ON THE ANALYTIC STRUCTURE OF A FAMILY OF HYPERBOLOIDAL BEAMS OF POTENTIAL INTEREST FOR ADV-LIGO

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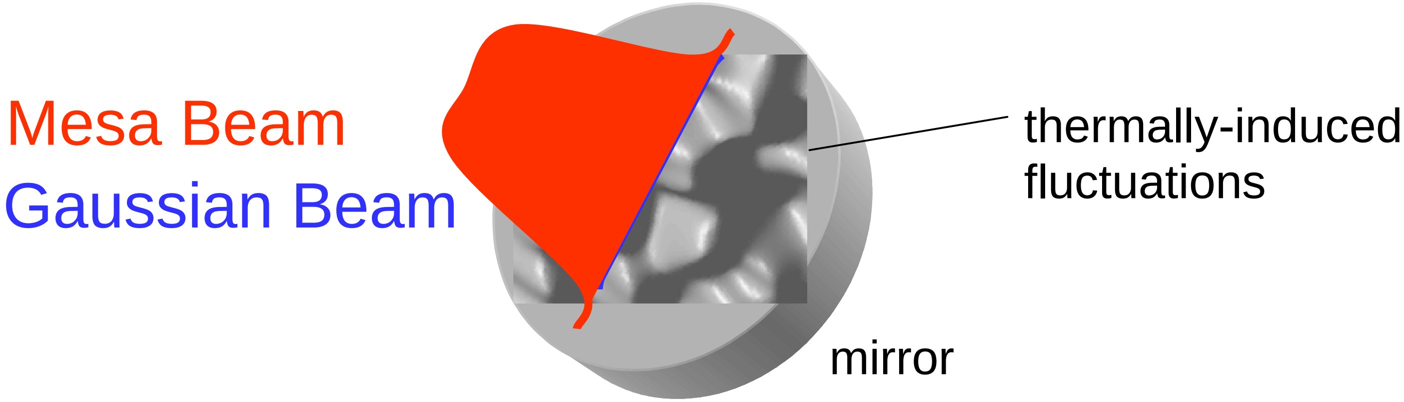
Agenda

- Background: *Non-spherical* mirrors and *flat-top* beams
- From *nearly-flat* (FM) and *nearly-concentric* mesa (CM) beams to Bondarescu-Thorne (BT) *hyperboloidal* beams
- Analytic structure of BT hyperboidal beams
  - Rapidly-converging Gauss-Laguerre (GL) expansion
  - Some results: Beam shapes and mirror corrections
  - Generalized duality relations (lowest-order mode)
    - *Complex-order* Fourier transform
- Conclusions and perspectives

Background

- Use of *flat-top* ("mesa") beams suggested for mitigating thermal noise effects in future LIGO interferometers [D'Ambrosio *et al.*, LIGO-G000223-00-D]



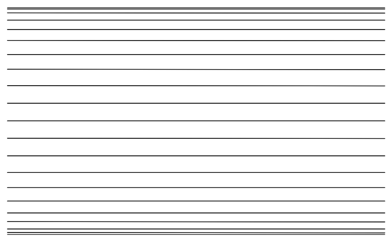
  - Better averaging of thermally-induced mirror surface fluctuations
  - Potential reduction by a factor 3 in thermoelastic noise power [D'Ambrosio *et al.*, gr-qc/0409075]

Background (cont'd)

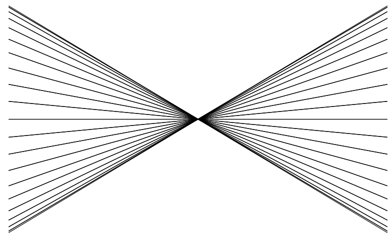
- *Nearly-flat* mesa (FM) beams
  - Synthesized via coherent superposition of minimum-spreading Gaussian beams (GB)
  - Supported by *nearly-flat* mirrors with "Mexican-hat" profile
    - Prototype cavity developed; experimental tests under way [Agresti *et al.*, LIGO-G0400412-00-D]
    - Concerns about *tilt-instability* [Savov & Vyatchanin, gr-qc/0409084]
- *Nearly-concentric* mesa (CM) beams
  - Same intensity distribution at the mirror
  - *Much weaker* tilt-instability [Savov & Vyatchanin, gr-qc/0409084]
- FM and CM configurations connected by *duality* relations
  - One-to-one mapping between eigenstates [Agresti *et al.*, gr-qc/0511062]
- More general ("hyperboloidal") beams [Bondarescu & Thorne, gr-qc/0409083] may be of interest

FM/CM vs. BT hyperboloidal beams

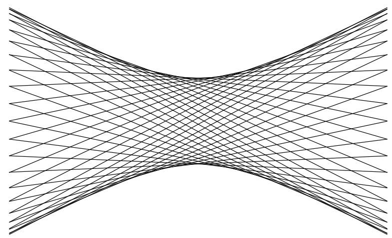
- FM beams
  - Optical axes are the generators of a *cylinder*



- CM beams
  - Optical axes are the generators of a *cone*

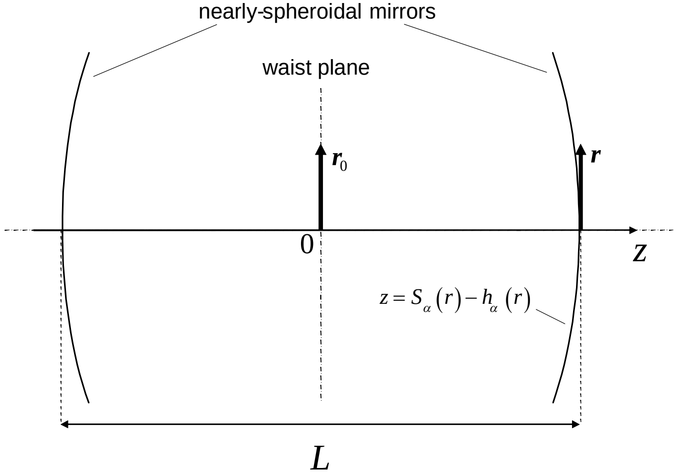


- BT hyperboloidal beams
  - Optical axes are the generators of a *hyperboloid*
  - Parameterized via “twist angle” $0 \leq \alpha \leq \pi$
  - Contain FM (α= 0) and CM ($\alpha=\pi$) as special cases
  - “Optimal” configuration expected in a neighborood of $\alpha=\pi$



BT hyperboloidal beams

- Supported by *nearly-spheroidal* mirrors



- Fiducial spheroid: $S_\alpha(r) \approx \frac{L}{2} - \frac{r^2 \sin^2(\alpha/2)}{L}$
- Waist-plane aperture radius: R_0
- GB spot size: $w_0 = \sqrt{\frac{L}{k_0}}$ (minimum spreading)

- Field distribution on fiducial spheroid [Bondarescu & Thorne, gr-qc/0409083]

$$U_\alpha(r, S_\alpha) = \Lambda \int_0^{R_0} dr_0 \int_0^{2\pi} d\theta_0 r_0 \exp\left[i \frac{r r_0}{w_0^2} \sin\theta_0 \sin\alpha - \frac{\left(r^2 + r_0^2 - 2 r r_0 \cos\theta_0\right)}{2 w_0^2} (1 - i\cos\alpha) \right]$$

  - θ_0-integral generally needs to be computed numerically
  - Closed-form (Gaussian) solution for $\alpha = \pi/2$

BT hyperboloidal beams (cont'd)

- Mirror profile correction

$$\arg\left[U_\alpha\left(r, S_\alpha - h_\alpha\right)\right] = \text{constant} \quad \Rightarrow \quad h_\alpha\left(r\right) = \frac{\arg\left[U_\alpha\left(r, S_\alpha\right)\right] - \arg\left[U_\alpha\left(0, S_\alpha\right)\right]}{k_0}$$

- Symmetry/duality relations
  - Field distribution

$$U_{-\alpha} = U_\alpha, \quad \frac{U_{\pi-\alpha}}{\Lambda} = \frac{U_\alpha^*}{\Lambda^*}$$

  - Mirror profile correction

$$h_{\pi-\alpha}\left(r\right) = -h_\alpha\left(r\right)$$

GL expansions

- FM and CM beams
  - Field distributions at the waist plane related via Fourier transform (FT)
  - Coincide with “flattened” beams in [Sheppard & Saghafi, *Opt. Comm.* **132**, 144, 1996]
    - Gauss-Laguerre (GL) expansions available

$$U_\pi(r,0)=\sum_{m=0}^{\infty}A_m^{(\pi)}\psi_m\left(\frac{\sqrt{2}r}{w_0}\right),\quad U_0(r,0)=\sum_{m=0}^{\infty}A_m^{(0)}\psi_m\left(\frac{\sqrt{2}r}{w_0}\right)$$

$$\psi_m(\xi)=\sqrt{2}\exp\left(-\frac{\xi^2}{2}\right)L_m\left(\xi^2\right)$$ GL (orthonormal) basis functions

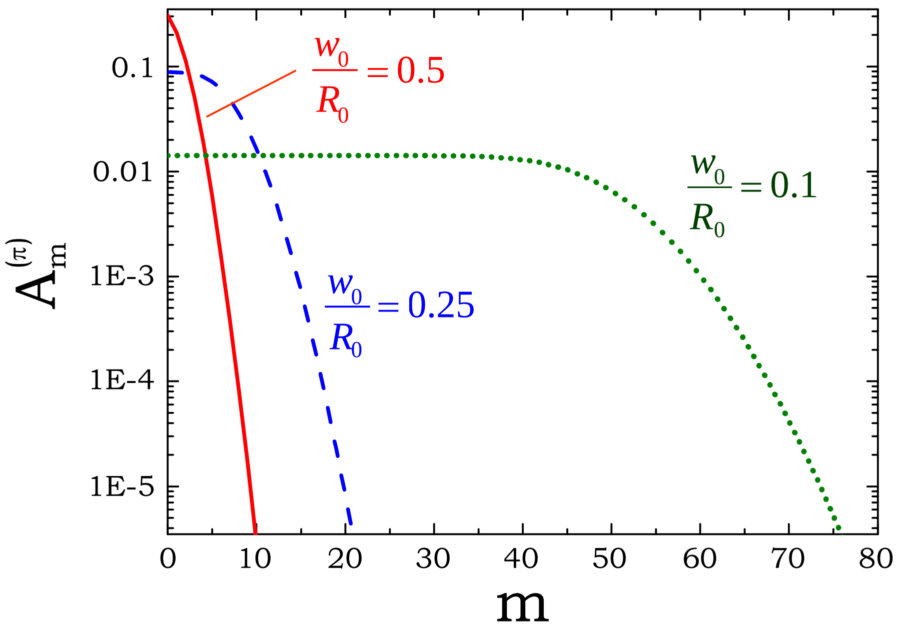
- FT $\Rightarrow$ $\boxed{A_m^{(0)}=(-1)^m A_m^{(\pi)}}$

GL expansions (cont'd)

- Expansion coefficients [Sheppard & Saghafi, *Opt. Comm.* **132**, 144, 1996]

$$A_m^{(\pi)} = \frac{\sqrt{2}w_0^2}{R_0^2} P\left(m+1, \frac{R_0^2}{2w_0^2} \right)$$

P: incomplete Gamma function



- Easily computable
- Abrupt fall-off for $m \geq \frac{R_0^2}{2w_0^2}$

⇨ **Rapid convergence**

GL expansions (cont'd)

- Extension to ***generic*** **BT hyperboloidal beams** [Galdi *et al.*, gr-qc/0602074]

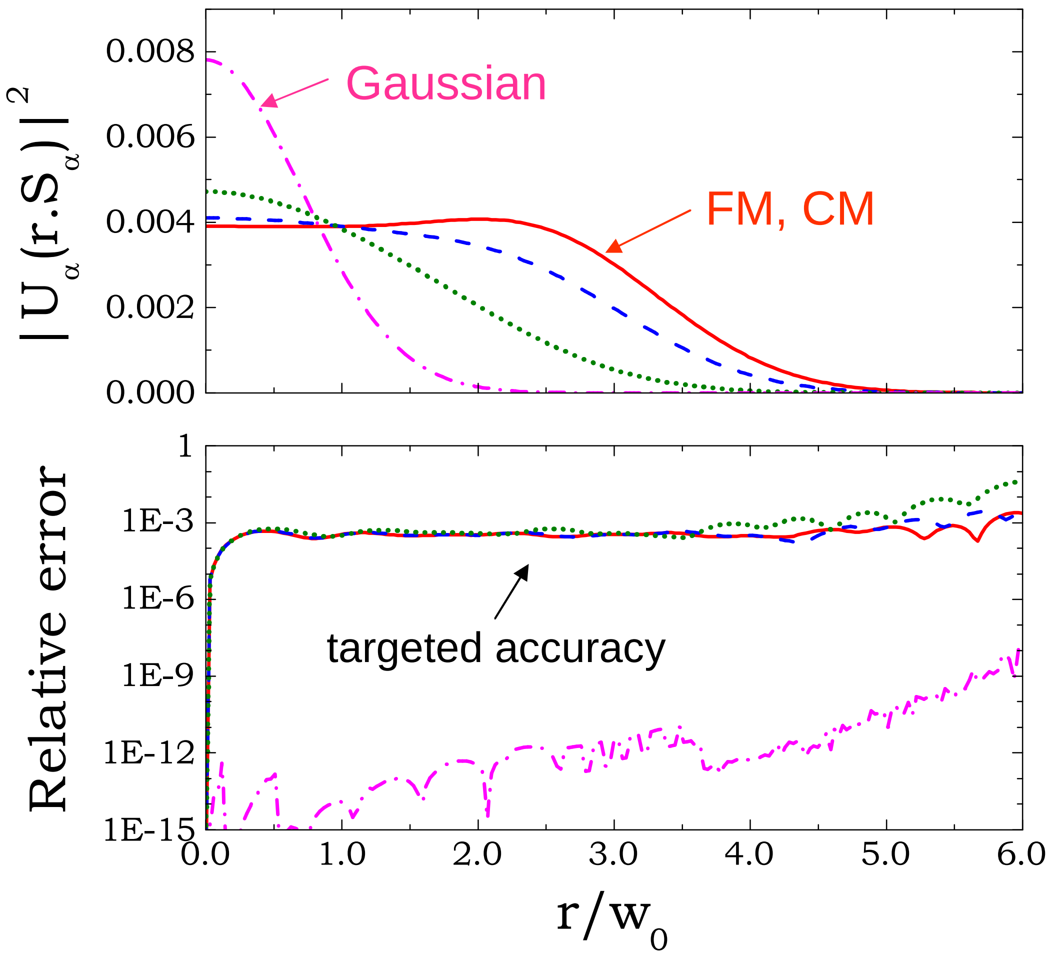
$$U_\alpha(r,0)=\sum_{m=0}^{\infty}A_m^{(\alpha)}\psi_m\left(\frac{\sqrt{2}r}{w_0}\right),\quad \boxed{A_m^{(\alpha)}=(-\cos\alpha)^m A_m^{(\pi)}}$$

- Allows field computation *at any point in space*
  - Use standard GL (paraxial) propagators
  - Field distribution on the fiducial spheroids

$$U_\alpha(r,S_\alpha)=\Omega\exp\left(ik_0\frac{r^2\cos\alpha}{2L}\right)\sum_{m=0}^{\infty}(-i)^m A_m^{(\alpha)}\psi_m\left(\frac{r}{w_0}\right)$$

  - Symmetry/duality relations verified

Results: Beam shapes



- Parameters

  $w_0 = 2.603cm,\ R_0 = 10.4cm,$
  $\lambda_0 = 1064nm$

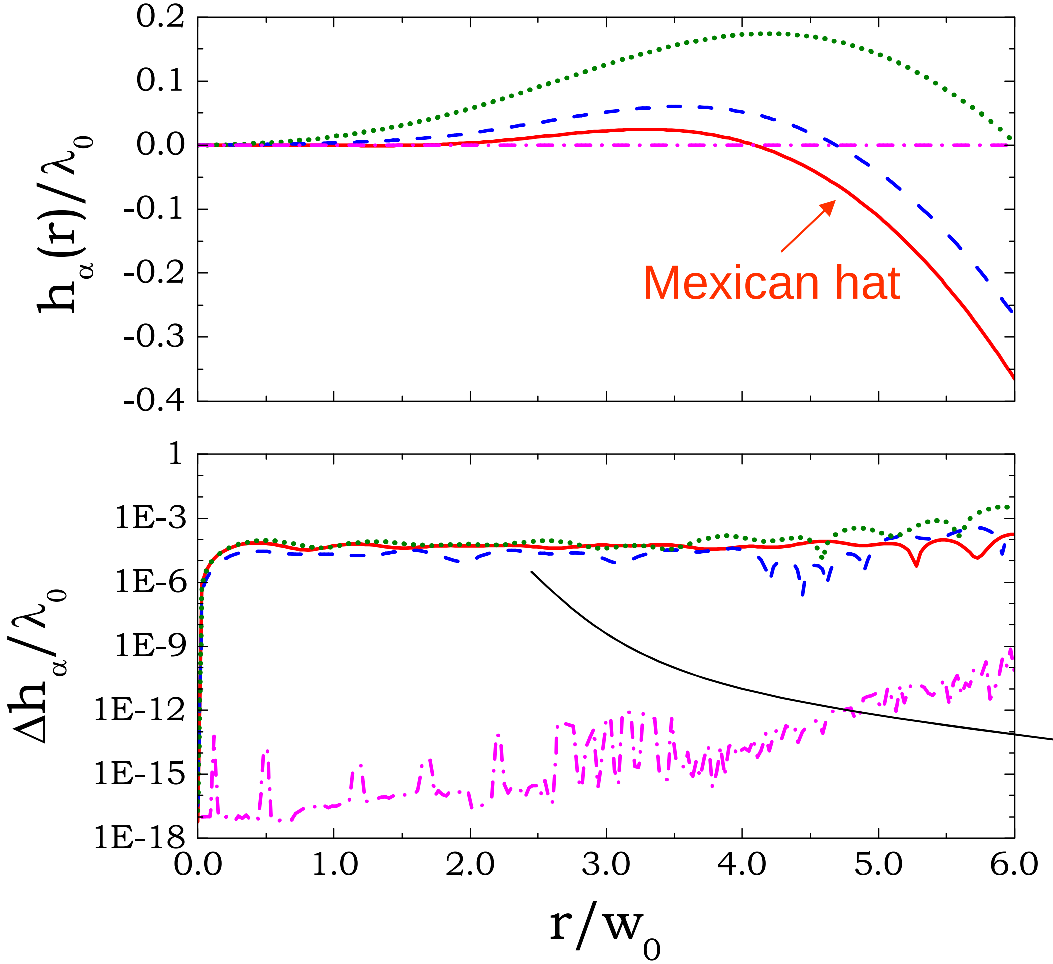
- Truncation criterion

  $m \le M, \quad \left| \frac{A_M^{(\alpha)}}{A_0^{(\alpha)}} \right| < 10^{-3}$

- Reference solution from [Bondarescu & Thorne, gr-qc/0409083]

$\alpha = 0, \pi \ (M = 18)$
$\alpha = 0.1\pi, 0.9\pi \ (M = 17)$
$\alpha = 0.2\pi, 0.8\pi \ (M = 14)$
$\alpha = 0.5\pi \ (M = 0)$

Results: Mirror corrections



- Same parameters

  $\alpha = \pi \ (M = 18)$

  $\alpha = 0.9\pi \ (M = 17)$

  $\alpha = 0.8\pi \ (M = 14)$

  $\alpha = 0.5\pi \ (M = 0)$

- For $0 \le \alpha < \pi/2$

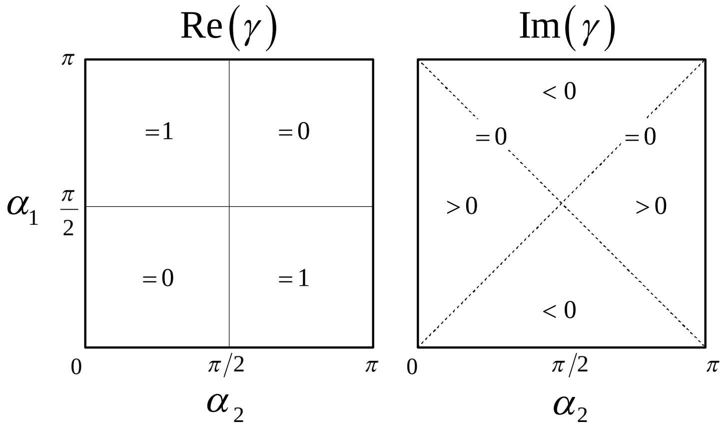
  $h_{\pi-\alpha}(r) = -h_\alpha(r)$

- Typical errors: ~0.1nm
  - Well within fabrication tolerances

Generalized duality relations

- Two *arbitrary* (α_1,α_2)-indexed BT hyperboloidal beams related by *fractional* FT operators of *complex order*

$$\gamma = 1 + \frac{i}{\pi}\log\left(-\frac{\cos\alpha_2}{\cos\alpha_1}\right)$$



Special cases

$\alpha_1 = \alpha_2 \quad \Rightarrow \quad \gamma = 0$ (identity operator)

$\alpha_1 = \pi - \alpha_2 \quad \Rightarrow \quad \gamma = 1$ (standard FT)

- Generalizes (for lowest-order mode) the FM-CM duality relations in [Agresti *et al.*, gr-qc/0511062]

Conclusions and perspectives

- Summary
  - Focus on the analytic structure of a class of hyperboloidal beams of interest for future LIGO interferometers
  - Rapidly-converging, physically-insightful GL expansions for *generic* BT hyperboloidal beams
    - Field computation at any point in space
    - Validation/calibration against reference solution
  - Generalized duality relations
    - *Complex-order* FT

- Current/future research
  - Thorough parametric analysis
    - Implications for Adv-LIGO
  - Extension to higher-order modes (HOM)
    - Techniques to depress HOM (parametric instabilities)

For more details …

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On the Analytic Structure of a Family of Hyperboloidal Beams of Potential Interest for Future LIGO Interferometers

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(Dated: February 20, 2006)

For the baseline design of advanced LIGO interferometers, use of optical cavities with *non-spherical* mirrors supporting flat-top ("mesa") beams, potentially capable of mitigating the thermoelastic noise of the mirrors, has recently drawn a considerable attention. To overcome the severe tilt-instability problems affecting the originally conceived *nearly-flat*, "Mexican-hat-shaped" mirror configuration, K. S. Thorne proposed a *nearly-concentric* mirror configuration capable of generating the same mesa beam profile on the mirror surfaces. Subsequently, Bondarescu and Thorne introduced a generalized construction that leads to a one-parameter family of "hyperboloidal" beams which allows continuous spanning from the nearly-flat to the nearly-concentric mesa beam configurations. This paper is concerned with a study of the analytic structure of the above family of hyperboloidal beams. Capitalizing on certain results from the applied optics literature on flat-top beams, a physically-insightful and computationally-effective representation is derived in terms of rapidly-converging Gauss-Laguerre expansions. Moreover, the functional relation between two *generic* hyperboloidal beams is investigated. This leads to a generalization (involving *fractional* Fourier transform operators of *complex* order) of some recently discovered *duality* relations between the nearly-flat and nearly-concentric mesa configurations. Possible implications and perspectives for the advLIGO optical cavity design are discussed.

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