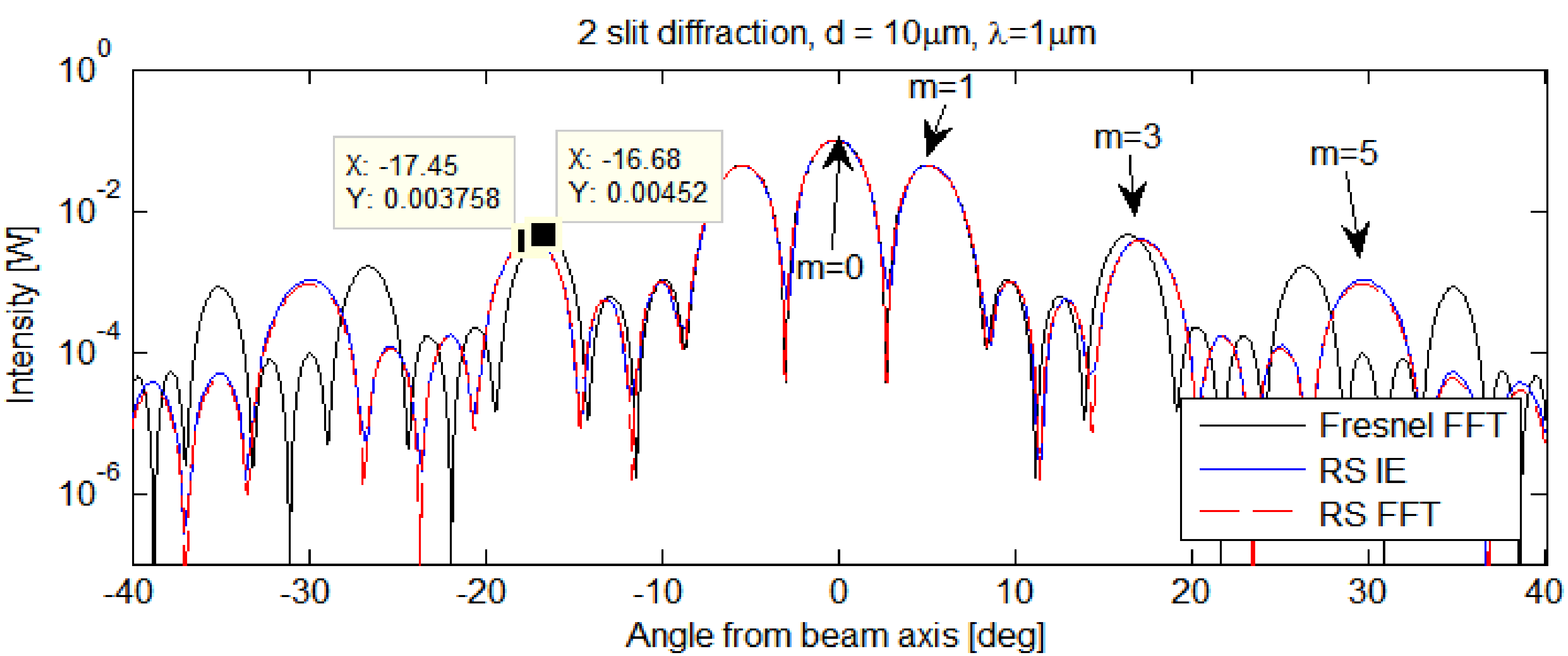
Largest power difference only seen when beam size is very large!



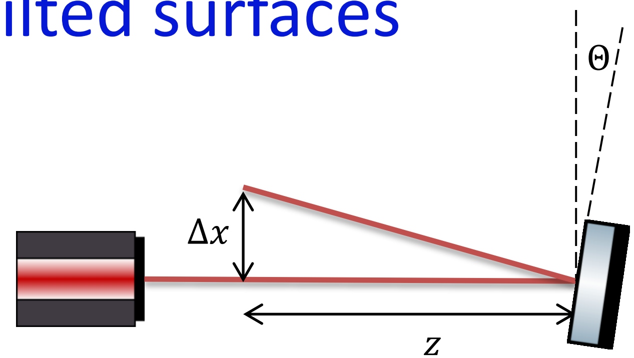
Fresnel Limit, θ_{max}	Max angle ϕ_{max}	N_{rs1}	$1 - N_{rs2}$	Power difference
$11°$	$\sim 14°$	$\sim 10^{-5}$	$\sim 10^{-5}$	**$< 10^{-1}$ppm**
$0.05°$	$\sim 0.005°$	$\sim 10^{-6}$	$\sim 10^{-9}$	**$< 10^{-3}$ppm**
$2°$	$\sim 9.6°$	$\sim 10^{-2}$	$\sim 10^{-3}$	**$< 10^{-1}$ppm**

But other things do not work

- Overall N_{rs1} and N_{rs2} limits chosen here are overly conservative
- Well it shows that the Fresnel approximations are very accurate for what we do
- The real difference comes at wide angles, like when we look at gratings (For example $\mathrm{m} = \pm 3$, θ should be $\pm 17.45°$ below)



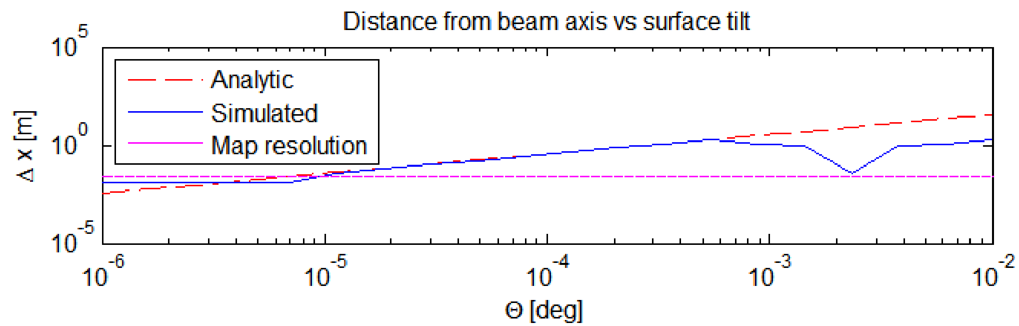
FFT Aliasing with tilted surfaces

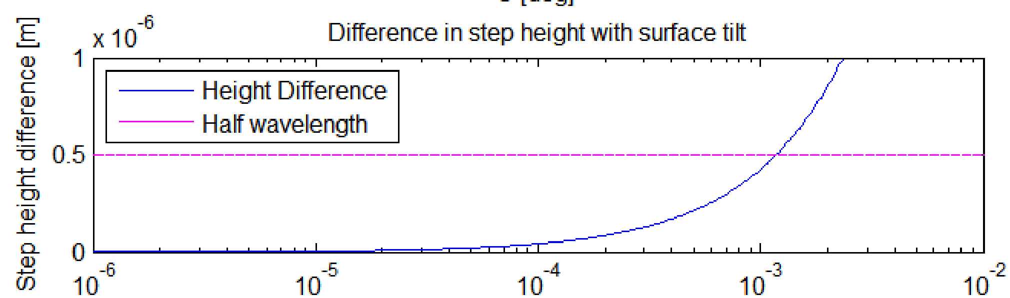


- Maximum difference in height between adjacent samples for reflection is $\frac{\lambda}{2}$

- This sets a maximum angle for calculating misalignment effects
  - $Max\,Tilt = arctan\left(\frac{\lambda}{2\delta}\right)$
  - Where δ is the FFT sampling size

- For LIGO ETM maps this is,
  - $\delta = \frac{L}{N} \approx \frac{0.34m}{1200} = 3 \times 10^{-4}m$
  - $Max\,Tilt \approx 0.1°$
  - Cavity field of view $\approx 0.005°$





In this example: $\lambda = 1\mu m,\ \delta \approx 0.02m, Max\,Tilt \approx 10^{-3°}$

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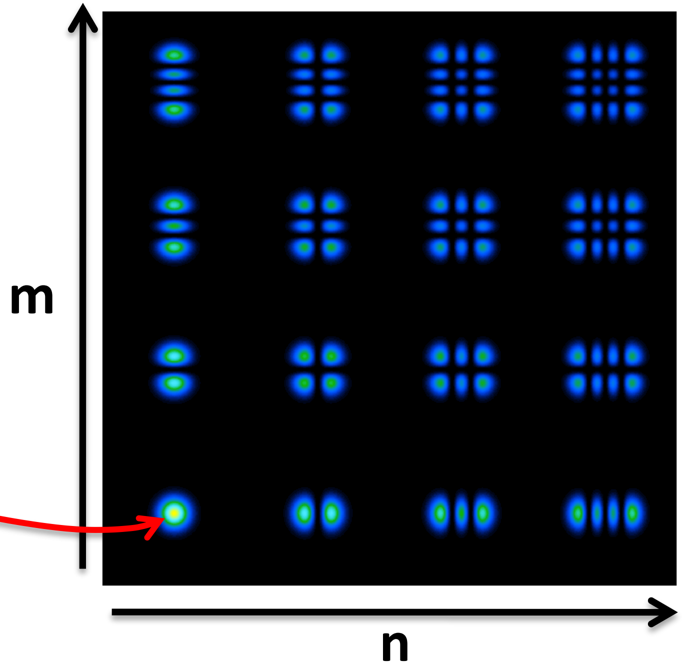
Modal models

- Modal models exploit the fact that a well behaved interferometer can be well described by cavity eigenmodes

- Represent our beam with different spatial basis functions by a series expansion:

$$E(x,y,z) = e^{-ikz} \sum^{\infty} u_{nm}(x,y)$$



- $u_{nm}(x,y)$ is our basis function choice, typically we use
  - Hermite-Gauss (HG) modes: Rectangular symmetric
  - Laguerre-Gauss (LG) modes: Cylindrically symmetric

- Modal model only deals with paraxial beams and small distortions, what we would expect in our optical systems

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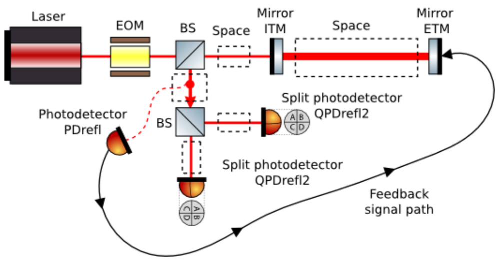
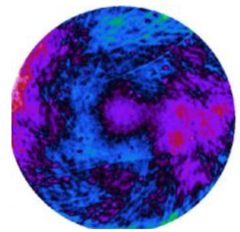
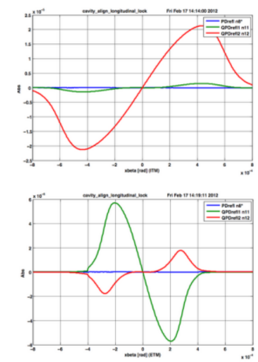
Why we use modal models



Mirror map
or
some distortion





- Model arbitrary optical setups
  - Can tune essentially every parameter
  - Quick prototyping
  - Fast computation for exploring parameter space
  - Undergone a lot of debugging and validation

- ***Finesse*** does all the hard work for you! Currently under active development
  - http://www.gwoptics.org/finesse/

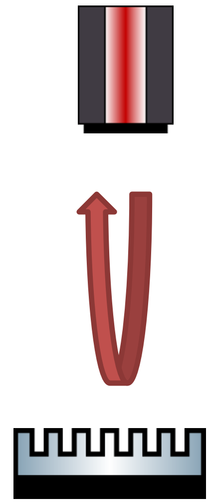
Sinusoidal grating with modes

- Common problem, what mode do I need to use to see a certain spatial frequency?
- Reflect beam from a sinusoidal grating and vary spatial wavelength, Λ, height of the grating to be 1nm
  - Have taken the range of spatial wavelengths in the LIGO ETM08 mirror map, $\sim 5 \times 10^{-4}$m to $\sim 64 \times 10^{-2}$m
- Calculate reflection using Rayleigh-Sommerfeld FFT and increasing number of modes
  - Modes 0 to 25
- Scattering from sinusoidal grating, most power goes into and around mode n_{max} along with 0^{th} mode [1]

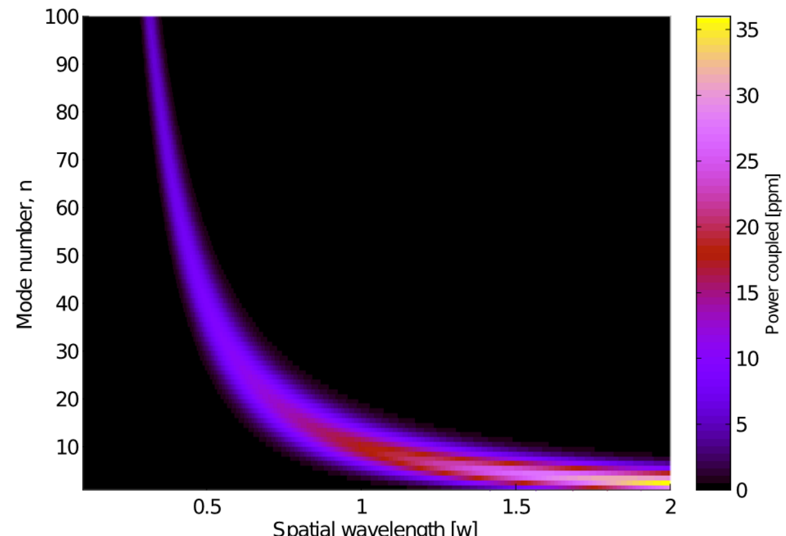
Λ is the grating period

$$n_{max} = \left(\frac{\pi w_o}{\Lambda}\right)^2$$

[1] Winkler 94, Light scattering described in the mode picture



$w_o = 6$cm



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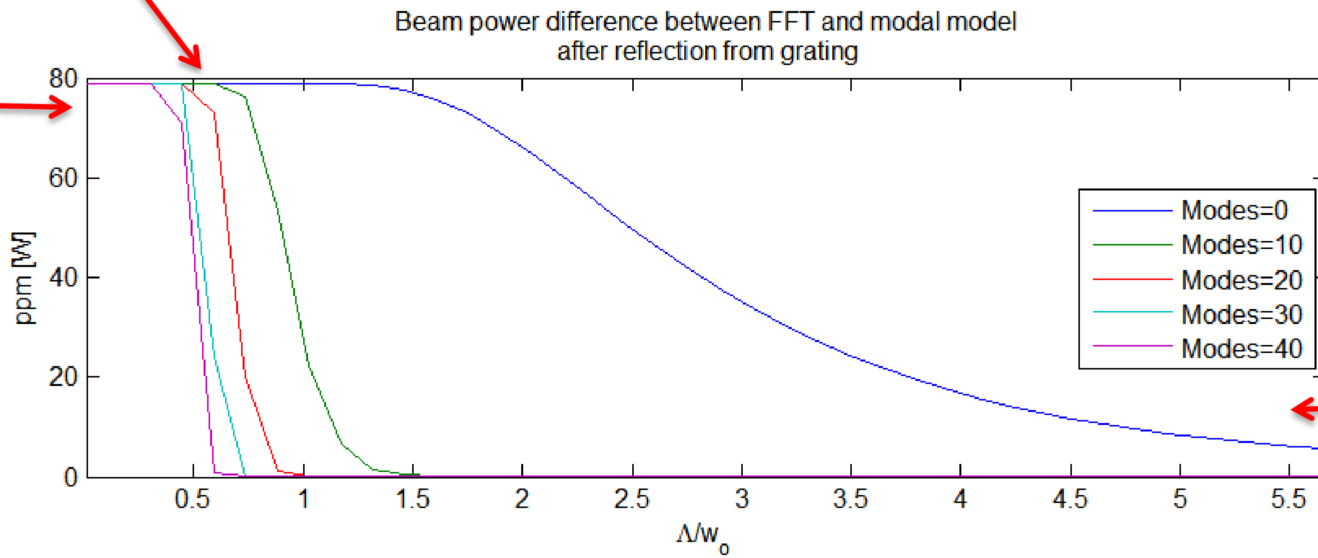
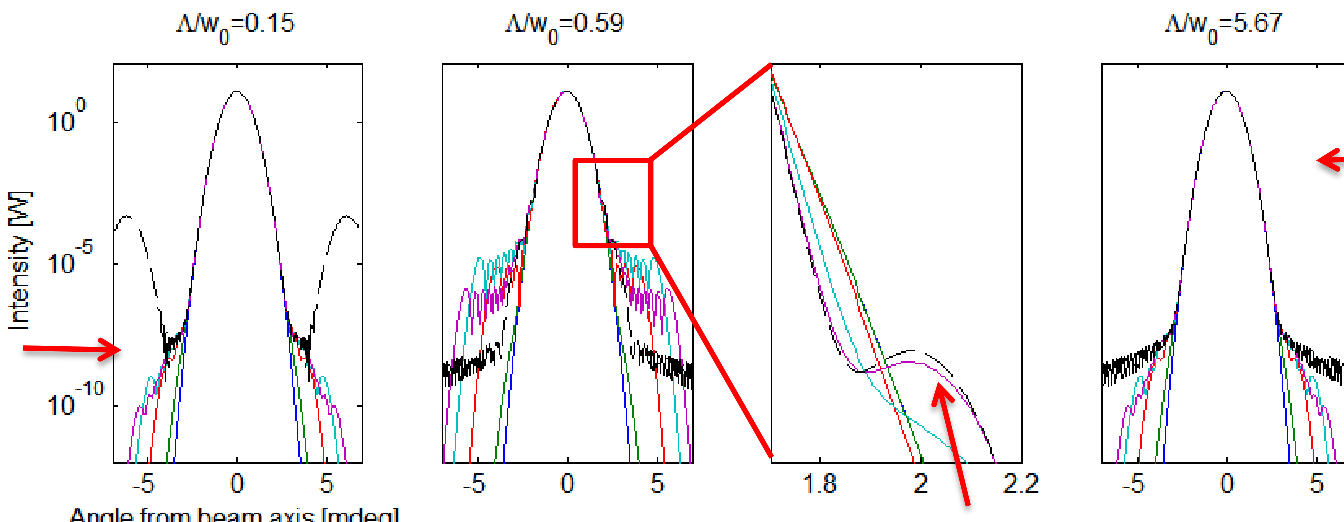
Sinusoidal grating with modes

80ppm is the power in the 1st orders



As spatial frequency becomes higher, higher diffraction orders appear which modal model can't handle. Only 0th order is accurately modeled

For low spatial frequency distortions only a few modes are needed



High spatial frequency compared to beam size

Low spatial frequency compared to beam size

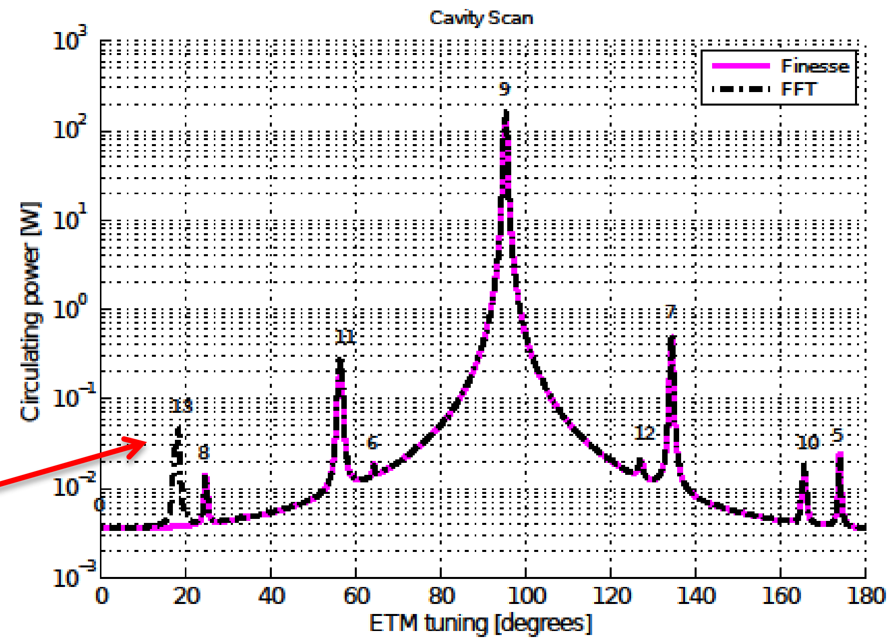
Modes struggle with complicated beam distortions, requires many modes

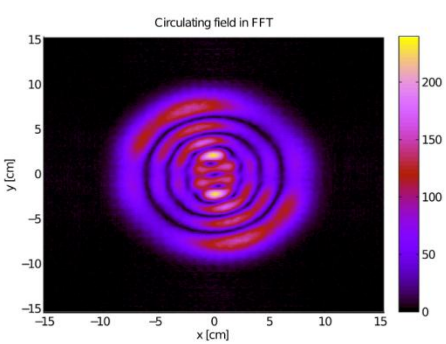
Real life modal model examples

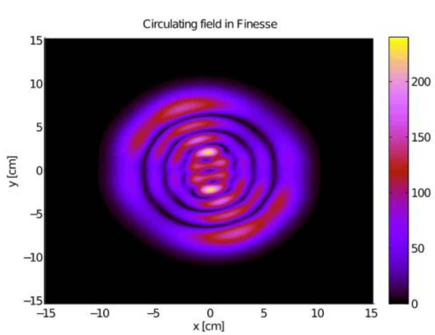
- A lot of work done on many topics:
  - Thermal distortions
  - Mirror maps
  - Cavity scans
  - Triangular mode cleaners

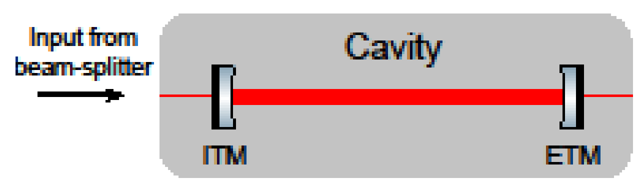
- Simulating LG33 beam in a cavity with ETM mirror map

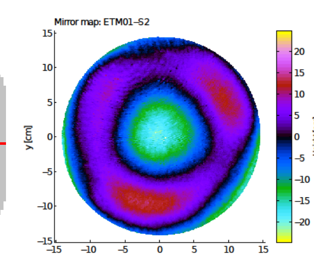
- Modal simulation done with only 12 modes hence missing 13th mode peak compared to FFT











Rigorous Simulations

- Scalar diffraction and modal methods can't do everything, so what next?

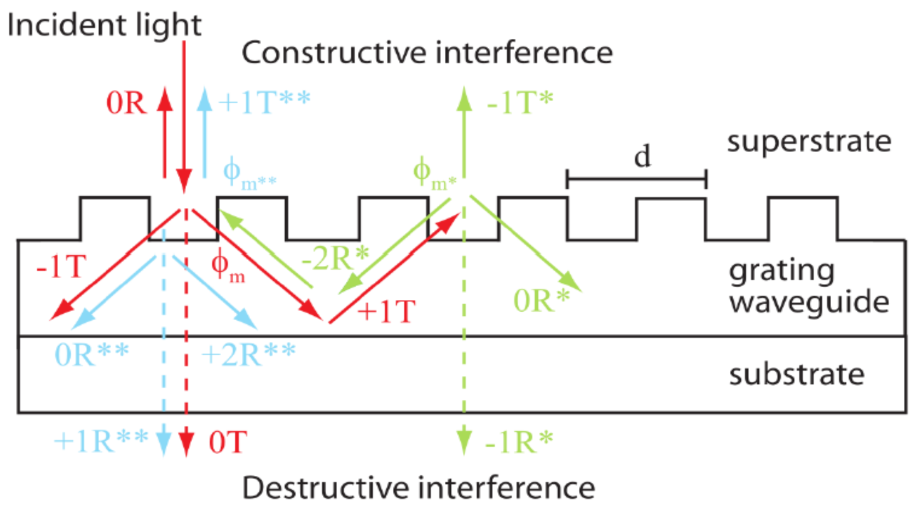
- Wanted to study waveguide coatings in more detail
  - Sub-wavelength structures
  - Polarisation dependant
  - Electric and magnetic field coupling
  - Not paraxial

- Requires solving Maxwell equations properly for beam propagation

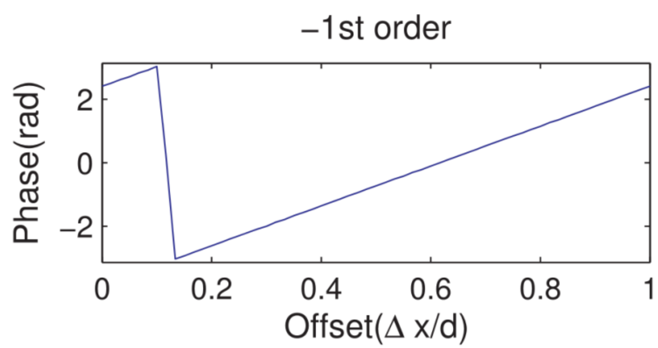
- Method of choice: Finite-Difference Time-Domain (FDTD)

Waveguide Coating – Phase noise

- Waveguide coatings proposed for reducing thermal noise
- Gratings however are not ideal due to grating motion coupling into beams phase [1]
- Wanted to verify waveguide coatings immunity [2] to this grating displacement phase variations with finite beams and higher order modes



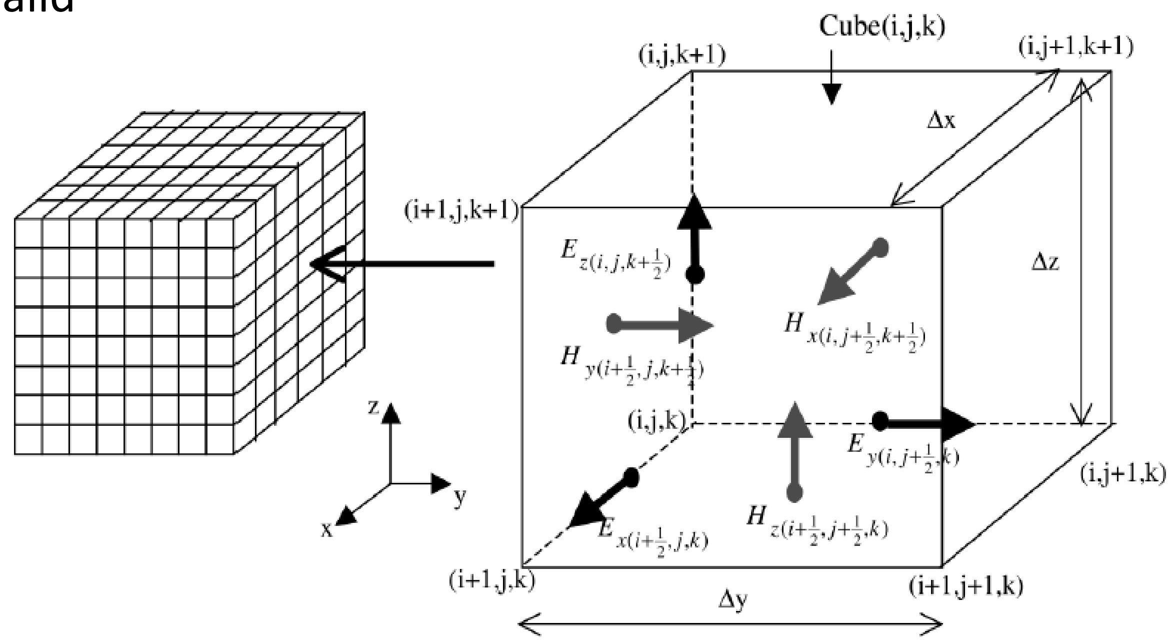
$$\Delta\phi = -\Delta x \frac{2\pi m}{d}$$



[1] Phase and alignment noise in grating interferometers, Freise et al 2007
[2] Invariance of waveguide grating mirrors to lateral beam displacement, Freidrich et al 2010

Finite-Difference Time-Domain (FDTD)

- The Yee FDTD Algorithm (1966), solving Maxwell equations in 3D volume
- Less approximations made compared to FFT and modal model
- Calculates near-field very accurately only, propagate with scalar diffraction again once scalar approximation is valid
- The idea is to…
  1. Discretise space and insert different materials
  2. Position E and B fields on face and edges of cubes
  3. Inject source signal
  4. Compute E and B fields using update equations in a leapfrog fashion. Loop for as long as needed



Taflove, Allen and Hagness, Susan C. Computational Electrodynamics: The Finite-Difference. Time-Domain Method, Third Edition

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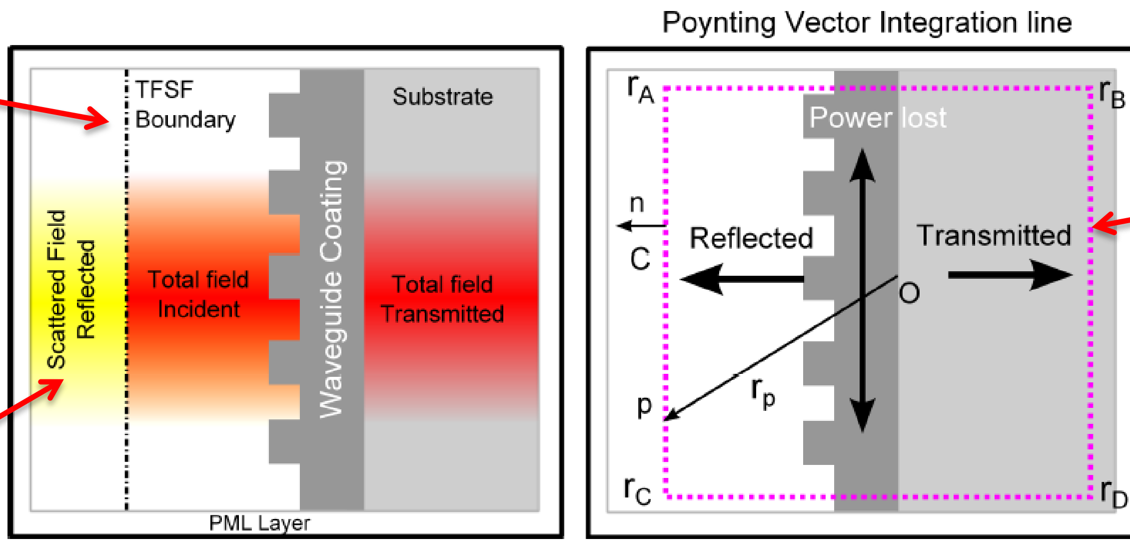
Waveguide coating simulation

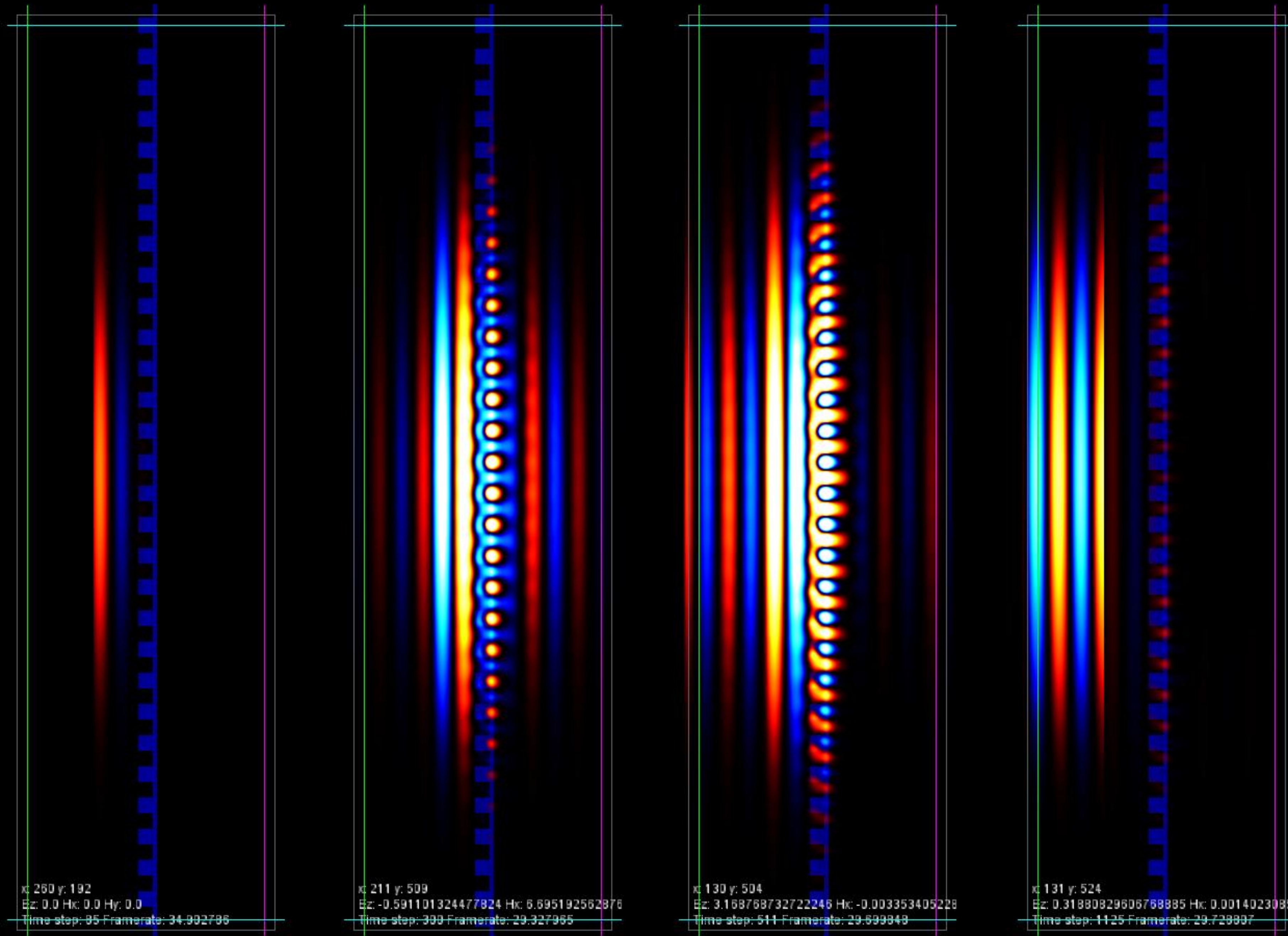
Incident field injected along TFSF boundary

Only beam reflected from grating here



Power flow measured across this boundary

- Simulation programmed myself, open to anyone else interested in it
- Validate simulation was working, compare to Jena work
- Analyse reflected beam phase front with varying grating displacements to look for phase noise

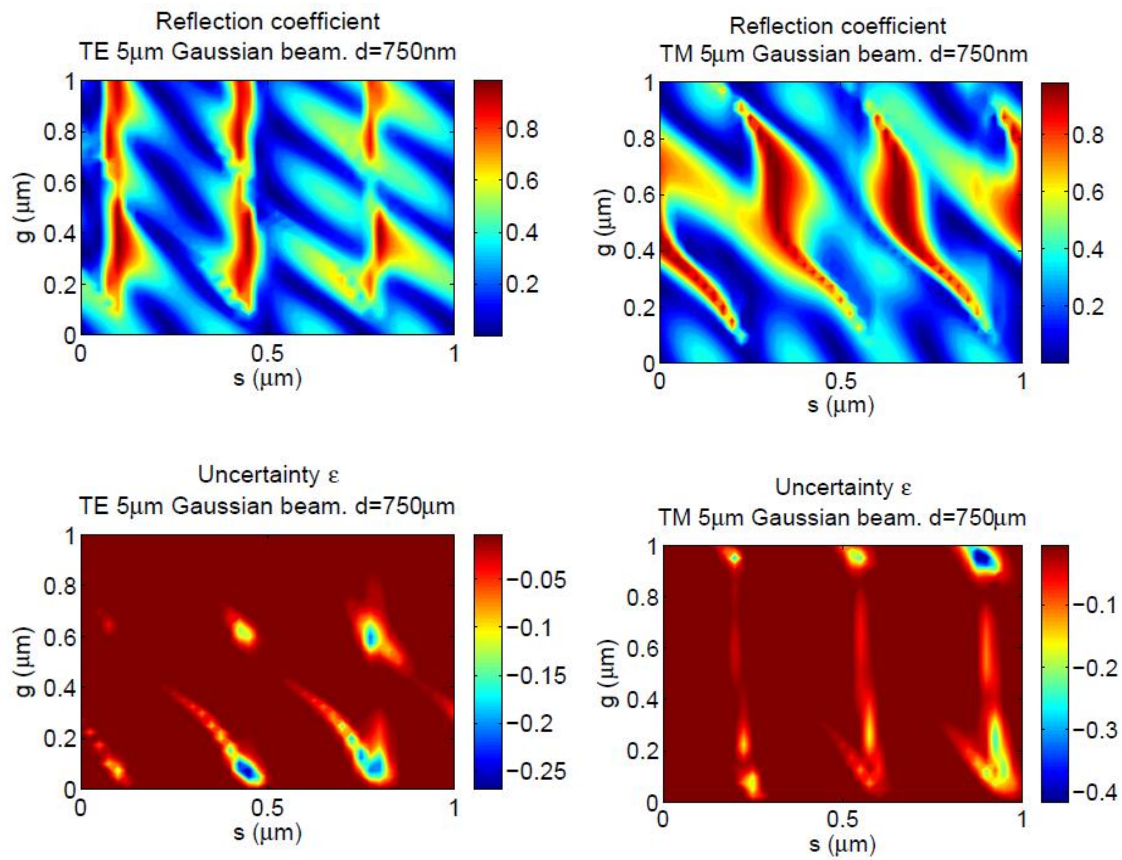


Timestep=85 Timestep=300 Timestep=511 Timestep=1125

Waveguide Coating Simulations

- Computed parameters that would allow for 99.8% reflectivity, agrees with Jena work on waveguide coatings[1]
- No phase noise was seen due to waveguide coating displacement, for fundamental beam or higher order modes
- Max phase variation across wavefront ≈ $10^{-6} \pm 10^{-4}$ radians
- Uncertainty plots shows difference between reflected and incident power due to finite size of simulation domain

TE Reflectivity *TM Reflectivity*



s – waveguide depth, g – grating depth

[1] High reflectivity grating waveguide coatings for 1064 nm, Bunkowski 2006

Conclusion…

- FFT’s work, just need to be careful, consider using RS in certain cases
- Modal model works, just need to ensure you use enough modes
- Modal model and FFT are identical for all real world examples
- Seen one option for rigorously simulating complex structures using the FDTD
    - FDTD code is available to anyone who wants to use/play with it!

…and finished…