Effect of parameter drift on Parametric Instability Threshold

W. Kells

LIGO Laboratory, Caltech

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New Parametric Instability issue ??

- Several studies to date (Phys Lett **A305**,111; **A354**,360) calculate the *threshold*, $\boldsymbol{R}(\{\boldsymbol{p}arameters\}) \geq 1$, of instability growth.
  - » All treat the contributing parameters $\{\boldsymbol{p}\}$ as fixed in time.
- May ask: will true dynamic system, $\{\boldsymbol{p}(t)\}$ alter threshold such that $\boldsymbol{R}_{eff} < \boldsymbol{R}(\langle\{\boldsymbol{p}(t)\}\rangle)$?
  - » Even slow drift may be suspect: $\tau_{PI} \sim \tau_m /(R-1)$ ~ 100-1000s in AdvLIGO for expected acoustic mode {m} natural ring down $\sim 2\pi Q_m / \omega_m$
  - » Narrow acoustic resonances $\delta_m = \omega_m / Q_m$ allow very small drifts to slew many line widths in $\Delta t \ll \tau_m$
  - » AdvLIGO cavity mirror ROC change of ~0.2% (via thermal effects), HTM *PI* coupling modes can shift resonance ω_1 by several widths ($\sim \delta_1/\pi$ = 250 Hz)
- For Advanced Ligo $\{\boldsymbol{p}\}$, $\boldsymbol{R}(\{\boldsymbol{p}_{static}\}) \approx 1$ for many acoustic {m}, so even small dilutions $\boldsymbol{R}_{eff} < \boldsymbol{R}(\{\boldsymbol{p}_{static}\})$ would be crucial.

Parameterization of ***R*** threshold

- Specific formulation: cavity field = $\text{sum}_{i,j}$ over well defined transverse mode Lorentzian resonances

Each Isolated Acoustic Mode "m"

$$R = \frac{4P_0 Q_m}{mcL{\omega_m}^2}\left(\sum_i \frac{Q_{1i}\Lambda_{1i}}{1+(\Delta\omega_{1i}/\delta_{1i})^2} - \sum_j \frac{Q_{1aj}\Lambda_{1aj}}{1+(\Delta\omega_{1aj}/\delta_{1aj})^2}\right)$$

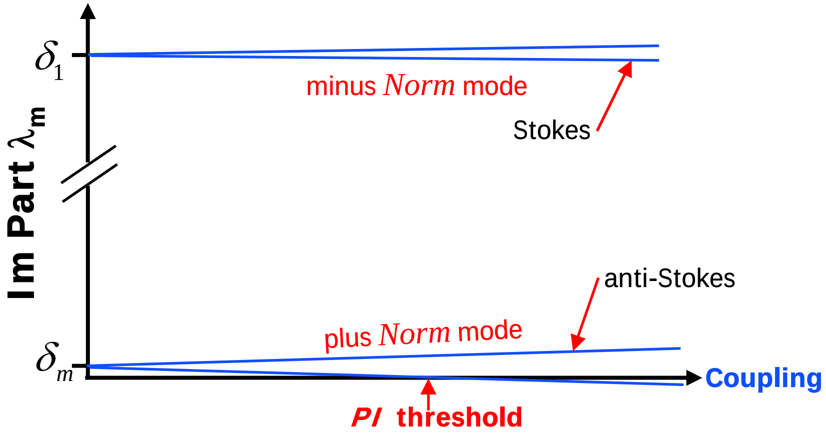
- Parameters:

P_0 = cavity circulating power
Q_m, ω_m = Acoustic mode Q, frequency
M= TM (mirror) mass
Q_{1j}= j^{th} cavity mode Q
Λ_{1j}= j^{th} cavity mode surface overlap with m
δ_{1j}= j^{th} cavity mode Lorentzian line width
$\Delta\omega = \omega_0 - \omega_m - \omega_{Gouy}$ depends on detailed cavity geometry
$\Delta\omega_a = \omega_0 + \omega_m - \omega_{Gouy}$

- Only "Stokes" excitations at $\omega_{1j} = \omega_0 - \omega_m$ contribute to ***R***>0
  - » Formula not apparently dependent on $\beta = \omega_{PI} - \omega_m$ (no acoustic Lorentzian factor) ??
  - » All cavity fields are expressed in stationary limit: acoustic source and cavity parameters cannot change (t)

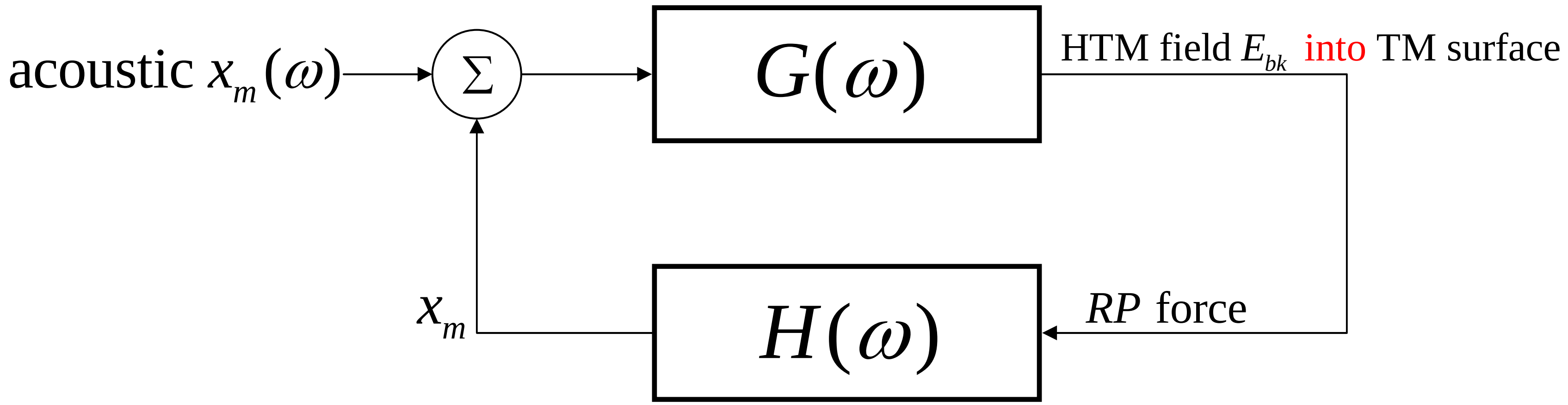
Nature of the *PI* "mode"

- Original concept (Phys. Lett A **287**, 331& **299**, 326) was eigenvalue, λ, solution to coupled, linearized 3 mode interaction equations.
  - » Normal modes of *coupled* acoustic + cavity SHOs
    - – Coupling via 3[d] mode, $E_0(\omega_0)$, but approximated as fixed parameter.
  - » Typical $\lambda_m^{\pm}$ solutions correspond to two distinct normal modes (NOT Stokes, a-Stokes !)
  - » Im[λ^-]<0 for any physical coupling: always damped.
    - – However this normal mode is ~ free cavity HTM, so not of interest.
  - » Im[λ^+]<0 in a-Stokes approximation. In Stokes Im[λ^+]->0 for sufficient coupling: unstable
  - » Thus only λ^+ mode of interest with Re[λ^+] =β shift w.r.t. ω_m .



Feedback model

- Linearized coupled SHOs work well ⇒ equivalent Feedback analysis:



- Split into two transfer functions:
  - » TM acoustic Amplitude → excited Stokes/a-Stokes HTM cavity field: $G(\omega)$
  - » Cavity field →Force on TM Surface (radiation pressure): **Const.**
  - » Force→ driven response of damped acoustic oscillator: $H(\omega)$

Feedback model

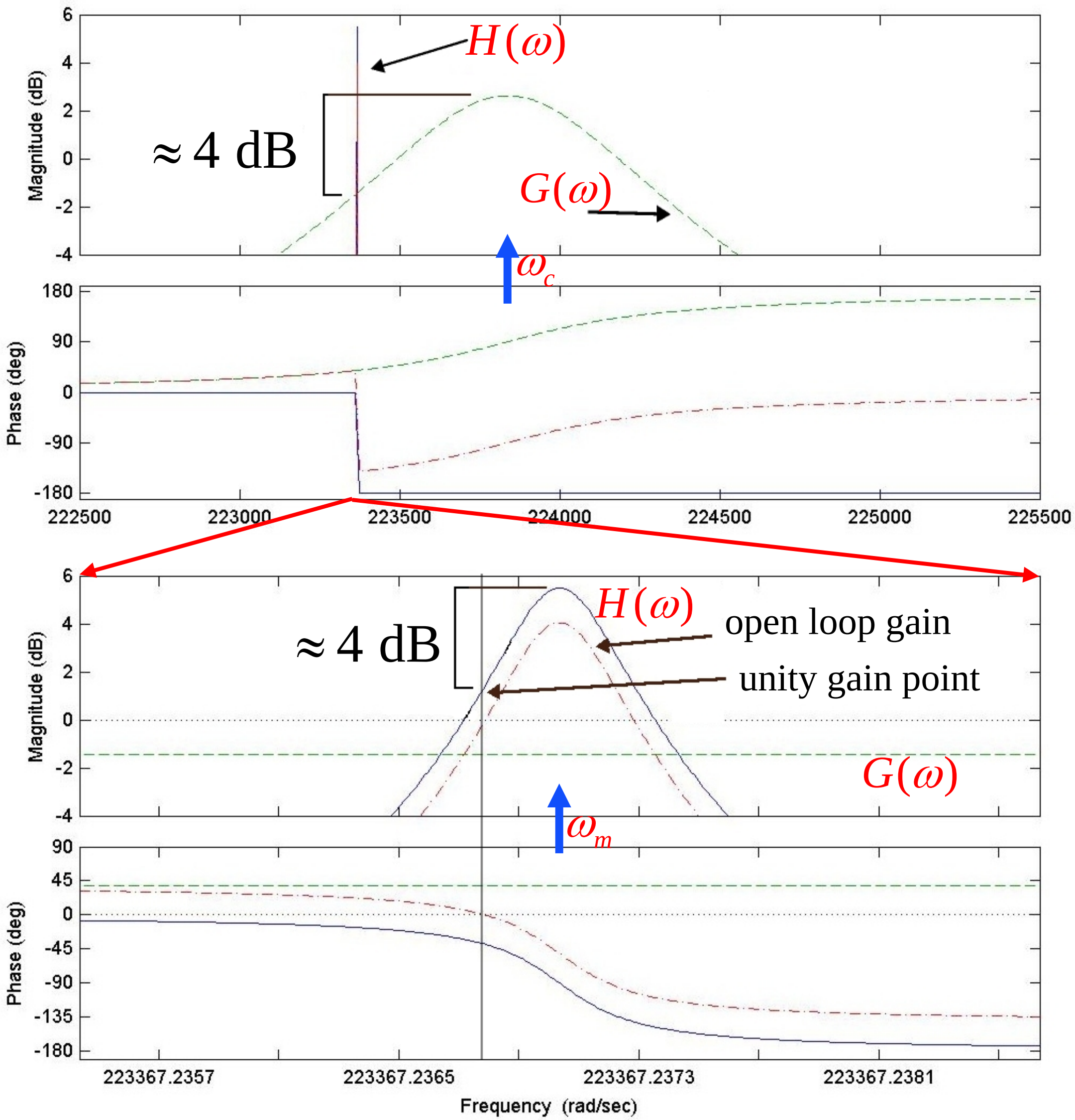
$$G(\omega) = i\alpha\left(\frac{-e^{-i\left(\frac{-2\omega L}{c}+\phi_{as}\right)}}{1-r_1 r_2 e^{-i\left(\frac{-2\omega L}{c}+\phi_{as}\right)}} + \frac{e^{i\left(\frac{2\omega L}{c}+\phi_s\right)}}{1-r_1 r_2 e^{i\left(\frac{2\omega L}{c}+\phi_s\right)}}\right) \rightarrow \frac{\alpha}{-\omega+\omega_c-\frac{1-r_1 r_2}{r_1 r_2}\frac{c}{2L}i}$$

Single pole & Stokes Approx.

$$H(\omega) = \frac{1}{-\omega^2+i\gamma\omega+\omega_m^2} \rightarrow \frac{1}{-\omega+\omega_m+i\frac{\gamma}{2}}\left(\frac{1}{2\omega_m}\right)$$

- $G(\omega)$ contains two terms: Stokes and anti-Stokes
  - » Stokes amplifies TM vibrations, anti-Stokes damps
  - » For well spaced cavity HTM resonances only one ~ coincides with $\omega_0 \pm \omega_m$
- Open loop *unity gain* ω_{UG} corresponds to $G(\omega_{UG})\, H(\omega_{UG})=$***R***
  - » Net phase shift =0 means $G(\omega_{UG})$, $H(\omega_{UG})$ must be phase conjugates.
  - » Eigen-frequency coupled system occurs at equal magnitude decrements from the two transfer function peaks

Feedback Bode analysis



- Open loop transfer function has unity gain, with 0 phase shift. Unstable!
- At freq. where open loop gain = 1, equal factor down peaks in both transfer functions

Acoustic mode Energy gain/dissipation

- Rate of work, $\dot{W}$ done on mirror by radiation pressure.
- Compared with natural dissipation rate, U_m/τ_m of acoustic {m}
  - » Since {m} strictly does no work on cavity field: pure *parametric* excitation.
- Instability identified by $\dot{W}\,\tau_m/U_m \geq 1$ (Kells, LIGO-T060296)
  - » Independent of specific eigen-frequency, $\beta << \delta$. Therefore assume $\beta=0$.

- Not dependent on specific cavity field formulation ~~modal~~
  - » $\dot{W}$ is real physical quantity: no phase condition, so β irrelevant
  - » Allows unambiguous meaning to ***R***({***p***}) away from threshold (LIGO-T060207)
  - » If field Approx. as modes: recover same coupled mode ***R*** formula, thus verifying previous interpretation as sum over modal parts

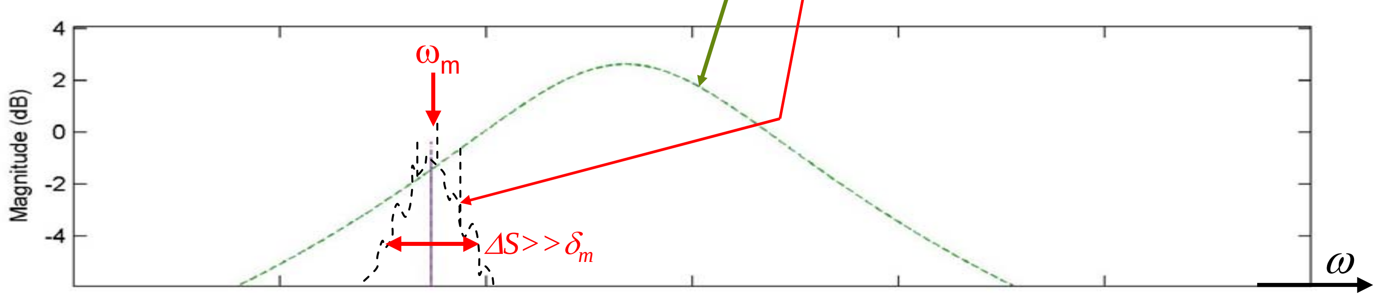
Drift in parametric spectrum

- Coupled mode *PI* analysis to date: necessarily static system {***p***}
- Anticipate significant drift in most {***p***}
  - » Drift of ω_m due to $T_{ambient}$, beam heat, stress, thermal deformation.
    - – Especially as cavity power is cycled to $\mathbf{P}_{0max}$
    - – Changes to ω_{1j} due to cavity deformations (mirror radius, beam position)
    - – Parametric induced change: β, τ_m
- Expected instability $\tau_{PI} \equiv \frac{\tau_m}{R-1} = \frac{2}{\delta_m (R-1)} \approx \frac{2}{.001\mathrm{s}^{-1}(<10)} \approx 10^3\ \mathrm{sec}$
- Expected thermal ω_m drift <1Hz/hr so that $\tau_{PI} \bullet \text{drift rate} / \delta_m \gg 1$
  - » Static analysis assumes drift < δ_m, β
  - » How do we know $\boldsymbol{R}_{static}$ reasonably represents drifting interferometer?
- Analysis via the linear feedback model resolves this:
  - » Fundamental problem is linear near threshold $\Rightarrow$ invoke Fourier decomposition!
  - » Drift described in terms of broader ($line \gg \delta_m$) spectrum acoustic excitation
  - » Interpret power spectrum as $\propto \dot{W}$ [non-linear!] via Parseval theorem.

$Work_m$: spectrum shape independent

- Stokes cavity $E_{1j}(\omega_0 - \omega)$ components linear in acoustic {m} $x_m(\omega)$, using $x_m(\omega) \equiv \tilde{x}_m$

$$\tilde{E}_{bk} \equiv E_{bk}(\omega_0 - \omega) = G(\omega)\tilde{x}_m \approx G(\omega_m)\tilde{x}_m$$



- $\dot{W}(\omega)$ density acting on {m} thus closely in proportion to spectral density of $x_m(\omega)$: $\dot{W}(\omega)d\omega \propto E_{bk}\omega\tilde{x}^*_m d\omega = G(\omega)\tilde{x}_m\omega\tilde{x}^*_m d\omega$

- Then, by Parseval, mean rate of work done on {m}:

$$\int \dot{W}(\omega)d\omega \propto \int_{\Delta S} G(\omega)\left|\tilde{x}_m\right|^2 \omega d\omega \approx G(\omega_m)\omega_m \int_{\Delta S}\left|\tilde{x}_m\right|^2 d\omega \propto \left\langle U_m(t)\right\rangle_{\Delta t \sim \Delta S^{-1}}$$

that is, independent of particular spectrum of $x_m(\omega)$

No *PI* dilution

- *Physical* assumption: that U_m(t) cannot change due to parameter drift
  - » E.g. isolated TM energy is conserved (excepting slow dissipation).
- Condition for this independence: $\delta_m \sim .002\text{rad/s} \leq \Delta S << \delta_{\text{Cav mode}} > 500\text{rad/s}$
  - » For example: LIGO TM "ambient" drift in ω_m is < 1Hz/hr
  - » Faster dithering ω_m? Strictly limited by smooth oscillator $U_P \leftrightarrow U_K$ continuity any short time SHO phase jumps << up conversion scale.
- Conclusion: plausible TM drifts, even by $>>\delta_m$, can't dilute *PI*
- Another category: cavity parameter (Guoy phase) change effect on $H(\omega)$.
  - » $H(\omega)$ treated as a static filter…… to be modified in dynamic reality.
  - » Static Approx. holds if $\tau_{1j}\,\Delta\omega_{1j} < \pi$ which holds for $\Delta\omega_{1j} < \delta_{1j}$ (~2π 85 Hz) during t~ τ_{1j} <10ms.