Error Function Filter

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Reducing the number of data points

Introduction

- Discrete Fourier Transform Definitions
- The Nyquist Theorem
- Ideal Filter/Error Function Filter
- Pulsar numbers
- Dramatically reducing the number of needed data points

DFT definitions

$$t_i \equiv iT/N = i\Delta_t \qquad f_m \equiv m/T \qquad j \equiv \sqrt{-1}$$

$$D\left[f_m\right] \equiv \frac{T}{N} \sum_{i=-N/2}^{N/2-1} d\left[t_i\right] \exp\left(j2\pi f_m t_i\right)$$

$$\cong \int_{-T/2}^{T/2} d\left(t\right) \exp\left(j2\pi f_m t\right) dt$$

$$d\left[t_i\right] = \frac{1}{T} \sum_{m=-N/2}^{N/2-1} D\left[f_m\right] \exp\left(-j2\pi f_m t_i\right)$$

$$\cong \int_{-1/2\Delta_t}^{1/2\Delta_t} D\left(f\right) \exp\left(-j2\pi f t_i\right) df$$

Convolution

Frequency

$$\frac{1}{T}\sum_{n=-N/2}^{N/2-1} X\left(f_n\right)S\left(f_m - f_n\right) = \Delta_t \sum_{i=-N/2}^{N/2-1} x\left(t_i\right)s\left(t_i\right)\exp\left(j2\pi f_m t_i\right)$$

$$= XS\left(f_m\right)$$

Time

$$\Delta_t \sum_{n=-N/2}^{N/2-1} x\left(t_n\right)s\left(t_m - t_n\right) = \frac{1}{T}\sum_{i=-N/2}^{N/2-1} X\left(f_i\right)S\left(f_i\right)\exp\left(-j2\pi f_m t_i\right)$$

$$= xs\left(t_m\right)$$

Nyquist Theorem[1]

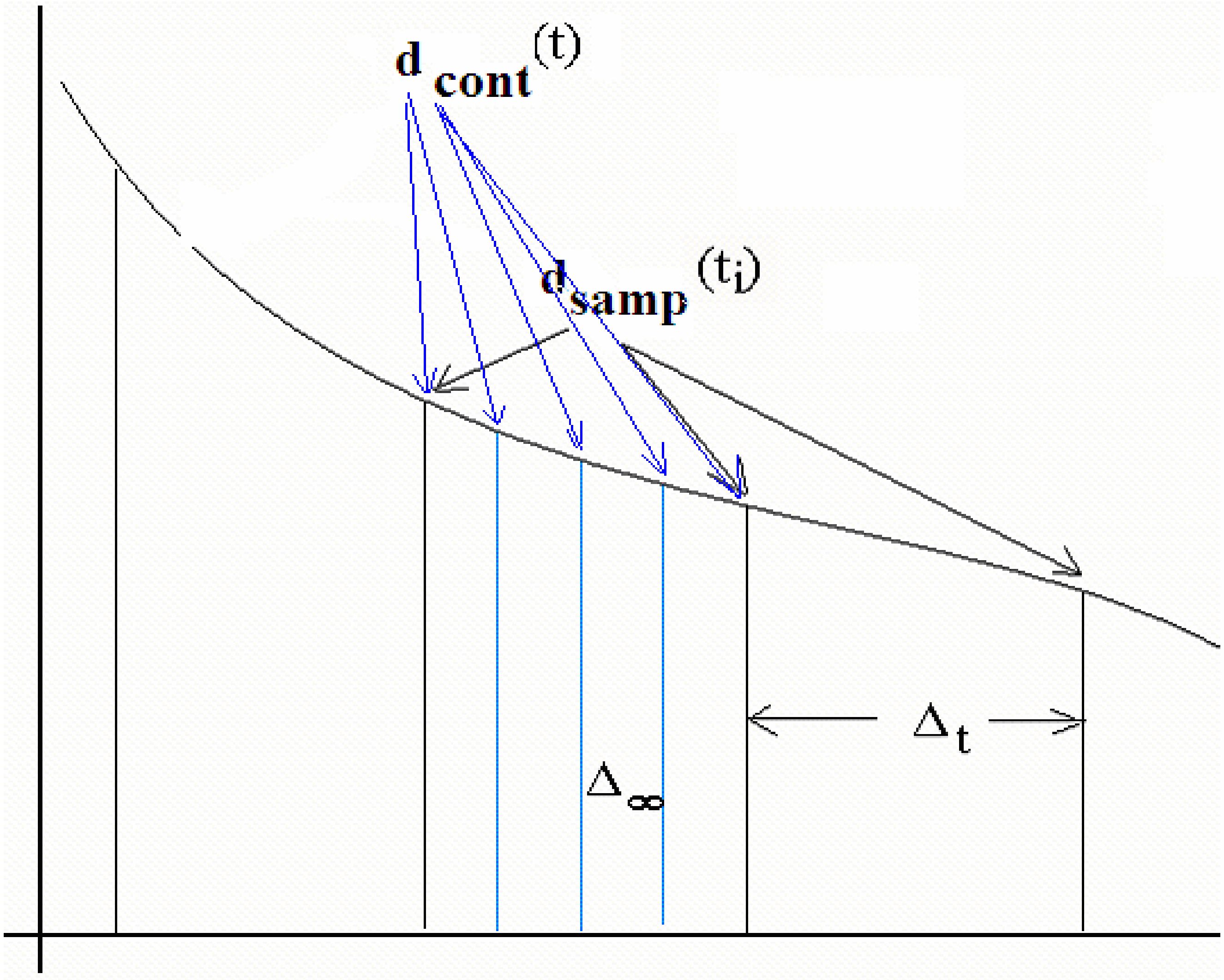
Define

$$N_\infty = MN$$

$$T = N_\infty \Delta_\infty = N \Delta_t$$

$$f_m = \frac{m}{T} \quad -\frac{N_\infty}{2} < m < \frac{N_\infty}{2}$$

[1] Loosely based on Alan V. Oppenheimer, Ronald W. Schafer with John R. Buck, **Discrete-time Signal Processing** – Prentice Hall Signal Processing Series – Alan V. Oppenheimer, editor, Second edition 1999 – first 1989



Data versus time

$$s\left(t_k\right) \equiv M \sum_{i=-N/2}^{N/2-1} \delta_{k,Mi}$$

$$S\left(f_m\right) = \Delta_\infty M \sum_{i=-N/2}^{N/2-1} \sum_{k=-N_\infty/2}^{N_\infty/2-1} \delta_{k,Mi} \exp\left(j2\pi f_m t_k\right)$$

$$= \Delta_t \sum_{i=-N/2}^{N/2-1} \exp\left(j2\pi \frac{m}{T} Mi \frac{T}{N_\infty}\right)$$

Note upper limit of N/2-1

$$= \Delta_t \sum_{i=-N/2}^{N/2-1} \exp\left(j2\pi i \frac{m}{N}\right)$$

This sum is N for m=kN, zero otherwise

The function s(t)

$$S(f_m) = \frac{T}{N} N \sum_{k=-M/2}^{M/2-1} \delta_{m,kN} = T \sum_{k=-M/2}^{M/2-1} \delta_{m,kN}$$

$$d_{samp}(t_k) = d_{cont}(t_k) s(t_k)$$

This is set up for convolution

$$D_{samp}(f_m) = \frac{1}{T} \sum_{n=-N_\infty/2}^{N_\infty/2-1} D_{cont}(f_n) S(f_m - f_n)$$

Inserting S(f)

$$D_{samp}\left(f_m\right) = \frac{1}{T} T \sum_{k=-M/2}^{M/2-1} \sum_{n=-N_\infty/2}^{N_\infty/2-1} D_{cont}\left(f_n\right) \delta_{(n-m),kN}$$

$$= \sum_{k=-M/2}^{M/2} \sum_{n=-N_\infty/2}^{N_\infty/2-1} D_{cont}\left(f_n\right) \delta_{(n-m),kN}$$

$$= \sum_{k=-M/2}^{M/2-1} D_{cont}\left(f_{m+kN}\right)$$

$$f_{kN} = \frac{kN}{T} = \frac{k}{\Delta_t}$$

D_{samp} is periodic by construction

Nyquist theorem in frequency

Back Transform

- The back transform over all N_∞ points is not wanted since it will produce the spiky function transformed forward.
- Define an ideal filter as

$$H(f)=\begin{cases} 1/2 & |f|=1/(2\Delta_t) \\ 1 & -1/(2\Delta_t)< f <1/(2\Delta_t) \\ 0 & |f|>1/(2\Delta_t) \end{cases}$$

$$D_{samp}\left(f_m\right)H\left(f_m\right)=\frac{1}{T}\sum_{k=-M/2}^{M/2-1}D_{cont}\left(f_{m+kN}\right)H\left(f_m\right)$$

The sum is over M as in $N\infty = M \times N$

With appropriate restrictions

$$D_{cont}\left(f_m\right)=D_{samp}\left(f_m\right)H\left(f_m\right)$$

But in any case define

$$D_H\left(f_m\right)=D_{samp}\left(f_m\right)H\left(f_m\right)$$

This form is a setup for convolution

Using the convolution theorem

$$d_H\left(t_m = m\Delta_\infty\right) = \Delta_\infty \sum_{i=-N_\infty/2}^{N_\infty/2-1} d_{samp}\left(t_i\right)h\left(t_m - t_i\right)$$

$$= M\Delta_\infty \sum_{i=-N_\infty/2}^{N_\infty/2-1} d_{cont}\left(t_i\right)\delta_{ik}h\left(t_k - t_i\right) \qquad t_i = i\Delta_\infty$$

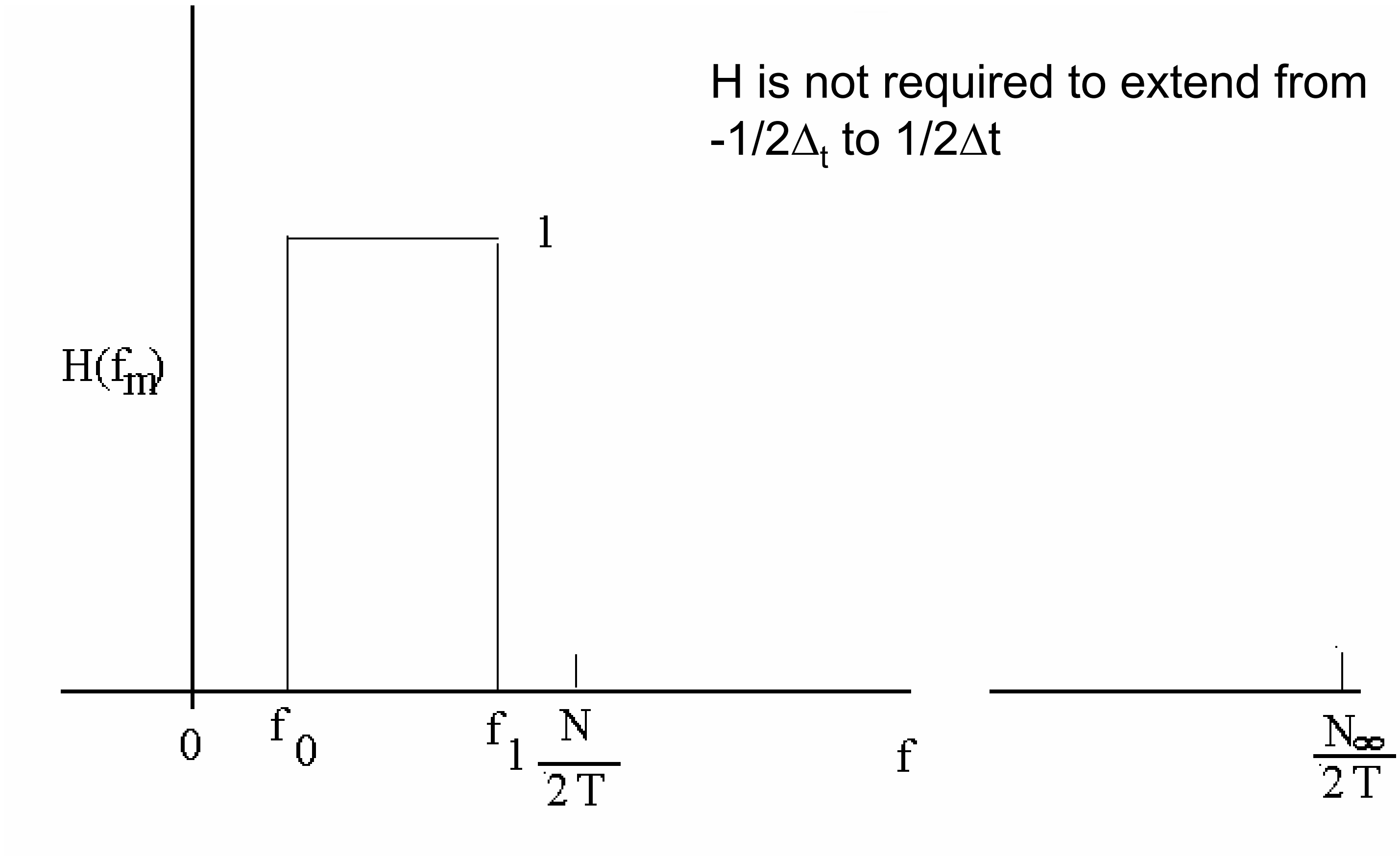
$$= \Delta_t \sum_{k=-N/2}^{N/2-1} d_{cont}\left(t_k\right)h\left(t_m - t_k\right) \qquad t_k = k\Delta_t$$

The time t_m is any time, the time t_k is for a data point.

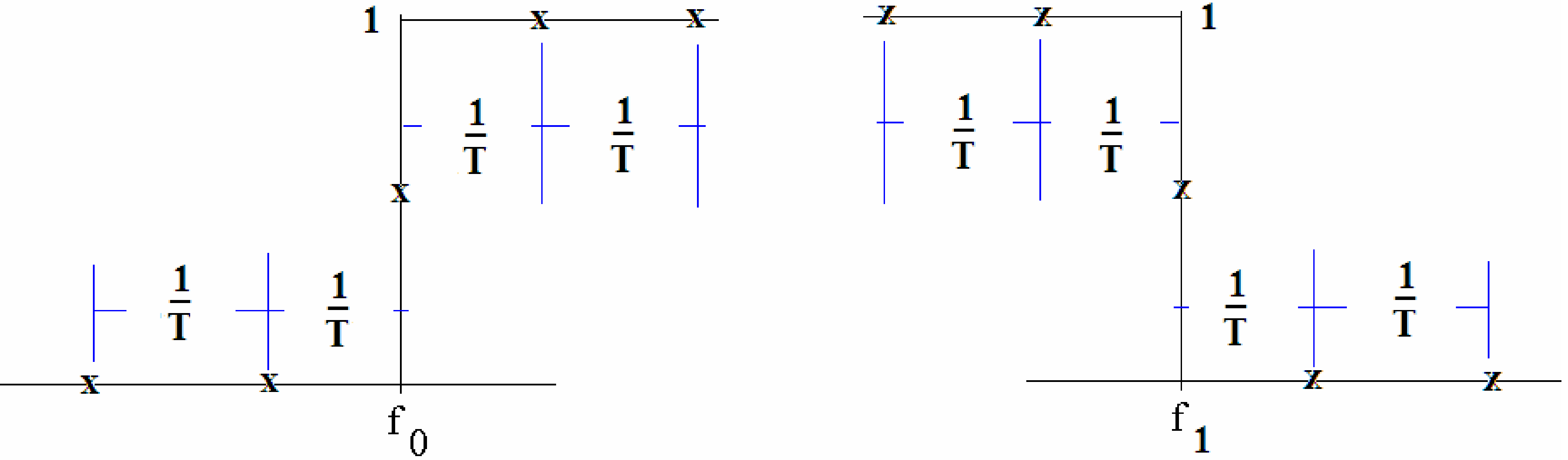
This is where the dramatic reduction in data points needed takes place.

The Nyquist Theorem in time

Ideal Filter/Error Function Filter



The ideal f_0 to f_1 filter



Details of the filter near the two ends

Ideal filter transformation

The fact that this sum is to N_∞ includes the extra point at N/2

$$h(t_i; f_0, f_1) = \frac{1}{T} \sum_{i=-N_\infty/2}^{N_\infty/2-1} H(f_m; f_0, f_1) \exp(-j2\pi f_m t_i)$$

$$h(t_i; f_0, f_1) = \frac{1}{T} \sum_{m=m_0}^{m=m_1-1} \exp(-j2\pi f t)$$

$$-\frac{1}{2T}\left(\exp\left(-j2\pi\frac{m_0}{T}t\right) - \exp\left(-j2\pi\frac{m_1}{T}t\right)\right)$$

Subrtacting ½ the first term ↑

The extra term ↑

A few steps are skipped involving 1/(1-exp(-j2π1/T)). All steps are rigorous for the sums

Ideal $h(t,f_0,f_1)$

$$h(t) = \exp\left(-j\pi\left(f_0 + f_1\right)t\right)\times \left[\frac{\sin\left(\pi\left(f_1 - f_0\right)t\right)\cos\left(\pi\dfrac{t}{T}\right)}{T\sin\left(\pi\dfrac{t}{T}\right)}\right]$$

The second term as T →∞ →
$$\frac{\sin\left(\pi\left(f_1 - f_0\right)t\right)}{\pi t}$$

Error function filter

$$AiGauss(f;f_0,w) \equiv \int_{-\infty}^{f} \frac{1}{w\sqrt{\pi}} \exp\left(-\left(\frac{x-f_0}{w}\right)^2\right) dx$$

Let $x=(f-f_0)/w$, then for $f<f_0$

$$AiGauss(f;f_0,w) = .5\left(1-erf\,(x)\right)$$

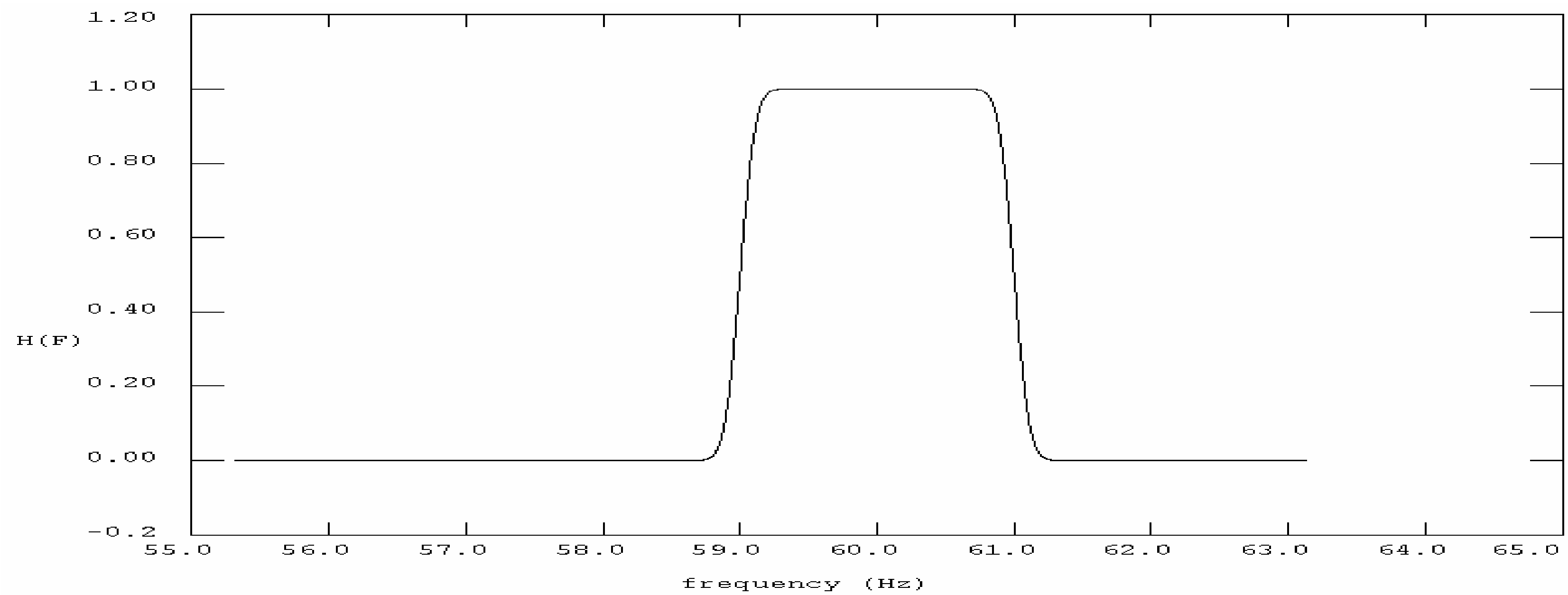
And for $f > f_0$

Equivalent erfs allow overlapping regions to exactly sum to 1

$$AiGauss(f;f_0,w) = .5\left(1+erf\,(x)\right)$$

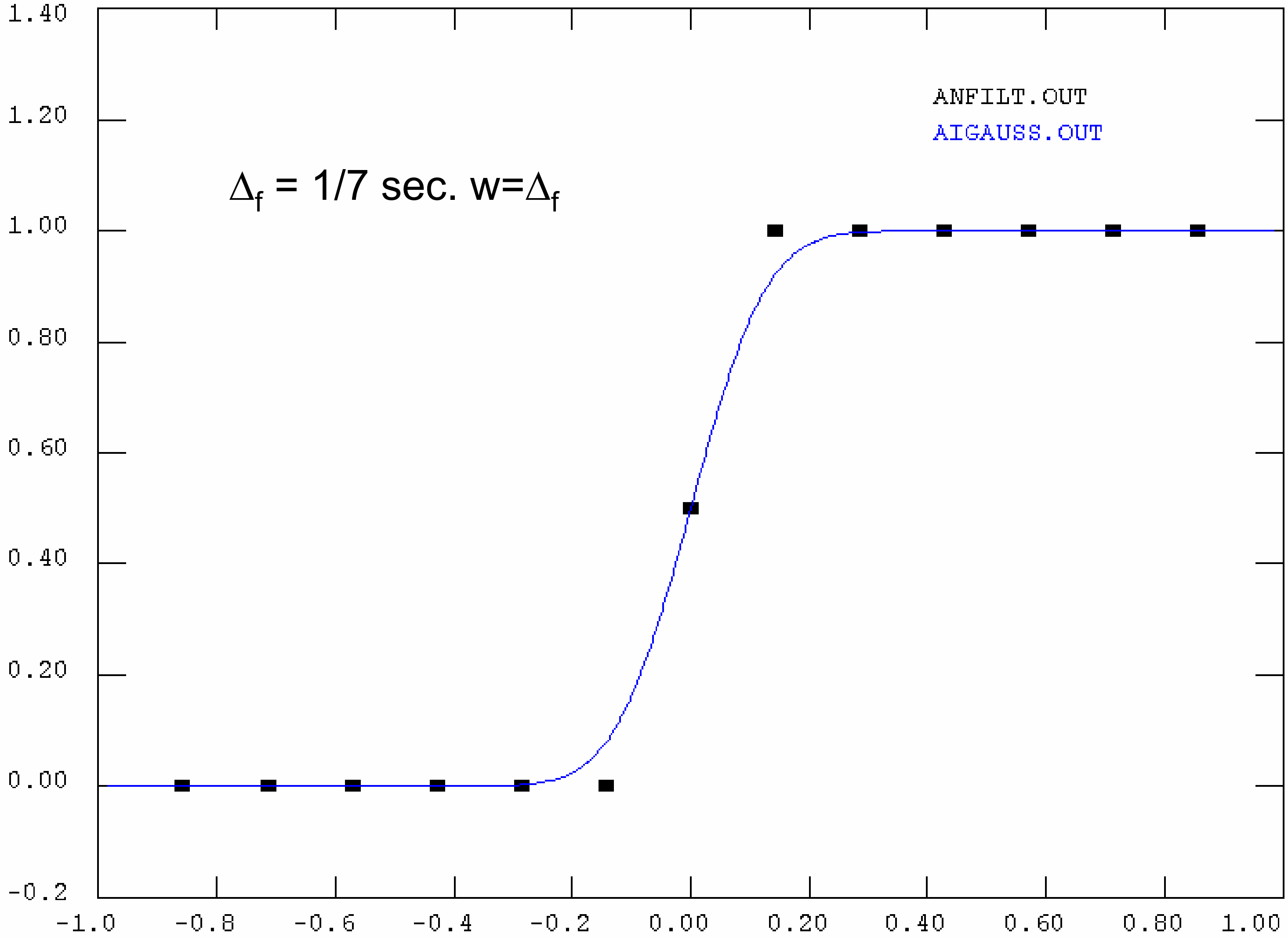
$H_{errf}(f, f_0, f_1)$

$$H_{errf}\left(f; f_0, f_1, w\right) \equiv AiGauss(f; f_0, w) - AiGauss(f; f_1, w)$$



The f_0 = 59 Hz, f_1=61 Hz w = 0.125 Hz.

Error function filter/ ideal filter



$h_{errf}(t,f_0,f_1,w)$

Approximation of the integral result requires integration by parts

$$h_{errfI}\left(t; f_0, f_1, w\right) = \exp\left(-\pi^2 w^2 t^2\right) \times$$

$$\exp\left(-j\pi\left(f_0 + f_1\right)t\right) \frac{\sin\left(\pi\left(f_1 - f_0\right)t\right)}{\pi t}$$

In a reversal of the Nyquist theorem, the correctly periodic version is

The $\exp(-(\pi wt)^2)$ term makes the sum rapidly convergent

$$h_{errf}\left(t; f_0, f_1, w\right) = \sum_{k=-\infty}^{\infty} h_{errfI}\left(t + kT; f_0, f_1, w\right)$$

This now differs from the ideal filter only by the exponential factor.

Limiting the convolution range

For $|t-tk| > 6/(\pi w)$, the exponential part of h(t,f0,f1,w) is less then

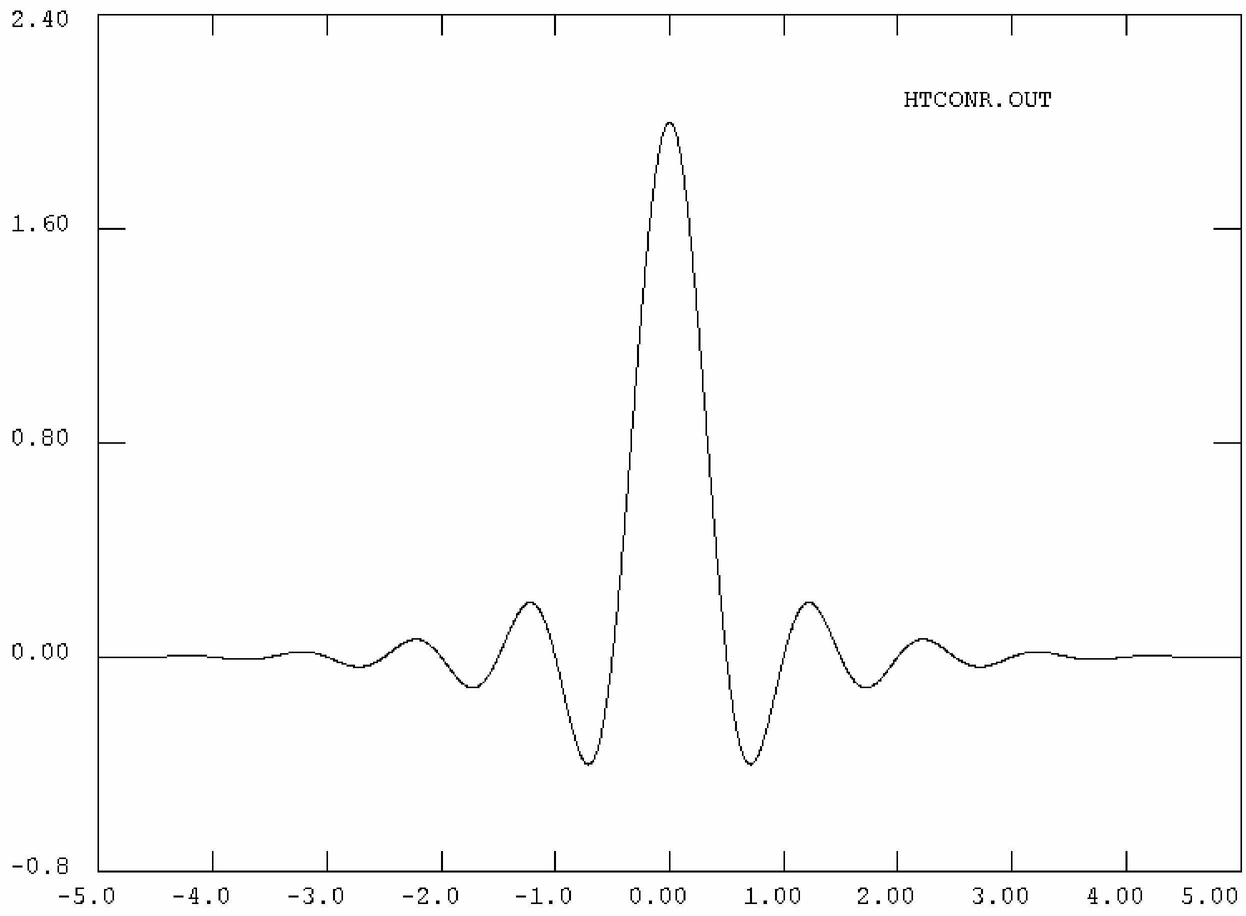
$$\exp_f = \exp\left(-\left(\pi w \frac{6}{\pi w}\right)^2\right) = \exp(-36) = 2.3\text{e-}16$$

This leads to a definition $k_{min}(t)=(t-6/(\pi w))/\Delta t$ such that

$$d_H(t) = \Delta_t \sum_{k_{\min}(t)}^{k_{\max}(t)} d_{cont}(t_k)\,h(t-t_k) \qquad t_k = k\Delta_t$$

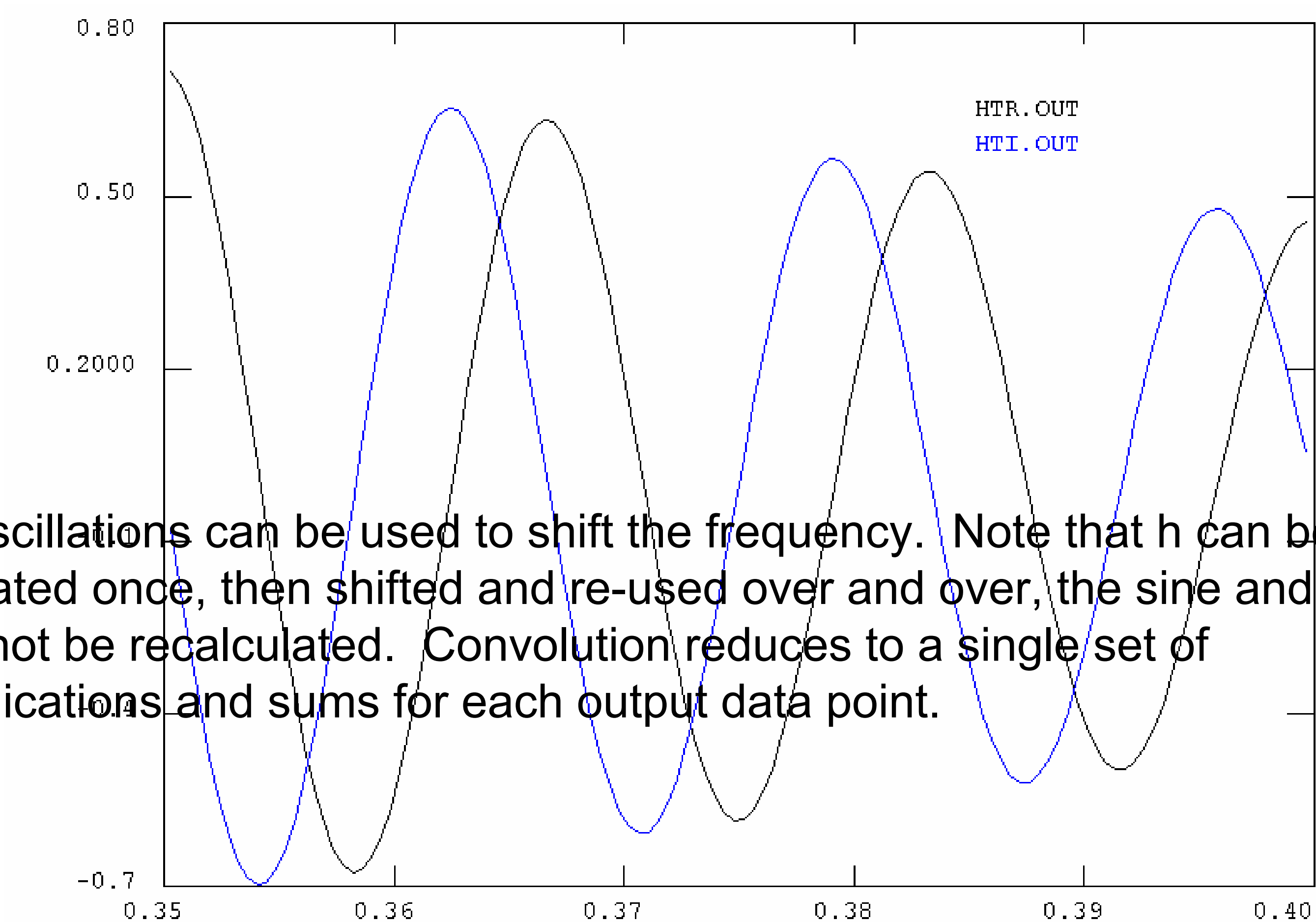
Even for infinite T, this sum is finite. For t such that $k_{min}(t) > -N/2$ and $k_{max}(t) < N/2$ $d_H(t)$ does not depend on T

$|h(t,f_0,f_1,w)|$



Time in seconds.

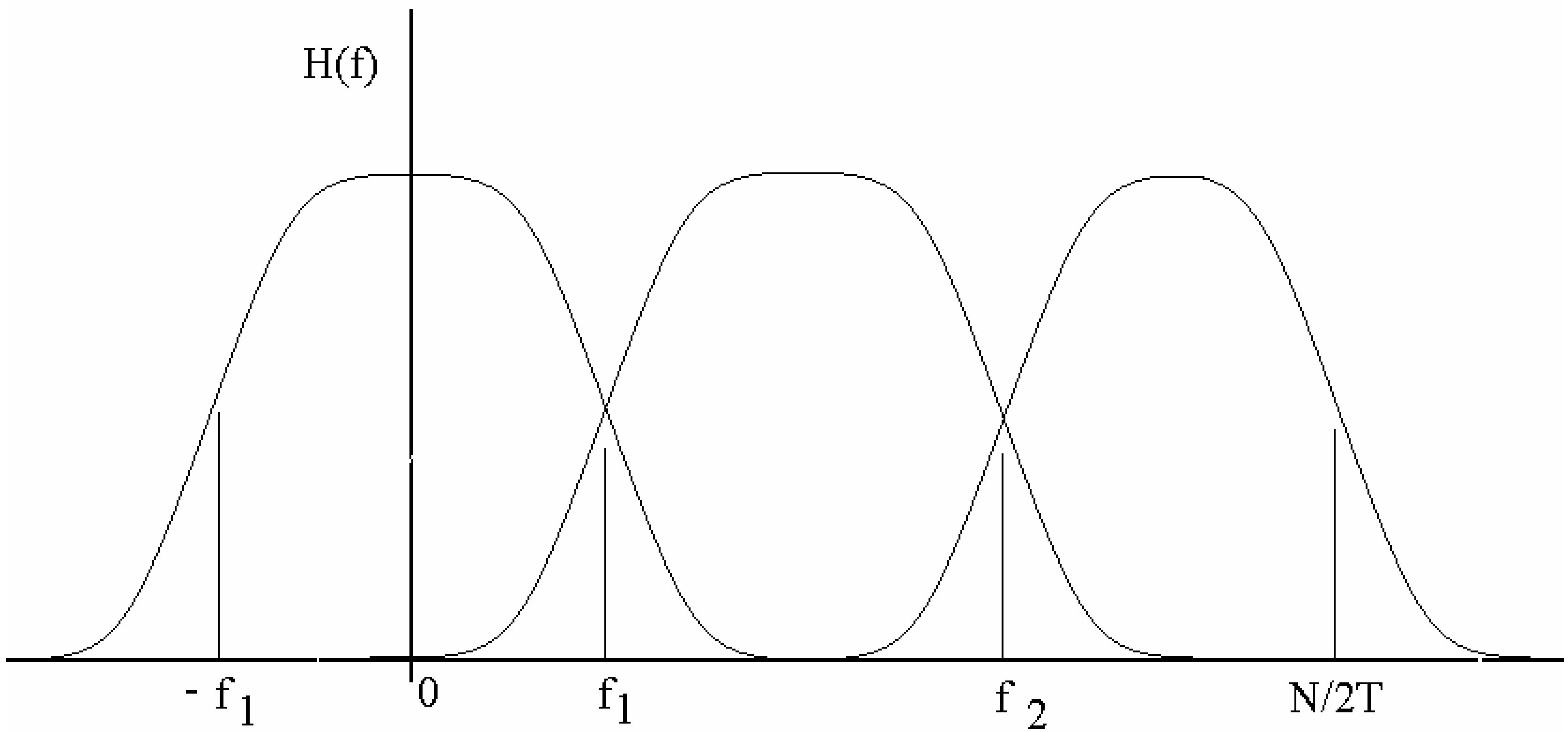
$h(t,f_0,f_1,w)$



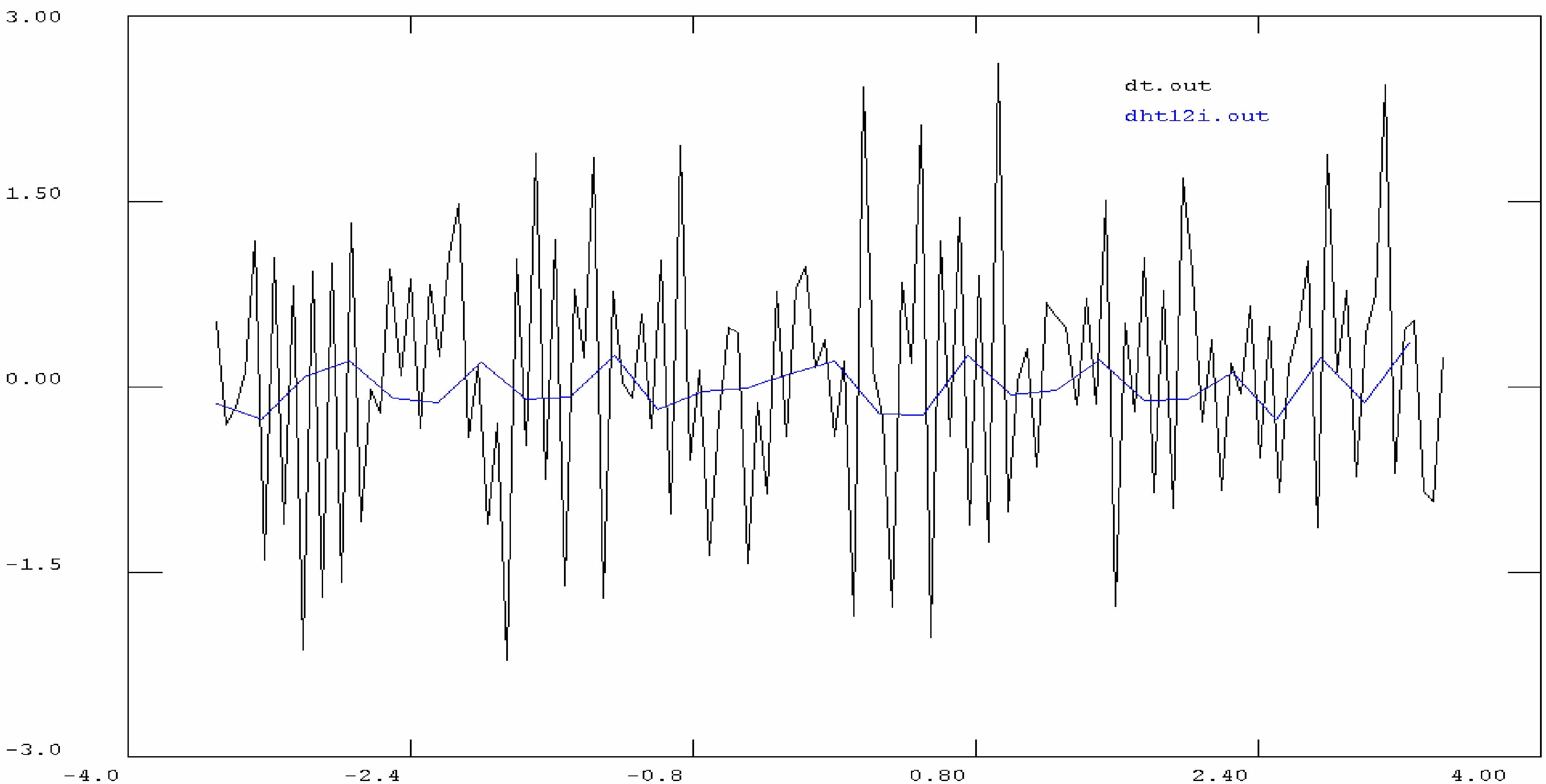
The oscillations can be used to shift the frequency. Note that h can be calculated once, then shifted and re-used over and over, the sine and cosines need not be recalculated. Convolution reduces to a single set of multiplications and sums for each output data point.

Small region of time showing the oscillations in real and imaginary h(t)

Splitting the Space

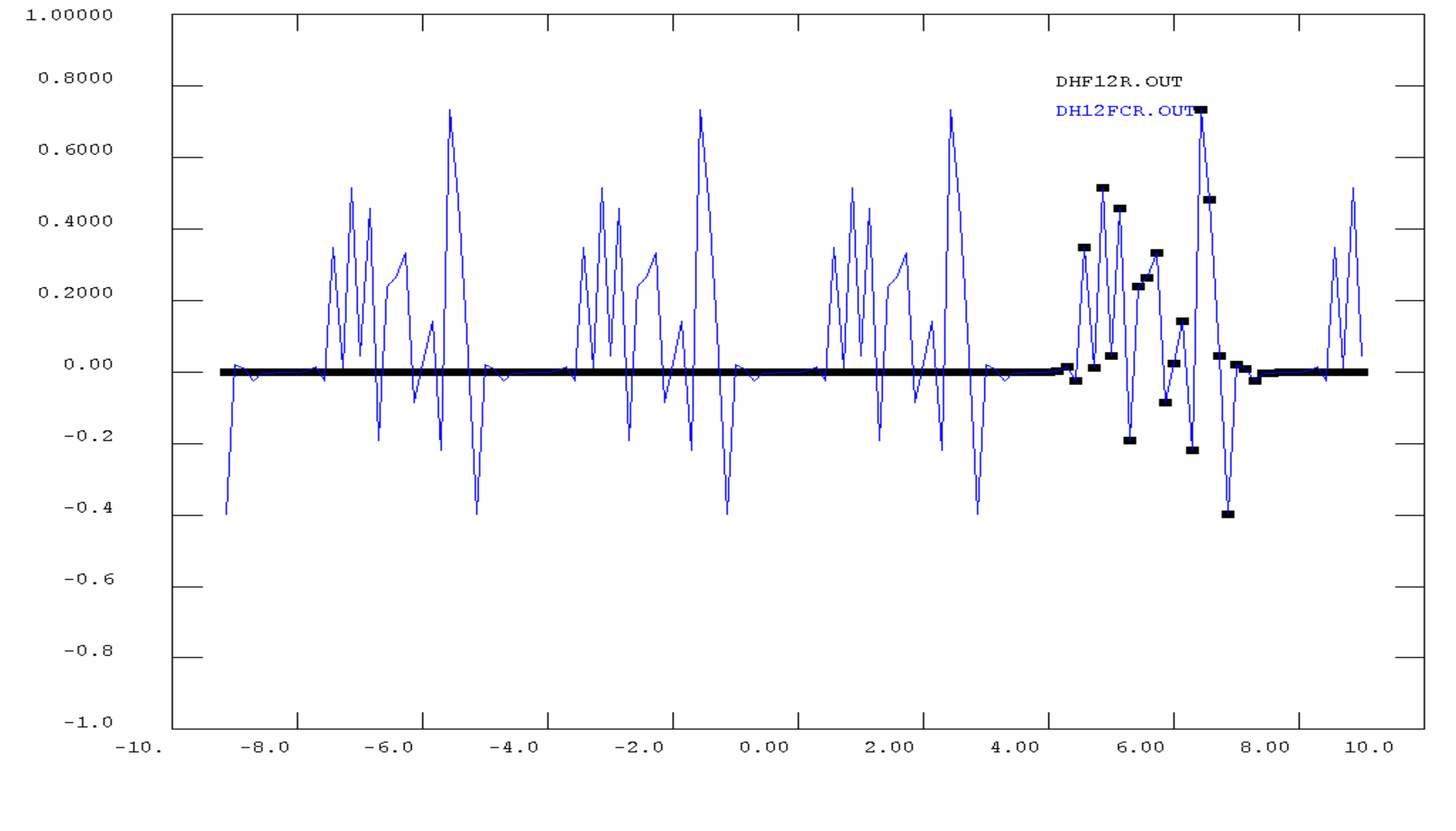


Second region



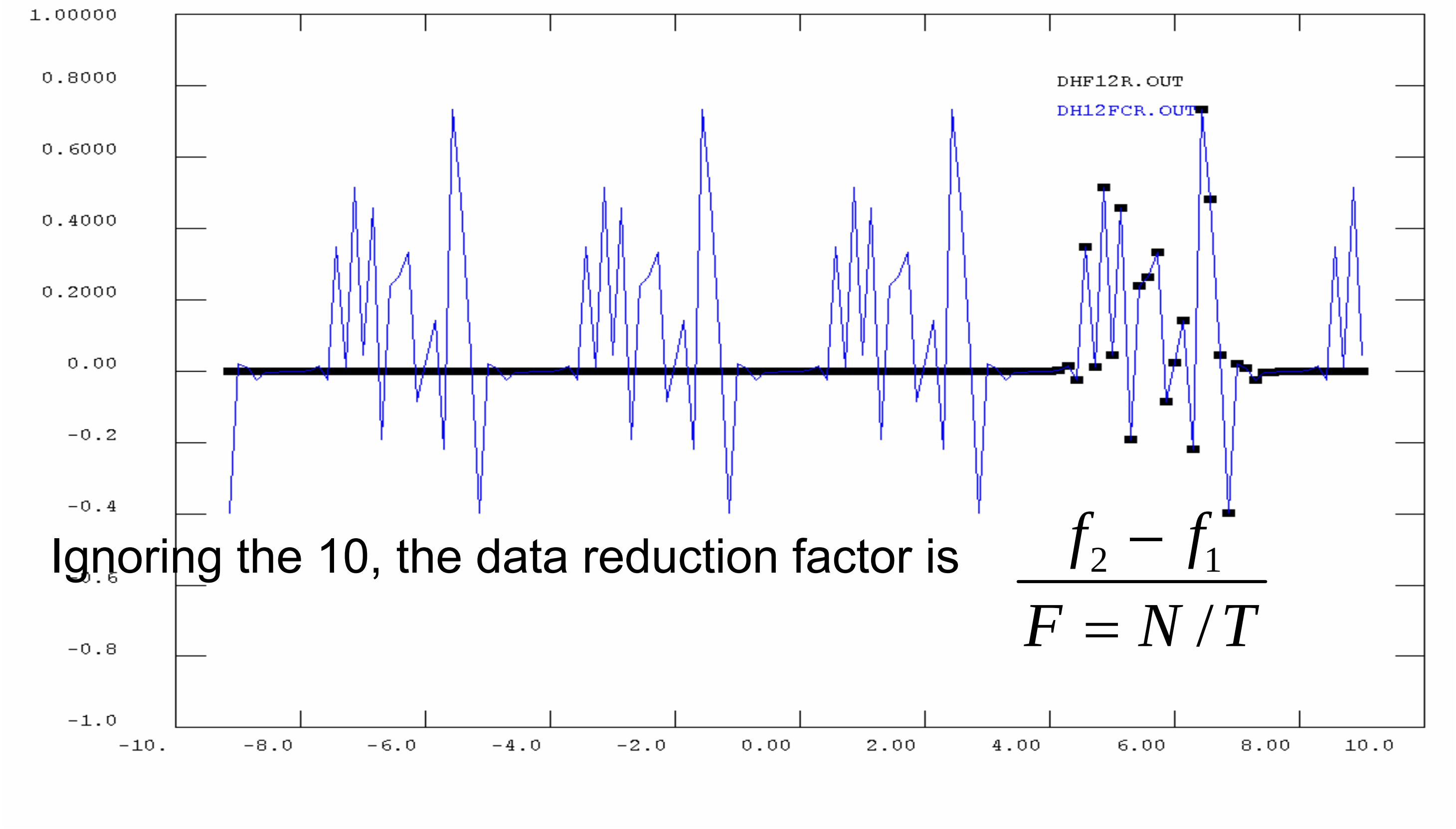
The convolution with $h(t,f_1,f_2,w)$ produces complex data. The imaginary part is shown above.

Second region in frequency



Real part of transform of convoluted data between 32/Time and 50/time using 50-32+10 data points

Second region in frequency



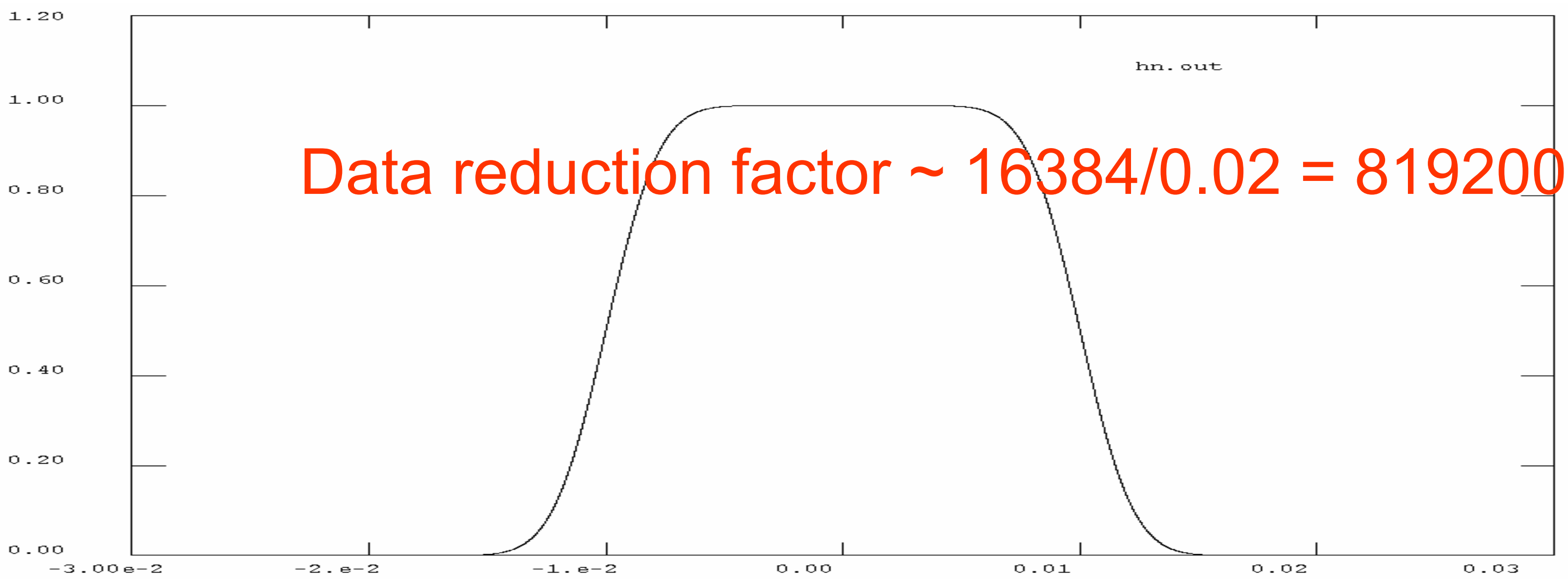
Ignoring the 10, the data reduction factor is $\dfrac{f_2 - f_1}{F = N / T}$

Real part of transform of convoluted data between 32/Time and 50/time using 50-32+10 data points

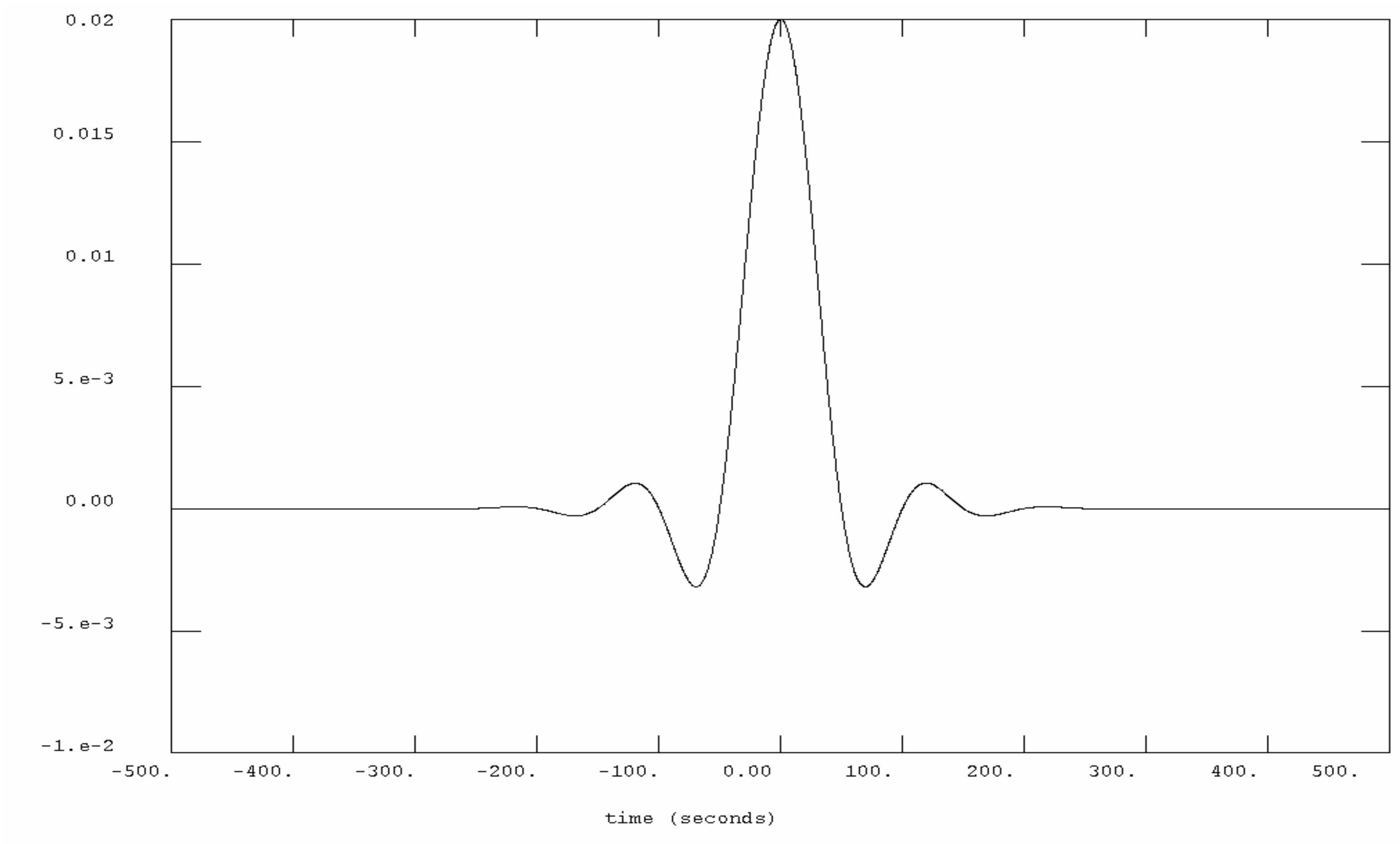
Pulsar numbers

The size of F was found by Cornish and Larson to be ~ 0.01 Hz[i], Thus there needs to be an output point every 10 seconds to follow the Doppler motion of B0531+21 which has a quadrupole frequency of 59.62±0.01 Hz.

[i] Neil J. Cornish and Shane L. Larson, "LISA data analysis: Doppler demodulation", Class. Quantum Grav. 20 (2003) S163-S170 – online at stacks.iop.org/CQG/20/S163



|h(t)| for 0.02 width signal



The time range on this plot is from –500 seconds to + 500 seconds.

Omissions

The phase will need to be monitored, if it drifts the signal will cancel to zero. – possibly the violin modes will help.

If the convolution went straight from the input data, noise in the region would in principle rise as $T^{1/2}$ while the signal would rise as T.

The noise is systematic and has many properties that identify it, an intermediate step in which known sources of frequencies that overlap the pulsar frequency are examined and removed will be investigated.