Optimal parameter-space covering for continuous-wave searches

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GWDAW11@Potsdam Dec 19, 2006

Matched filtering: parameter-space covering

Templates parametrized by $\lambda \equiv \{\lambda_1, \lambda_2, ...\lambda_n\} \in \mathbb{P}_n$
Detection statistic: $\mathcal{F}(\lambda)$.

"Distance" $m \equiv \frac{E[\mathcal{F}(\lambda_s) - \mathcal{F}(\lambda)]}{E[\mathcal{F}(\lambda_s)]} = g_{ij}(\lambda_s)\, \Delta\lambda^i\, \Delta\lambda^j + \mathcal{O}(\Delta\lambda^3)$

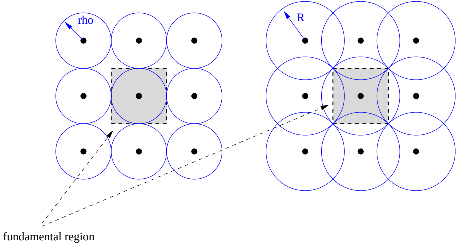
Optimal template bank:

- Set of "templates" $\lambda_{(i)}$ such that for ANY point $\lambda \in \mathbb{P}_n$, the mismatch m to the *nearest* template satisfies $m \le m_{\mathrm{max}}$.
$\Longrightarrow$ spheres of "covering radius" $R = \sqrt{m_{\mathrm{max}}}$.

- Find the covering $\{\lambda_{(i)}\}$ with the smallest number of templates $\Longrightarrow$ "sphere covering problem" ('dual' of sphere-packing problem!)

Sphere "Packing" versus "Covering"

Conway, Sloane, *Sphere packings, lattices and groups* (1998)



Packing density: $\Delta \equiv \frac{\text{Volume of sphere}}{\text{fundamental volume}} < 1$

☞ fraction of space occupied by spheres

Covering thickness: $\Theta \equiv \frac{\text{Volume of sphere}}{\text{fundamental volume}} > 1$

☞ average number of spheres covering a point

Packing problem: given ρ, *maximize* density Δ

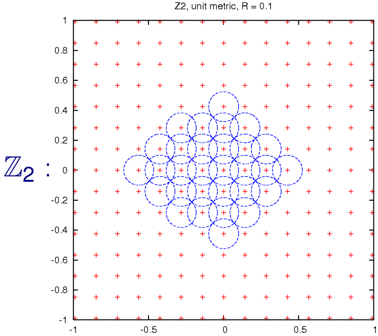
Covering problem: given R, *mimimize* thickness Θ

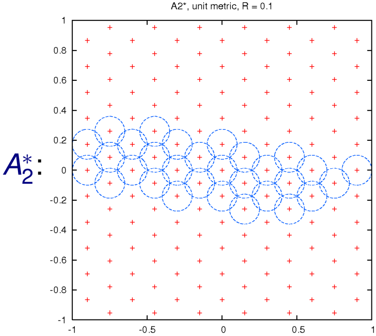
What is know about (Euklidean) Covering?

n	lattice	Θ	alt.	Θ
2	A_2^* (hexagonal)	1.21		
3	A_3^* (bcc)	1.46		
4	A_4^*	1.77		
5	A_5^*	2.12		
6	L_6^{c1}	2.46	A_6^*	2.55
7	L_7^c	2.90	A_7^*	3.06
8	L_8^c	3.14	A_8^*	3.67
9	L_9^c	4.27	A_9^*	4.39

- Today L_n^c best covering known for up to $n = 15$
- Generally A_n^* (previous record-holder) not much thicker than the best currently known lattice.

Advantage of A_n^* covering over $\mathbb{Z}^n$





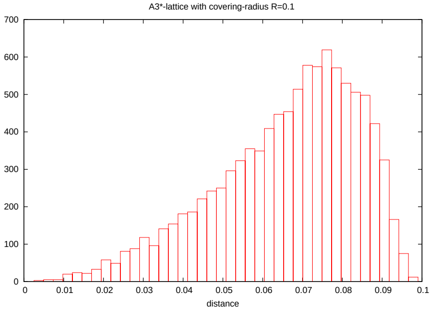
"gain" of A_n^* with respect to $\mathbb{Z}_n$: $\kappa(n) = \frac{\Theta(\mathbb{Z}_n)}{\Theta(A_n^*)} \sim e^n$

n	2	3	4	5	11	17
$\kappa(n)$	1.30	1.86	2.80	4.33	78.23	1691.6

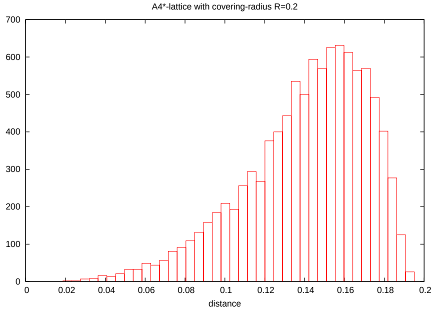
Monte Carlo - check of covering-radius

A_n^* lattice covering implemented in `LALLatticeCovering()`

- generate A_n^* lattice for given covering radius $R = \sqrt{m_{\max}}$
- pick random points and determine their minimal match m
  - ☞ histogram m



$n = 3, \quad R = 0.1$



$n = 4, \quad R = 0.2$

Flat metric approximation

We need explicitly flat metric to use lattice covering, but the continuous-wave metric g_{ij} in $\lambda^i = \left(\alpha, \delta, f, \dot{f}, \ldots\right)$ is curved.

- Approx. I: "phase metric": $g_{ij} \sim g^{\phi}_{ij} = \langle \partial_i \phi \, \partial_j \phi \rangle - \langle \partial_i \phi \rangle \langle \partial_j \phi \rangle$
$$\phi(t) = 2\pi \left[f \left(t + \frac{\boldsymbol{r}(t) \cdot \boldsymbol{n}}{c} \right) + \frac{1}{2} \dot{f} \left(t + \frac{\boldsymbol{r}(t) \cdot \boldsymbol{n}}{c} \right)^2 + \ldots \right]$$
- Approx. II: "JKS simplified phase": (linear Roemer delay)
$$\phi(t) \approx 2\pi \left[f\,t + \frac{1}{2} \dot{f}\,t^2 + \ldots + \frac{\boldsymbol{r}(t) \cdot \boldsymbol{n}}{c} f(t) \right]$$
- Approx. III: "Orbital metric": neglect $\boldsymbol{r}_{\mathrm{spin}}(t)$: (true for LISA)
$$\phi_{\mathrm{orb}}(t) \approx 2\pi \left[f\,t + \frac{1}{2} \dot{f}\,t^2 + \ldots + \frac{\boldsymbol{r}_{\mathrm{orb}}(t) \cdot \boldsymbol{n}}{c} f(t) \right]$$

☞ g^{orb}_{ij} can be shown to be flat!

Plans

- Improve LatticeCovering() to generate lattice "point-by-point" instead of "all-at-once"
- Use flat-metric approximation to build A_n^* covering for continuous-wave searches (LIGO and LISA)
- Test this by Monte-Carlo injection and signal recovery
- Alternative approach [with X. Siemens]: use (strong) degeneracy of orbital metric to reduce the "effective dimensionality" of the parameter space