LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

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1D PSD of mirror maps

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1 Introduction

Definition of PSDs used for mirror phase maps are discussed. When 1D PSD is calculated based on different kinds of measurements, one using 2D phasemap for long spatial wavelength measurement, and the other from data along a line for short wavelength measurement, a proper choice of the calculation from 2D PSD to 1D PSD is necessary to make those two PSDs overlap smoothly.

2 1D PSD and 2D PSD

The structure of the mirror surface in the spatial frequency domain is measured differently in the long wavelength region and in the short wavelength region. The long wavelength region, λ > 0.1mm, is derived from the two dimensional surface phasemap measured by an interferometer covering the entire mirror surface. The short wavelength region, λ < 1mm, is derived from measurements of multiple one dimensional measurements along lines with large magnification using PMM, for example.

The 1D PSD measured along a line from (x,y)=(-L/2,y) to (+L/2,y) is calculated as follows by using 1D amplitude spectral density (ASD_{1D}).

$$ASD_{1D}(f_x, y) = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx \delta(x,y) e^{i2\pi f_x x} \tag{1}$$

In this equation, δ(x,y) is the height of the mirror surface at location (x,y), and 1D PSD is calculated as the squared amplitude of ASD, i.e., PSD=$|ASD|^2$.

With this definition, 1D PSD is related to the spatial variation rms in a unit length as follows.

$$\begin{aligned} \int_{-\infty}^{\infty} df_x PSD_{1D}(f_x, y) &= \frac{1}{L} \int_{-\infty}^{\infty} df_x \int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dx_2 \delta(x_1, y) \delta(x_2, y) e^{i2\pi f_x (x_1 - x_2)} \\ &= \frac{1}{L} \int_{-L/2}^{L/2} dx \delta(x,y)^2 \end{aligned} \tag{2}$$

For later discussion, 2D PSD is introduced here. By using 2D ASD defined as

$$ASD(f_x, f_y) = \frac{1}{L} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \delta(x,y) e^{i2\pi(f_x x + f_y y)} \tag{3}$$

1D ASD_{1D} can be expressed as

$$ASD_{1D}(f_x, y) = \sqrt{L} \int_{-\infty}^{\infty} df_y ASD(f_x, f_y) e^{-i2\pi f_y y} \tag{4}$$

The 2D PSD is related to the spatial variation rms in a unit area as follows:

$$\int_{-\infty}^{\infty} df_x \int_{-\infty}^{\infty} df_y PSD(f_x, f_y)$$
$$= \frac{1}{L^2} \int_{-\infty}^{\infty} df_x \int_{-\infty}^{\infty} df_y \int_{-L/2}^{L/2} dx_1 \int_{-L/2}^{L/2} dy_1 \int_{-L/2}^{L/2} dx_2 \int_{-L/2}^{L/2} dy_2 \delta(x_1, y_1)\delta(x_2, y_2) e^{i2\pi\{f_x(x_1-x_2)+f_y(y_1-y_2)\}} \tag{5}$$
$$= \frac{1}{L^2} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \delta(x, y)^2$$

In general, it is assumed that the spatial structure at short wavelength or high spatial frequency region does not depend on the location on the mirror surface nor on the direction of the line. In order to reduce the statistical fluctuation, measurements are done along several lines at different locations with different directions on a mirror surface and the final 1D PSD is calculate by taking the average of those PSDs.

This averaging is carried out by calculating the average along the y direction, i.e.,

$$\begin{aligned} PSD_{1D}(f_x) &= \frac{1}{L} \int_{-L/2}^{L/2} dy PSD_{1D}(f_x, y) \\ &= \int_{-L/2}^{L/2} dy \int_{-\infty}^{\infty} df_{y1} \int_{-\infty}^{\infty} df_{y2} ASD(f_x, f_{y1}) ASD(f_x, f_{y2})^* e^{-i2\pi(f_{y1}-f_{y2})y} \\ &= \int_{-\infty}^{\infty} df_y \left|ASD(f_x, f_y)\right|^2 \\ &= \int_{-\infty}^{\infty} df_y PSD(f_x, f_y) \end{aligned} \tag{6}$$

where Eq. (4) is used to replace 1D ASD by 2D ASD.

This derivation shows that the 1D PSD measured using data sets along lines is related to the 2D PSD by the following integral:

$$PSD_{1DX}(f_x) = \int_{-\infty}^{\infty} df_y PSD(f_x, f_y) \tag{7}$$

It is assumed that the spatial structure is orientation independent, and the choice of the x-axis can be arbitrary. So, the 1D PSD is calculated from the 2D PSD by integrating over the orthogonal frequency axis. See LIGO-T070082.

For realistic mirrors, the surface structure is not cylindrically symmetric and the choice of the x-axis in the above equation affects the final 1D PSD shape. There can be another definition of 1D PSD derived from 2D PSD of arbitrary shape, which is calculated by integrating over azimual angle at a given frequency:

$$\begin{aligned} PSD_{1DA}(f) &= f \int_{0}^{2\pi} d\varphi_f PSD(f_x, f_y) \\ f &= \sqrt{f_x^2 + f_y^2},\ \varphi_f = a\tan(f_y / f_x) \end{aligned} \tag{8}$$

This 1D PSD has an extra factor f, which comes from the volume element in radial direction,

$$df_x df_y = f\,df\,d\varphi_f \tag{9}$$

These two definitions of PSDs have same relationship with the spatial rms:

$$\int_{-\infty}^{\infty} df_x PSD_{1DX}(f_x) = \int_0^{\infty} df PSD_{1DA}(f) = \frac{1}{L^2}\int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \delta(x,y)^2 \tag{10}$$

The definition of PSD_{1DR}, Eq. (8), is aesthetically cleaner and can be defined unambiguously for an arbitrary surface without any symmetry. The problem is that the 1D PSD measured using data along lines does not match with this PSD.

3 Comparison of two 1D PSDs

In order to compare two definitions, an explicit analytic expression for 2D PSD is used:

$$\begin{aligned} PSD(f,\varphi_f) &= \frac{A\cdot B\cdot D}{(1+(B\cdot f)^2)^{(C+1)/2}} \\ D &\equiv \Gamma(\frac{C+1}{2})/(2\sqrt{\pi}\Gamma(\frac{C}{2})) \\ &= \frac{1}{4}\ for\ C=2, \frac{1}{2\pi}\ for\ C=1 \end{aligned} \tag{11}$$

This is a 2D PSD which is flat at f=0 and falls off as $1/f^{C+1}$ at high frequency.

Two definitions of 1D PSD come out to be as follows:

$$PSD_{1DX}(f) = \frac{A}{(1+(B\cdot f)^2)^{C/2}} \tag{12}$$

$$PSD_{1DA}(f) = \frac{2\pi\cdot A\cdot B\cdot D\cdot f}{(1+(B\cdot f)^2)^{(C+1)/2}} \tag{13}$$

One change of the definition is that this PSD_{1DX} is factor 2 larger than the original definition, and the range of f is limited to positive value range, i.e., new PSD_{1DX}(f)=old PSD_{1DX}(f)+old PSD_{1DX}(-f). This way, the range of f use for the two PSDs are the same.

The ratio of the two PSDs is

$$\begin{aligned} \frac{PSD_{1DA}(f)}{PSD_{1DX}(f)} &= 2\pi\cdot D\frac{B\cdot f}{\sqrt{1+(B\cdot f)^2}} \\ &\underset{f\to\infty}{\to} 2\pi\cdot D \end{aligned} \tag{14}$$

Because PSD_{1DA} goes to 0 at f=0 and the areas of two PSDs are the same, Eq.(10), PSD_{1DA} becomes larger than PSD_{1DX} at large f region. With C=2, this limiting ratio is π/2 ~ 1.6.

4 Numerical example

In order to quantify these results, a test mirror map was generated and two 1D PSDs are compared. A simulated phasemap data are generated for a mirror with aperture of 34cm with a spacing of 34cm/1024 = 0.332mm.

The procedure is based on the algorithm developed by F. Bondu. The amplitude of the 2D PSD is calculated using the formula shown in Eq.(11), then the complex 2D ASD is generated using the following formula.

$$ASD(f_x, f_y) = \sqrt{PSD(f_x, f_y)}(a + i \cdot b) \tag{15}$$

where a and b are random numbers with normal distribution with mean of 0 and sigma of 1/√2.

Values of A, B and C in the formula are the one representing typical aLIGO test mass polished surfaces, A = 2.1 nm^2 mm, B = 0.05m^{-1}, and C = 2.

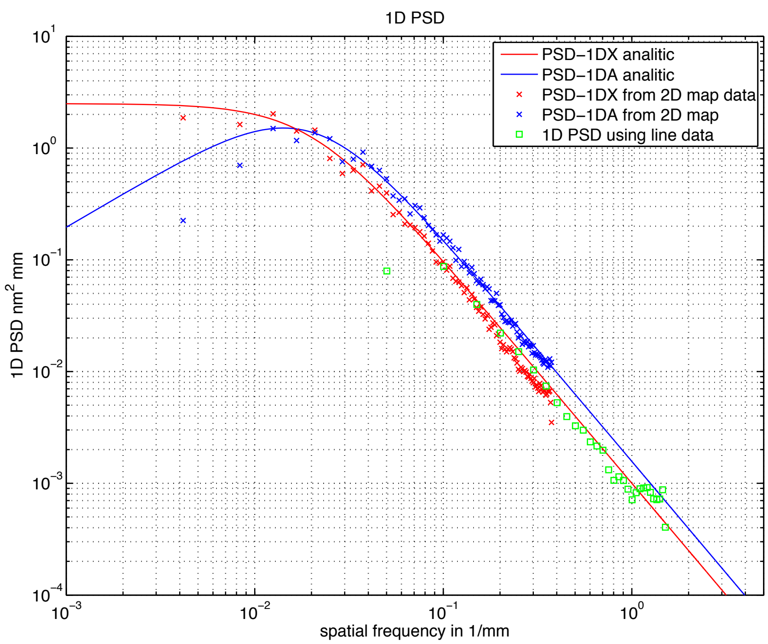

Figure 1 Comparison of 1D PSD

In this figure, two solid lines correspond to analytic expressions of 1D PSDs, Eq.(12) and Eq.(13).

The phasemap data is analyzed using two different resolutions. One is a 24 independent, separate line segments of 2cm long, with the native spacing of 0.322mm, mimicking TOPO kind measurement. The other is based on a two dimensional phasemap with coarse resolution, 1.33mm, mimicking IFO measurement.

Green data points are calculated by averaging 24 PSDs, each of which is a 1D PSD calculated from each line segment data with fine resolution.

From the second coarse data set, a 2D PSD was first calculated using a square region in the mirror with a size of 24cm x 24cm. From this 2D PSD, two definitions of 1D PSDs are calculated, which are shown with x marks in the figure. 1DX in the figure is an average of PSD-1DX calculated along x and that along y. Both 1DX and 1DA match well with their respective analytic formulas.

As was discussed in previous sections, the greed PSD smoothly connects to the red PSD-1DX, while the blue PSD-1DA is larger than the green PSD at the overlapping region.

If two PSDs measured are to be combined, the definition of 1D PSD derived from 2D PSD needs to be carefully chosen so that two sets of PSDs are smoothly connected.