Derivation of minimum lifetime after proof testing (addendum to LIGO-E1101226-v3)

In the document, “Proof Test Levels for Advanced LIGO Viewports”, https://dcc.ligo.org/LIGO-E1101226

The following equation is given for the minimum lifetime after proof testing:

$$t_{min} = B\sigma_p^{N-2}\sigma_a^{-N}$$

This equation is given as equation (3) in this reference:

> J.E. Ritter, Jr. and J.A. Meisel, "Strength and Failure Predictions for Glass and Ceramics", Journal of the American Ceramic Society, Vol. 59, No, 11-12, Nov-Dec 1976, pg. 478-481. https://ceramics.onlinelibrary.wiley.com/doi/10.1111/j.1151-2916.1976.tb09412.x

It is not derived, but sourced to the following 3 references:

> A. G. Evans and S. M. Wiedehorn, “Proof Testing of Ceramic Materials. Analytical Basis for Failure Prediction,” Int. J. Fract. Mech., 10 [31379-92 (1974).

> S. M. Wiederhom; pp. 613-46 in Fracture Mechanics of Ceramics, Vol. 2. Edited by R. C. Bradt, D. P. H. Hasselman, and F. F. Lange. Plenum Press, New York, 1974.

> S. M. Wiederhorn; pp. 633-63 in Ceramics for High-Performance Applications. Edited by John J. Burke, Alvin E. Gorum, and R. Nathan Katz. Brook Hill Publishing Co., Chestnut Hill, Mass., 1974.

The following derivation follows closely the derivation of proof testing theory in the following reference:

> E. R. Fuller Jr, S. M. Wiederhorn, J.E. Ritter Jr., P. B. Oates, Proof Testing of Ceramics: Part 2 Theory, Journal of Materials Science, 15 (1980), 2282-2295. https://link.springer.com/article/10.1007/BF00552318

This reference includes the effect of proof test loading rate, hold time and unloading rate. I've used a simpler formulation/approximation wherein the post proof test inert strength is simply the proof test load.

Subcritical crack growth can be expressed as a power function of the stress intensity factor:

$$V = AK_I^N \qquad [1]$$

V = crack velocity

K_I = stress intensity factor

A, N are material constants that depend on environment and material

For a uniform applied stress, the stress intensity factor is given by:

$$K_I = \sigma Y \sqrt{a} \qquad [2]$$

σ = applied stress

Y = a geometric constant

a = crack length

Failure occurs when the stress intensity factor reaches a critical value for rapid fracture. The fracture strength can be defined in terms of the crack length and the critical stress intensity factor from equation [2]:

$$S = K_{IC}/Y\sqrt{a} \qquad [3]$$

S = fracture strength

K_{IC} = critical stress intensity factor

The rate of strength degradation can be obtained by differentiating equation [3] with respect to time:

$$\frac{dS}{dt} = -\left(\frac{K_{IC}}{2Y}\right) a^{-3/2} \left(\frac{da}{dt}\right) \qquad [4]$$

Substitute V = da/dt (equation 1) and use (equation 3) to substitute for a:

$$\frac{dS}{dt} = -\left(\frac{Y^2}{2K_{IC}^2}\right) S^3 V \qquad [5]$$

From [2] and [3]:

$$K_I = \sigma \left(\frac{K_{IC}}{S}\right) \qquad [6]$$

Substituting [6] into [1] gives:

$$V = AK_{IC}^N \left(\frac{\sigma}{S}\right)^N \qquad [7]$$

Substituting [7] into [5] gives:

$$\frac{dS}{dt} = -\left(\frac{AY^2 K_{IC}^{N-2}}{2}\right) \left(\frac{\sigma}{S}\right)^N S^3 \qquad [8]$$

Equation [8] can be integrated to provide a relation between initial strength, S_i, and the strength S at any other time:

$$\int_{S_i}^{S_f} S^{N-3} dS = \int_0^{t_f} \left(\frac{-AY^2 K_{IC}^{N-2}}{2}\right) \sigma^N dt \qquad [9]$$

$$S_f^{N-2} - S_i^{N-2} = \left(\frac{-1}{B}\right) \int_0^{t_f} \sigma^N \, dt \qquad [10]$$

$$B = \frac{2}{AY^2 (N-2) K_{IC}^{N-2}} \qquad [11]$$

At failure $S_f = 0$ and for a constant applied stress σ_a:

$$-S_i^{N-2} = \frac{-\sigma^N t_f}{B} \qquad [12]$$

$$t_f = B S_i^{N-2} \sigma_a^{-N} \qquad [13]$$

The proof test is expected to ensure that the minimum inert strength, $S_{i,min}$, is set equal to the proof stress, σ_p, so that the minimum time to failure is:

$$t_{min} = B \sigma_p^{N-2} \sigma_a^{-N} \qquad [13]$$