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Solution of the Inverse Problem for Gravitational Wave Bursts

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Y. Gursel & M. Tinto, *Phys. Rev. D*, **40**, 3884 (1989)

Introduction

- Coincidence experiments with three (or more!) wide-band detectors of gravitational waves (GW) are going to be performed soon.
- They will enhance the likelihood of detection
  - For a given false-alarm probability, the detection threshold can be lowered => more volume of space can be searched for => more sources can potentially been seen!
- They will provide a self-consistency check for a positive observation => they will allow us to

Make Astronomical Observations!

Statement of the Problem

- Making astronomical observations by using the detectors data means:
  - being able to estimate the direction to the source (θ, ϕ), and
  - reconstruct the wave's two independent amplitudes ($h_+ (t)$, $h_x (t)$).
- The determination of these four unknowns provides the solution of the so called *Inverse Problem in Gravitational Wave Astronomy*.
- In what follows we will investigate the Inverse Problem for GW Bursts, i.e. signals that last only for a few milliseconds and do not have "a well defined" waveform.

Detector Response

$$h_{XX} = -h_{YY} = h_+(t), \quad h_{XY} = h_{YX} = h_\times(t)$$

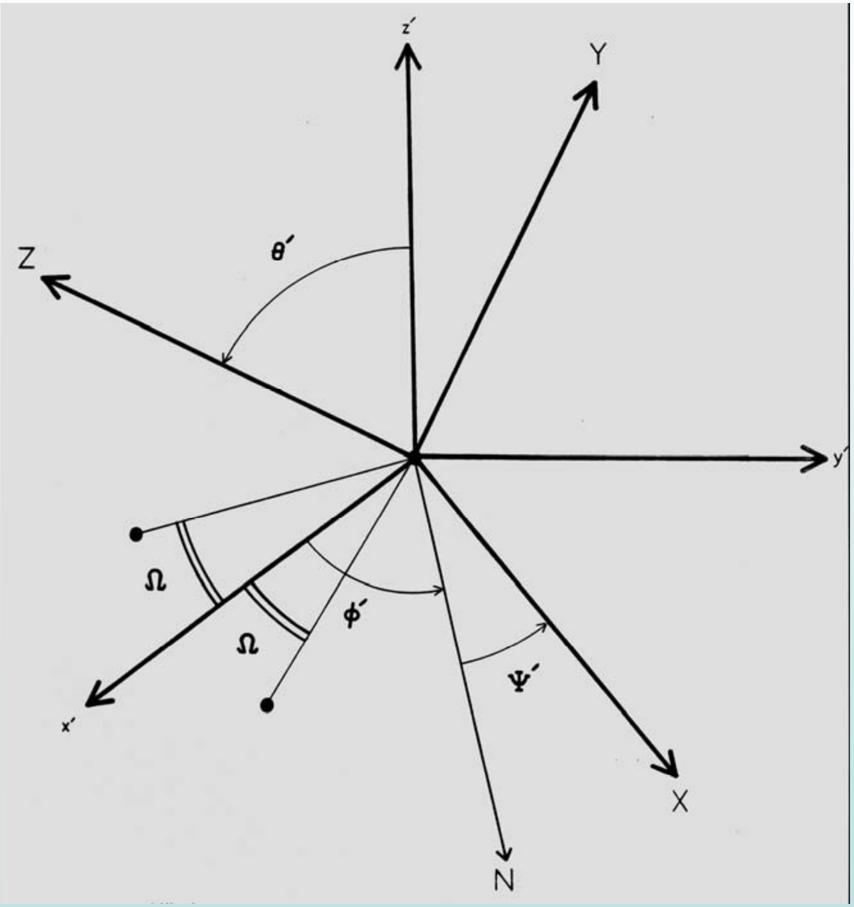
$$\mathbf{m} = \frac{1}{\sqrt{2}}(\mathbf{e}_X + i\,\mathbf{e}_Y)$$

$$\mathbf{m} = \frac{1}{\sqrt{2}}[(\cos\phi' - i\cos\theta'\sin\phi')\mathbf{e}_{x'} + (\sin\phi' + i\cos\theta'\cos\phi')\mathbf{e}_{y'} + (i\sin\theta')\mathbf{e}_{z'}]$$

$$W_{ij}(t) = h_+(t)\,\mathrm{Re}(m_i m_j) + h_\times(t)\,\mathrm{Im}(m_i m_j)$$

$$D_{ij}^{\mathrm{int}} = l_{1i}l_{1j} - l_{2i}l_{2j}$$

$$R(t) = D_{ij}W^{ij}(t)$$



Note: Ψ' can be taken to be equal to zero!

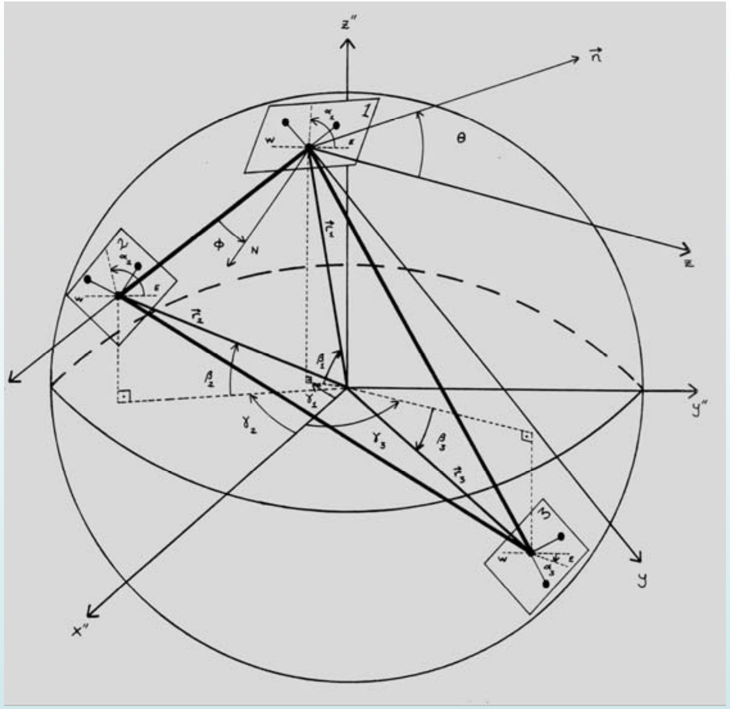
$$[h_+(t) + ih_\times(t)]_{\mathrm{new}} = [h_+(t) + ih_\times(t)]_{\mathrm{old}}\, e^{2i\psi'}$$

S.V. Dhurandhar & M. Tinto, *MNRAS*, **234**, 663 (1988)

Network Coordinates

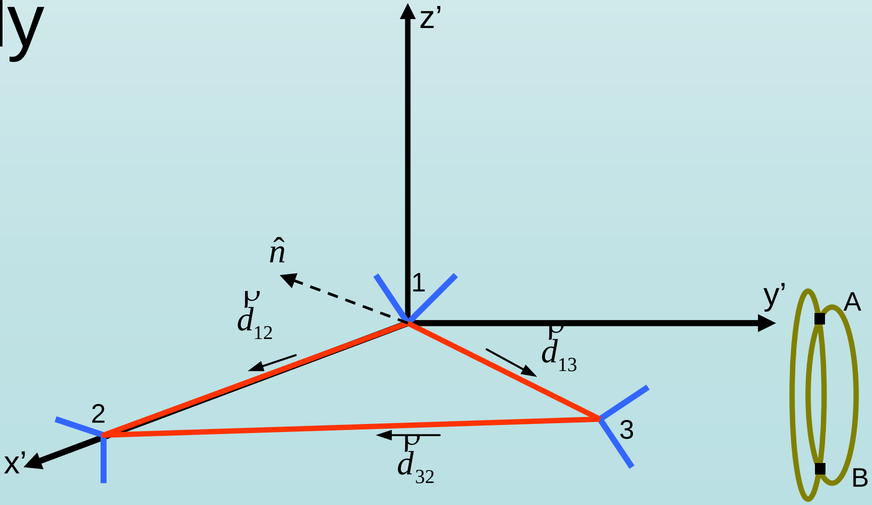
$$R(t)=F_+(\theta,\phi,\alpha,\beta,\gamma)h_+(t)+F_\times(\theta,\phi,\alpha,\beta,\gamma)h_\times(t)$$

- A network of 3 wide-band detectors gives 3 functions of time $R_k(t)$ (k=1, 2, 3), and two independent time delays.
- They provide enough information for uniquely identify the source location.



$$\tau_{12}=\frac{\hat{n}\cdot\vec{d}_{12}}{c}\quad;\quad\tau_{13}=\frac{\hat{n}\cdot\vec{d}_{13}}{c}\quad;\quad\tau_{32}=\frac{\hat{n}\cdot\vec{d}_{32}}{c}$$

$$\tau_{13}+\tau_{32}=\tau_{12}$$



Time-Delays and Antenna Patterns

1. How do we compute the time delays?
2. How can we take advantage of the asymmetry of the detectors’ antenna patterns w.r.t. the symmetry plane in order to uniquely identify the location of the source in the sky?

One could compute the cross-correlations between pairs of detectors:

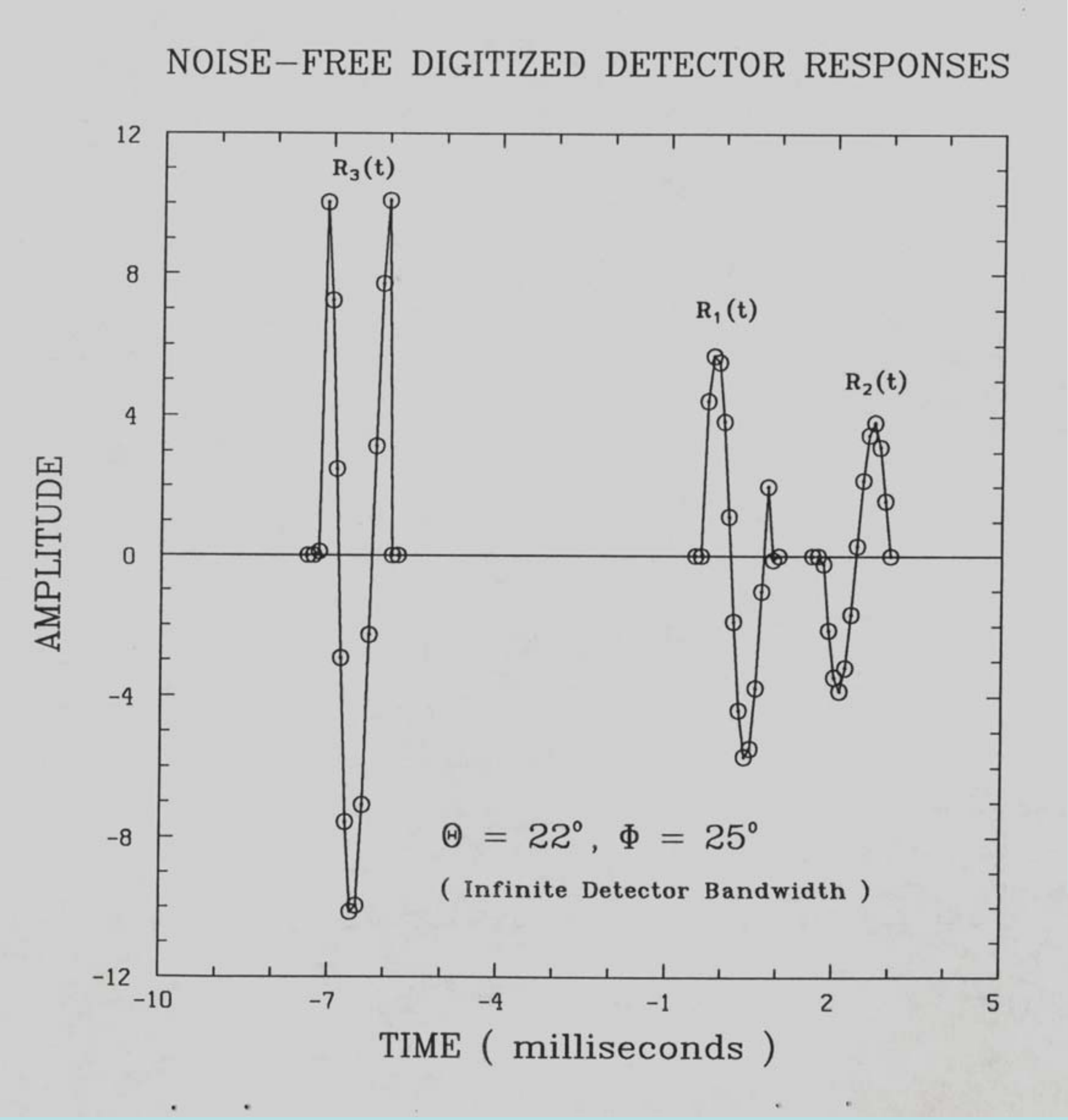
$$C(\tau) = \int_0^T R_1(t) R_2(t+\tau)\, dt$$

<u>However:</u> Because of the Earth curvature, the detectors will see two different linear combinations of $h_+(t)$ & $h_x(t)$

Three-Detector Responses

$$h_+(t) = A\sqrt{2}\sin(2\pi ft)$$

$$h_\times(t) = A\sqrt{2}\cos(2\pi ft)$$



Time-Delays and Antenna…(cont.)

<u>Example</u>:

$$R_1(t_1) = A_1(h_o,\theta,\phi)\cos(2\pi f t_1 + \Phi_1(\theta,\phi) - \hat{n}\cdot\vec{r}_1)$$
$$R_2(t_2) = A_2(h_o,\theta,\phi)\cos(2\pi f t_2 + \Phi_2(\theta,\phi) - \hat{n}\cdot\vec{r}_2)$$

$$\Longrightarrow \quad t_1 - t_2 = \tau_{12} + \frac{\Phi_2 - \Phi_1}{2\pi f}$$

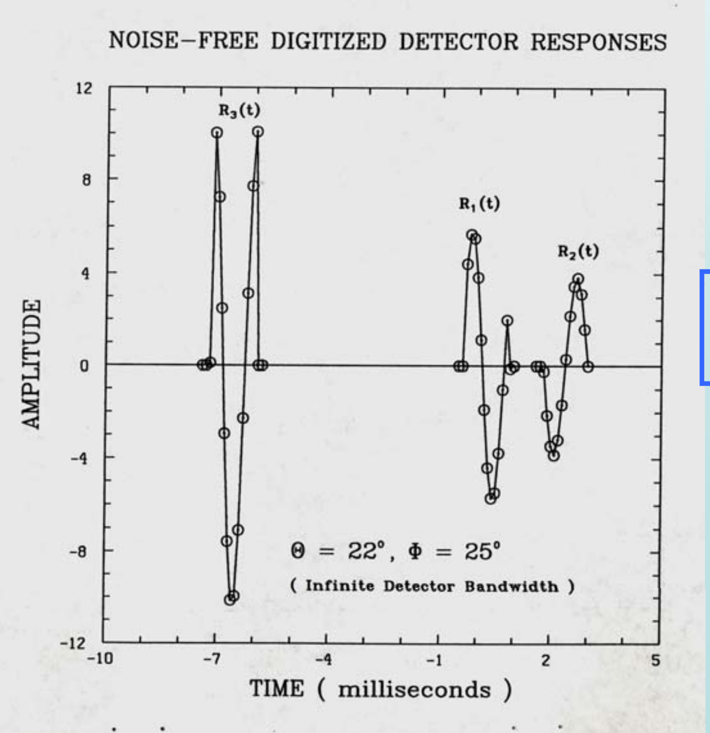
Wave travel time from detector 1 to 2

Extra delay due to Earth-curvature

- The extra-delay can be significant!
- For the detectors' locations considered in '89, the extra delay can be ~ 15% of the exact time delay => significant inaccuracy in source location.

Determination of the Time-Delays and Solution of the Inverse Problem

- Let us assume the clocks at each site to be perfectly synchronized!
- Let us also assume for the moment:
  - Noiseless detectors.
  - To know the location of the source in the sky.



$$R_1(t) = F_{1+}(\theta_s,\phi_s)h_+(t) + F_{1\times}(\theta_s,\phi_s)h_\times(t)$$

$$R_2(t+\tau^s{}_{12}) = F_{2+}(\theta_s,\phi_s)h_+(t) + F_{2\times}(\theta_s,\phi_s)h_\times(t)$$

$$R_3(t+\tau^s{}_{13}) = F_{3+}(\theta_s,\phi_s)h_+(t) + F_{3\times}(\theta_s,\phi_s)h_\times(t)$$

$$I(t,\theta_s,\phi_s) \equiv K_1(\theta_s,\phi_s)R_1(t) + K_2(\theta_s,\phi_s)R_2(t+\tau^s{}_{12}) + K_3(\theta_s,\phi_s)R_3(t+\tau^s{}_{13}) = 0$$

$$K_1(\theta,\phi) = F_{2+}F_{3\times} - F_{2\times}F_{3+}$$

$$K_2(\theta,\phi) = F_{3+}F_{1\times} - F_{3\times}F_{1+}$$

$$K_3(\theta,\phi) = F_{1+}F_{2\times} - F_{1\times}F_{2+}$$

Determination of the…(cont.)

The following two-parameter function

$$I(t,\theta,\phi) \equiv K_1(\theta,\phi)R_1(t) + K_2(\theta,\phi)R_2(t+\tau_{12}) + K_3(\theta,\phi)R_3(t+\tau_{13})$$

becomes identically null as $(\theta, \phi) \rightarrow (\theta_s, \phi_s)$

$$\Rightarrow \quad L(\theta,\phi) = \int_{-\infty}^{+\infty} I^2(t,\theta,\phi)\,dt$$

The function $L(\theta, \phi)$ also becomes identically null as $(\theta, \phi) \rightarrow (\theta_s, \phi_s)$!

Least-Squares Method with Noisy Detector Responses

$$R_{1\Lambda}(t) = R_1(t) + \Lambda_1(t)$$

$$R_{2\Lambda}(t+\tau_{12}) = R_2(t+\tau_{12}) + \Lambda_2(t+\tau_{12})$$

$$R_{3\Lambda}(t+\tau_{13}) = R_3(t+\tau_{13}) + \Lambda_3(t+\tau_{13})$$

Λ_i (t) = Random processes representing the noise in each detector.

$$I_\Lambda(t,\theta,\phi) \equiv K_1(\theta,\phi)R_{1\Lambda}(t) + K_2(\theta,\phi)R_{2\Lambda}(t+\tau_{12}) + K_3(\theta,\phi)R_{3\Lambda}(t+\tau_{13})$$

$$L_\Lambda(\theta,\phi) = \frac{1}{t_1 - t_2}\int_{t_0}^{t_1} I_\Lambda{}^2(t,\theta,\phi)\,dt$$

$$L_\Lambda(\theta,\phi) = L(\theta,\phi) + \frac{1}{t_1 - t_2}\int_{t_0}^{t_1}[K_1\Lambda_1(t) + K_2\Lambda_2(t+\tau_{12}) + K_3\Lambda_3(t+\tau_{13})]^2\,dt$$

$$+\frac{2}{t_1 - t_2}\int_{t_0}^{t_1} I(t,\theta,\phi)[K_1\Lambda_1(t) + K_2\Lambda_2(t+\tau_{12}) + K_3\Lambda_3(t+\tau_{13})]\,dt$$

Least-Squares Method with Noisy Detector Responses (cont.)

- If the detectors' noises are Gaussian distributed, the minimization procedure can be optimized by normalizing the function I_Λ(t, θ, φ) in the following way

$$I_\Lambda(t,\theta,\phi) \equiv \frac{K_1(\theta,\phi)R_{1\Lambda}(t) + K_2(\theta,\phi)R_{2\Lambda}(t+\tau_{12}) + K_3(\theta,\phi)R_{3\Lambda}(t+\tau_{13})}{[K^2{}_1(\theta,\phi)\sigma_1{}^2 + K^2{}_2(\theta,\phi)\sigma_2{}^2 + K^2{}_3(\theta,\phi)\sigma_3{}^2]^{1/2}}$$

$$L_\Lambda(\theta,\phi) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} \frac{[K_1(\theta,\phi)R_{1\Lambda}(t) + K_2(\theta,\phi)R_{2\Lambda}(t+\tau_{12}) + K_3(\theta,\phi)R_{3\Lambda}(t+\tau_{13})]^2}{K^2{}_1(\theta,\phi)\sigma_1{}^2 + K^2{}_2(\theta,\phi)\sigma_2{}^2 + K^2{}_3(\theta,\phi)\sigma_3{}^2}\, dt$$

How accurate is the determination of the source location with this method?

- For good SNR we expect the solution found by the minimizer to be close to the actual source location.
- We approximate $L_\Lambda(\theta, \phi)$ with a quadratic form in a nhbh of the minimum:

$$L_\Lambda(\theta,\phi) \cong L_\Lambda(\theta_m,\phi_m) + \cancel{(\vec{\nabla} L_\Lambda)}|_m \cdot(\theta-\theta_m,\phi-\phi_m)^T + \frac{1}{2}(\theta-\theta_m,\phi-\phi_m)\cdot \overleftrightarrow{H} \cdot(\theta-\theta_m,\phi-\phi_m)^T$$

- Since at the source location the least-squares function is normalized to unity, from the equation above we get:

$$1 - L_\Lambda(\theta_m,\phi_m) \cong \frac{1}{2}(\theta_s-\theta_m,\phi_s-\phi_m)\cdot \overleftrightarrow{H} \cdot(\theta_s-\theta_m,\phi_s-\phi_m)^T$$

How accurate is…(cont.)

- One can solve for $(\Delta\theta, \Delta\phi)$ by diagonalizing the Hessian matrix.
- By making an orthogonal transformation to diagonal coordinates (x_1, x_2) we get:

$$1 - L_\Lambda(\theta_m, \phi_m) \cong \frac{(x_{1s} - x_{1m})^2}{(\sqrt{1/\lambda_1})^2} + \frac{(x_{2s} - x_{2m})^2}{(\sqrt{1/\lambda_2})^2}$$

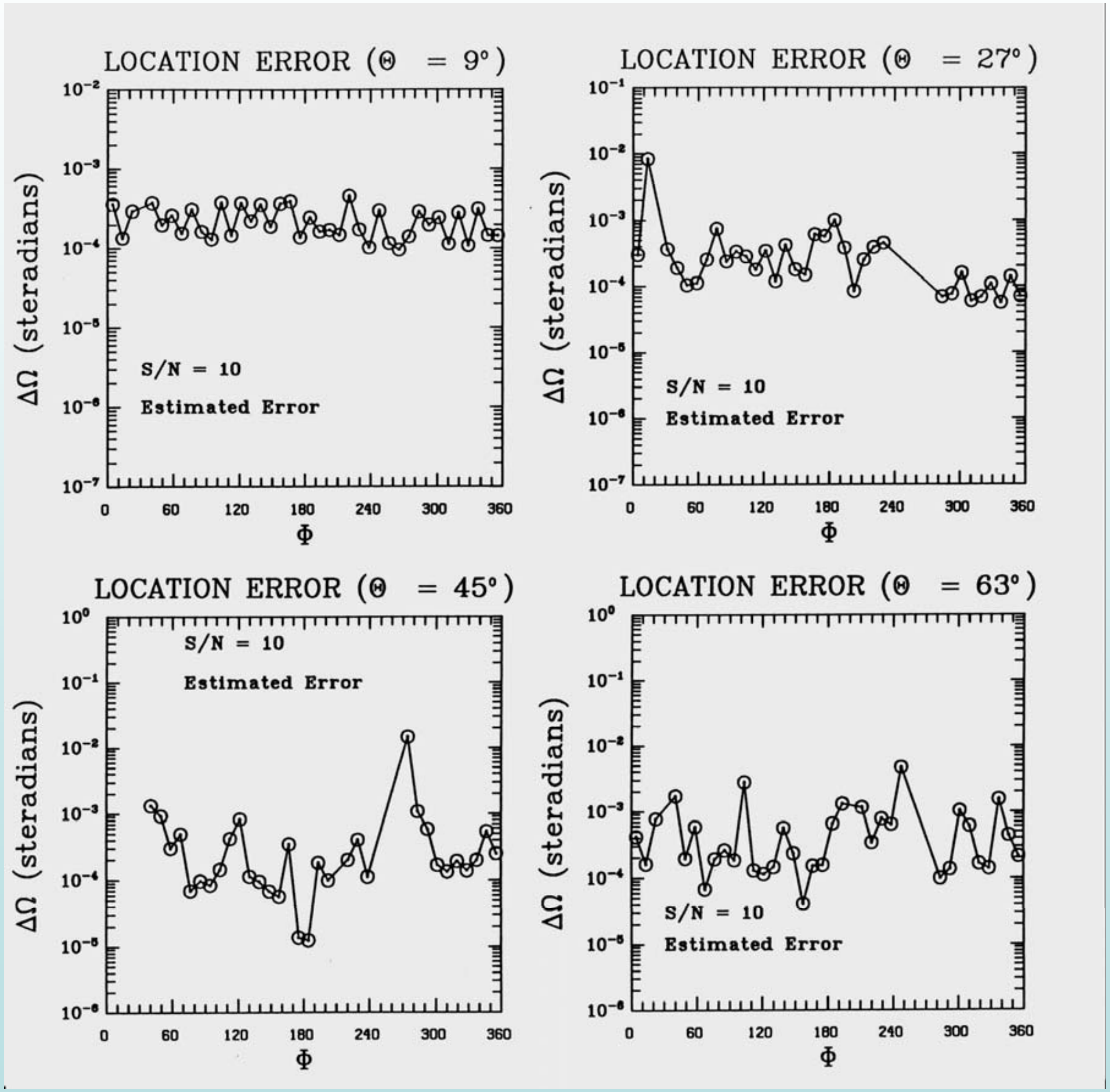
$$\Delta\theta = a\sqrt{1/\lambda_1} + b\sqrt{1/\lambda_2}$$
$$\Delta\phi = c\sqrt{1/\lambda_1} + d\sqrt{1/\lambda_2}$$

$$\Delta\Omega = \pi \frac{1 - L_\Lambda(\theta_m, \phi_m)}{(\lambda_1 \lambda_2)^{1/2}}$$

Error box in locating the source in the sky

Accuracy of the Estimated Source Location

$$\Delta\Omega = \frac{2c^2}{\pi^2 (\hat{n} \cdot A)\, f_0{}^2 (SNR)^2}$$



Wave Amplitudes Reconstruction

- Once the source location has been identified, one can reconstruct in 3 distinct ways the two wave amplitudes:

$$h_{+1}(t) = \frac{F_{3\times}R_{2\Lambda}(t+\tau_{12}) - F_{2\times}R_{3\Lambda}(t+\tau_{13})}{F_{2+}F_{3\times} - F_{2\times}F_{3+}}$$

$$h_{+2}(t) = \frac{F_{1\times}R_{3\Lambda}(t+\tau_{13}) - F_{3\times}R_{1\Lambda}(t)}{F_{3+}F_{1\times} - F_{3\times}F_{1+}}$$

$$h_{+3}(t) = \frac{F_{2\times}R_{1\Lambda}(t) - F_{1\times}R_{2\Lambda}(t+\tau_{12})}{F_{1+}F_{2\times} - F_{1\times}F_{2+}}$$

- There exists an Optimal Linear Combination of the above three expressions for the reconstructed wave amplitudes.
- It is optimal in the sense that it minimizes the root-mean-squared noise in the reconstructed waveform.

Wave Amplitudes Reconstruction (cont.)

$$h_{+opt.}(t) = a_{+1}h_{+1}(t) + a_{+2}h_{+2}(t) + a_{+3}h_{+3}(t)$$

$$a_{+1} + a_{+2} + a_{+3} = 1$$

- This constraint follows from the condition

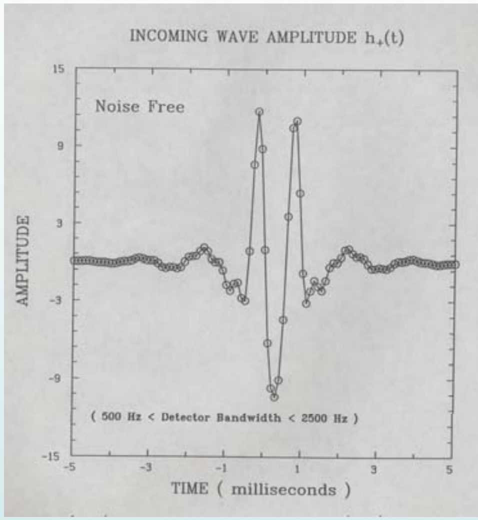
$$h_{+opt.}(t) \Rightarrow h_{+}(t) \quad when \quad \Lambda_{1,2,3}(t) \Rightarrow 0$$

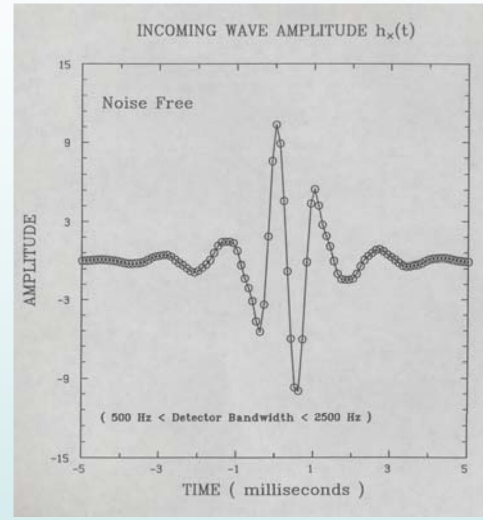
- The root-mean-squared error in the reconstructed waveform is equal to

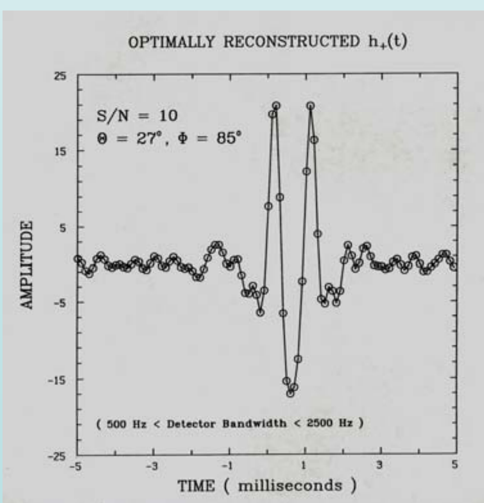
$$\delta h_{+} = \left\{ \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} [h_{+opt.}(t) - h_{+}(t)]^2 \, dt \right\}^{1/2}$$

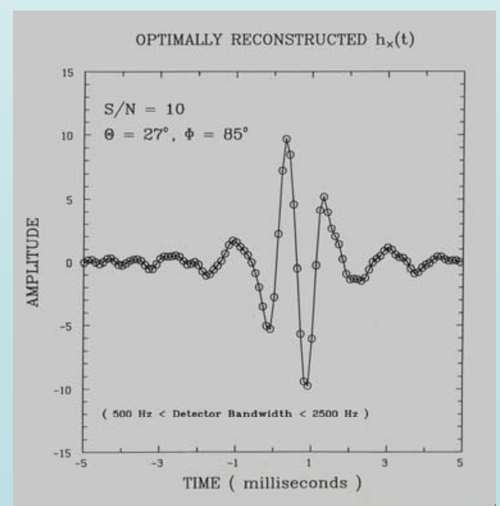
- We minimize δh_+ with respect to a_{+i} subject to the constraint $a_{+1} + a_{+2} + a_{+3} = 1$.
- The solution exists and is unique!

Optimally Reconstructed Wave Amplitudes









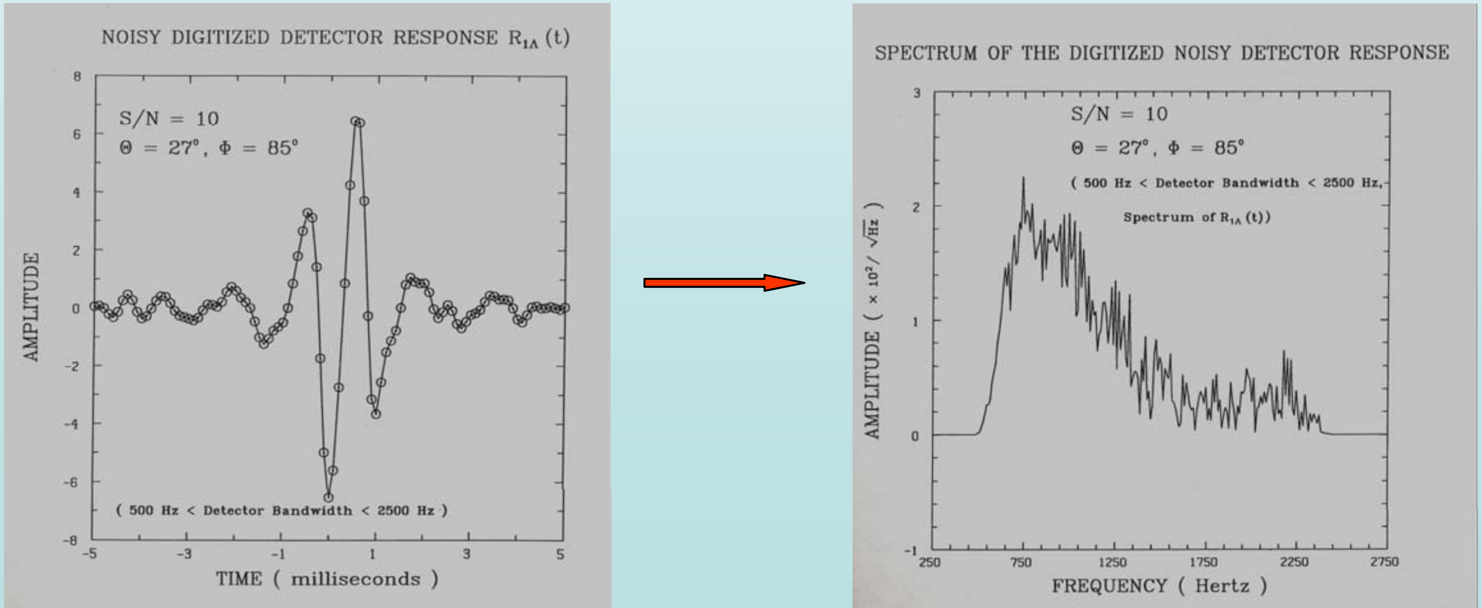
Can the accuracy of the method be improved?

- If we could further reduce the contribution of the noise to the Least-Squares function, the accuracy in the location of the source would improve!
- Optimal Filtering could be applied if *a priori* knowledge of the detector responses would be available (not our case!)
- There exists, however, methods that enable us to construct a near-optimal filter for the detector responses from the data themselves!
- These methods take advantage of the fact that the signal has a spectrum distinguishable from that of the noise.
- A fairly crude determination of the optimal filter can still perform well!!

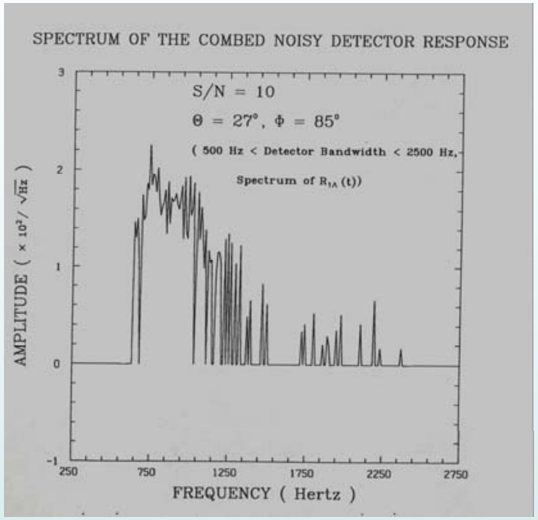
Numerical Recipes, CUP, 1986, p.417

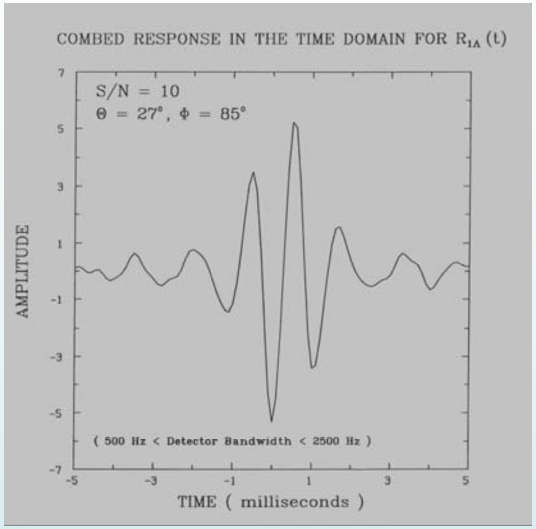
Can the accuracy of…(cont.)

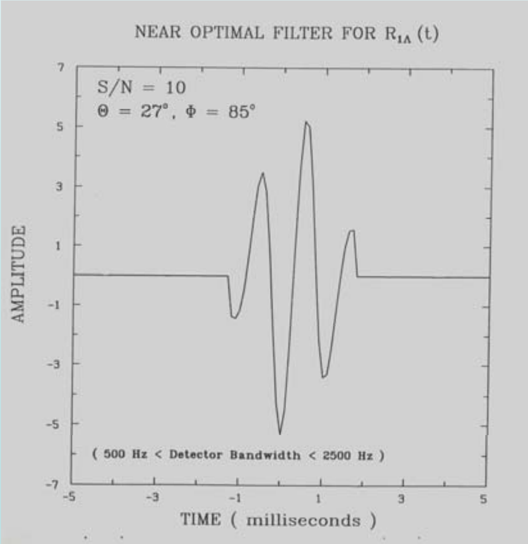
- Note that, in order to incorporate a filtering procedure within the Least-Squares method, one can not simply apply filtering to each of the responses: THE TIME DELAY INFORMATION WOULD BE ALTERED!
- Since the source location is determined by minimizing the integral of $I^2_\Lambda(t,\theta,\phi)$, filtering has to be applied to $I_\Lambda(t,\theta,\phi)$.

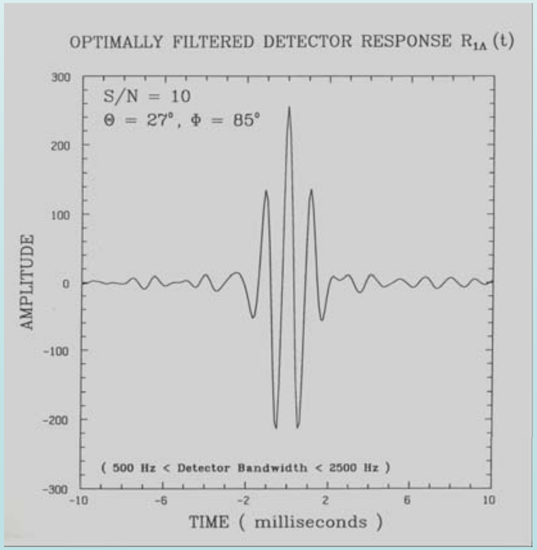


Near-Optimal Filter

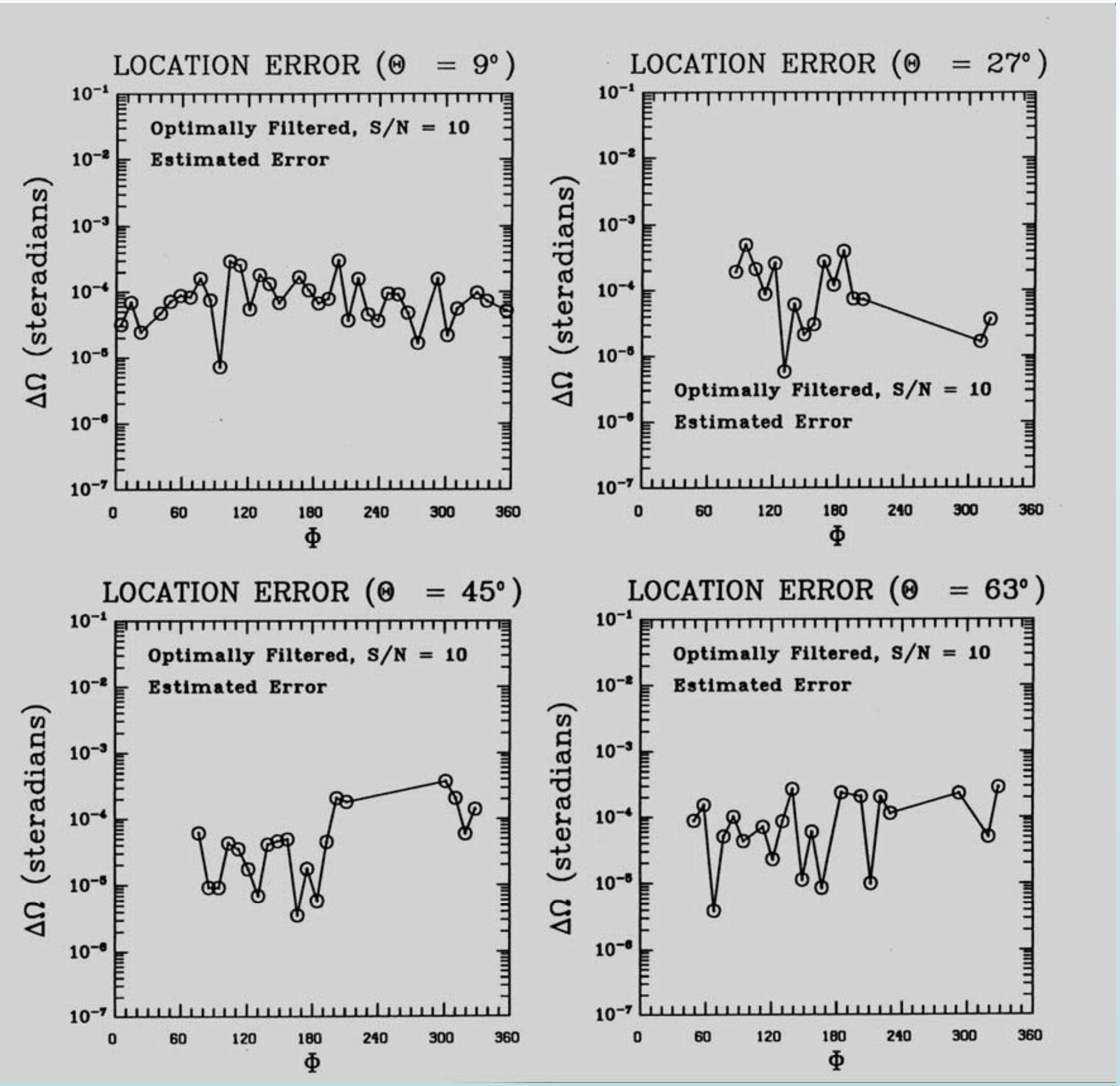








Accuracy of the Estimated Source Location (with Near-Optimal Filter)



Summary of Results and Conclusions

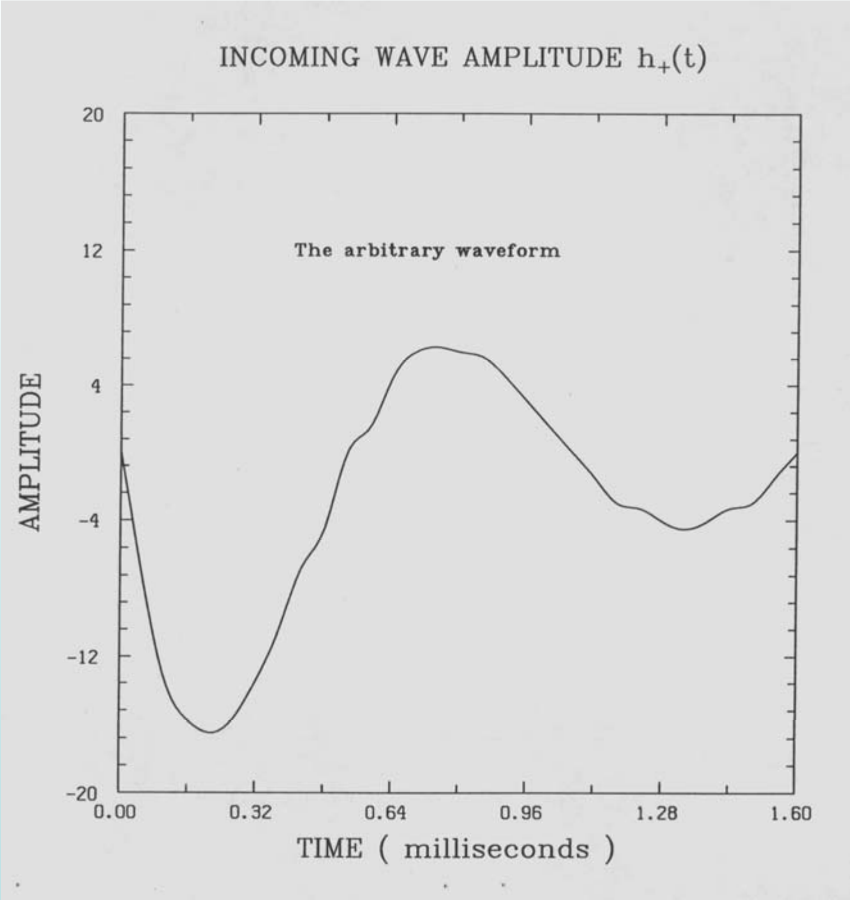
- Numerical simulations of the method showed that:
  - For broadband bursts of dominant frequency equal to ~1 kHz, with a SNR~10, the source could be located within a solid angle of ~ 10^{-5} sr.
  - For SNRs significantly lower than 10 the method could not distinguish between the two points in the sky.
  - For SNRs equal to 1 or less the method looses its resolution completely.

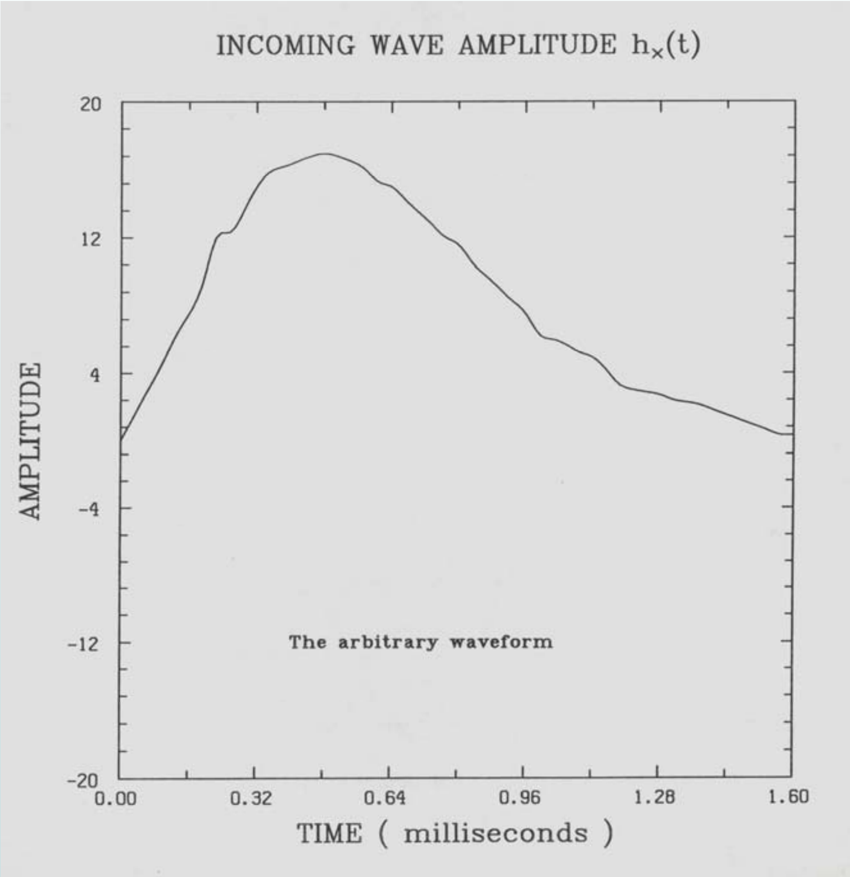
- This method could be applied to data from triple coincidence experiments that are planned to take place in the near future.
- It can be used as “veto” to coincident events, i.e. it can actually be used for testing the detection hypothesis!

Summary of…(cont.)

- Networks with 4 detectors running in coincidence have also been analyzed within the “GT” method.
- Better sky coverage and angular resolution.
- Results published in a conference proceedings (1st VIRGO Meeting, 1996).

Arbitrary Waveforms





$(F_{i+}F_{jx} - F_{ix}F_{j+})$ Functions