LIGO-G030529-00-D

7 October, 2003

Angular Correlation of LIGO Hanford and Livingston Interferometers

Malik Rakhmanov

Department of Physics, University of Florida, Gainesville, FL 32611

LIGO Science Seminar
Caltech, Pasadena

Gravitational Wave Polarizations

Gravitational waves are perturbations of metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

$\eta_{\mu\nu}$ – Minkowski metric,
$h_{\mu\nu}$ – GW strain tensor.

In the transverse traceless coordinates:

$$h_{\mu\nu} = \begin{pmatrix} 0 & & & \\ & h_+ & h_\times & \\ & h_\times & -h_+ & \\ & & & 0 \end{pmatrix}. \tag{2}$$

3-dimensional representation:

$$\mathbf{h} = h_+ \mathbf{m} + h_\times \mathbf{n}, \tag{3}$$

where $\mathbf{m}$ is a traceless and $\mathbf{n}$ is a transverse unit tensor

$$\mathbf{m} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{n} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Gravitational Wave Signal

A laser interferometer defines its own coordinate system:

$$\mathbf{x}' = \mathbf{R}^T \mathbf{x}. \tag{5}$$

The g.w. strain in this coordinates is

$$\mathbf{h}' = \mathbf{R}^T \mathbf{h}\, \mathbf{R}. \tag{6}$$

The signal is proportional to

$$V \equiv \frac{1}{2}\, \mathrm{Tr} \left\{ \mathbf{m}\, \mathbf{h}' \right\} = F_+ \, h_+ + F_\times \, h_\times . \tag{7}$$

Directional sensitivity (antenna patterns) is defined by

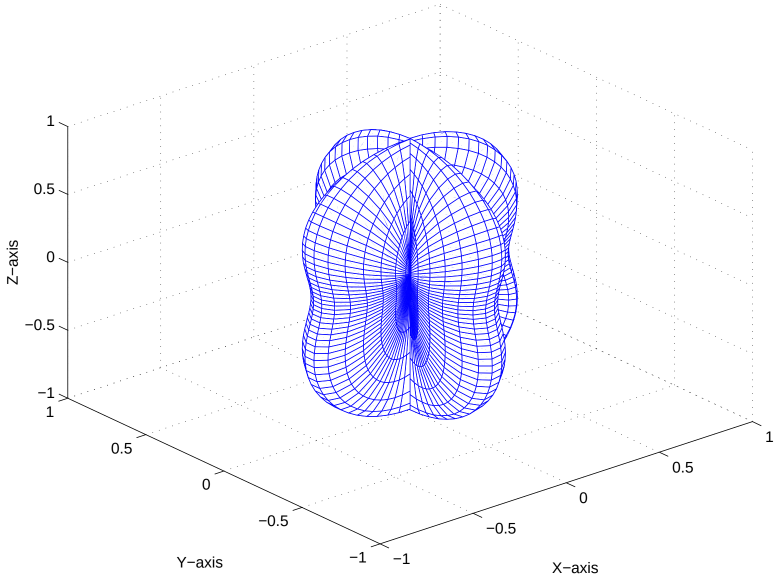
$$\begin{aligned} F_+ &= \frac{1}{2}\, \mathrm{Tr} \left\{ \mathbf{m}\, \mathbf{R}^T \mathbf{m}\, \mathbf{R} \right\}, & (8) \\ F_\times &= \frac{1}{2}\, \mathrm{Tr} \left\{ \mathbf{m}\, \mathbf{R}^T \mathbf{n}\, \mathbf{R} \right\}. & (9) \end{aligned}$$

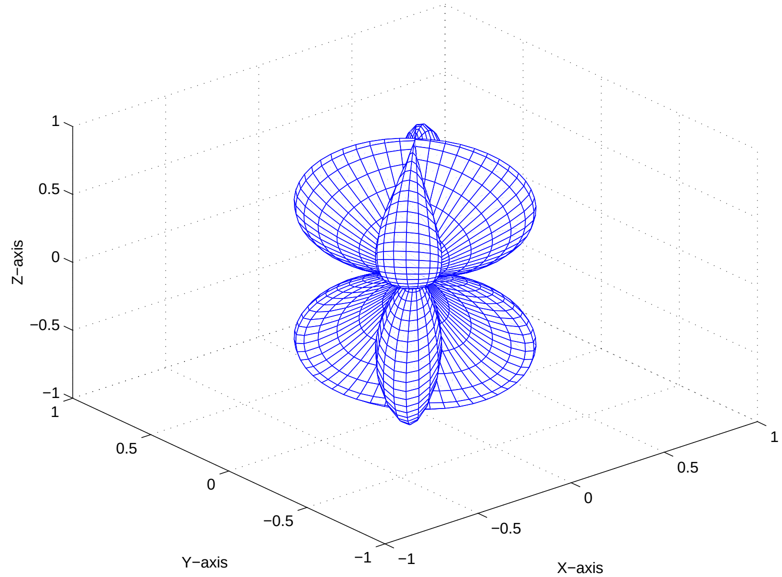
Standard Antenna Patterns

The standard antenna patterns can be obtained by using Euler angles:

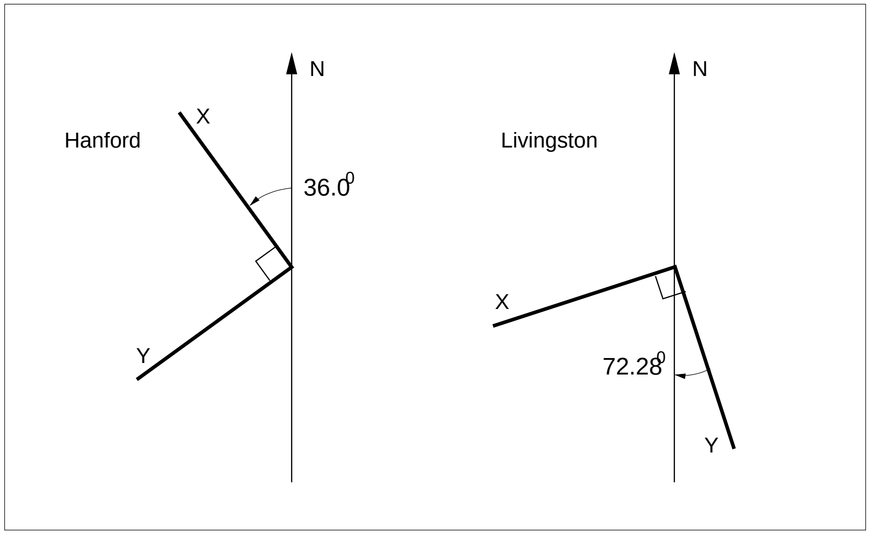
$$\mathbf{R} = \mathbf{R}_z(\psi)\, \mathbf{R}_y(\theta)\, \mathbf{R}_z(\phi). \tag{10}$$

For $\psi = 0$, F_+ and $F_\times$ are





LIGO Hanford and Livingston Detectors



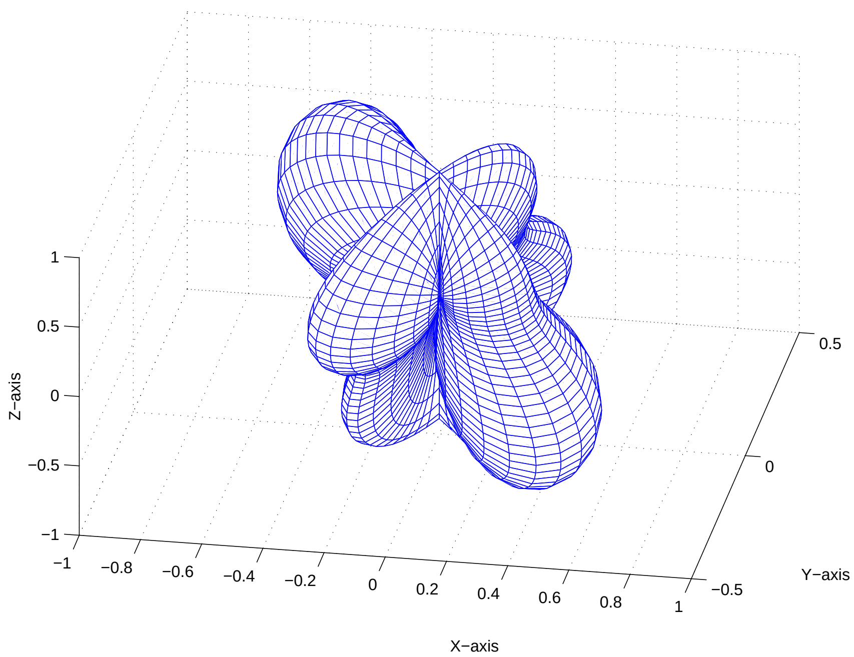
coordinate	Hanford	Livingston
latitude	$46^0\ 27'\ 18.53''$ N	$30^0\ 33'\ 46.42''$ N
longitude	$119^0\ 24'\ 27.57''$ W	$90^0\ 46'\ 27.27''$ W
azimuth	35.9993^0 N	72.2836^0 S

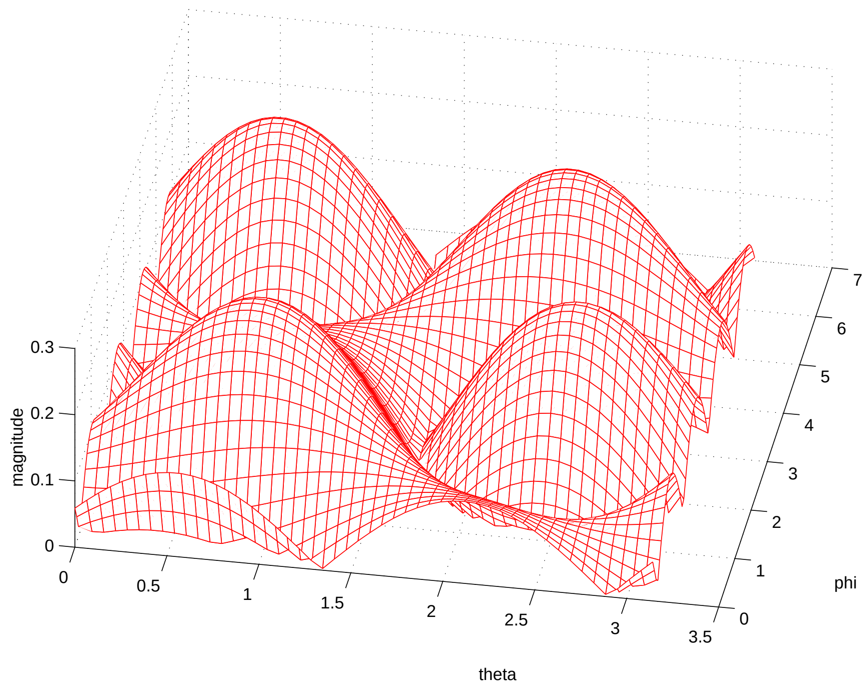
Angular separation on the Earth and maximum delay:

$$2\alpha = 27.2^0, \qquad \tau_0 = 9.997 \text{ ms}.$$

Comparison of the Antenna Patterns

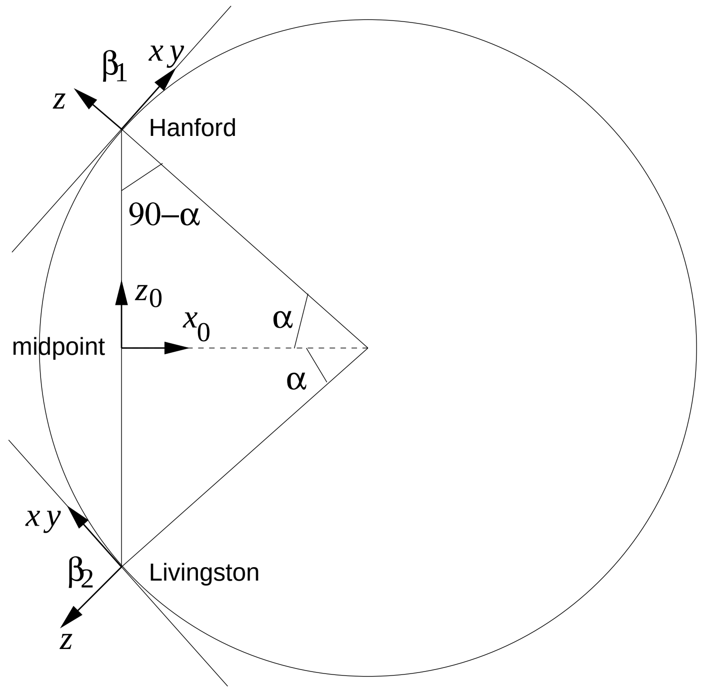
Left: The Livingston's F_+ as seen from the Hanford site.
Right: The difference between the two antenna patterns.





Transformation to Midpoint Frame

The midpoint and the detector coordinate frames on Earth:

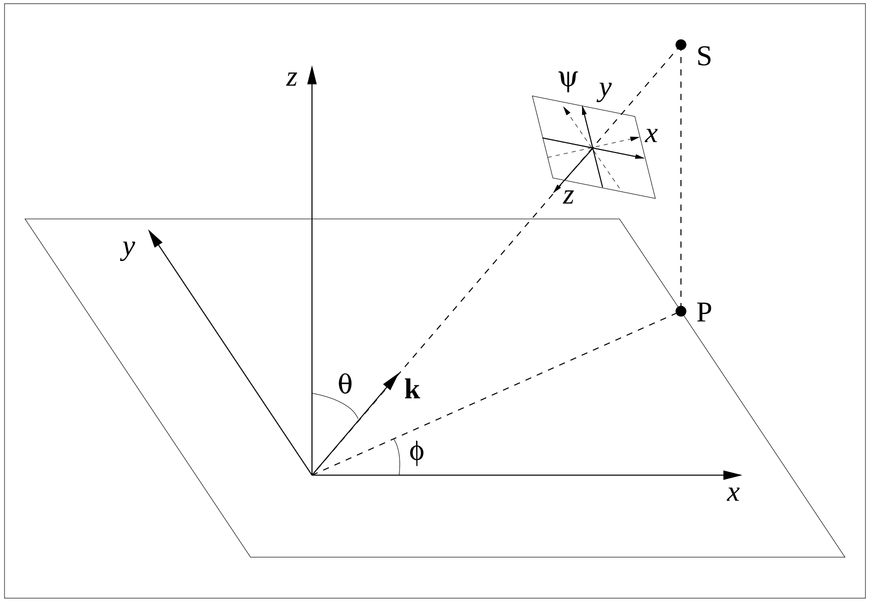


The two transformations are

$$\mathbf{R}_y(\pi/2-\alpha)\,\mathbf{R}_z(\beta_1), \qquad \mathbf{R}_y(\pi/2+\alpha)\,\mathbf{R}_z(\beta_2). \tag{11}$$

Transformation to G.W. Frame

GW coordinate system and the midpoint coordinate system



$$\mathbf{R}_1 = \mathbf{R}_z(\psi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_z(\phi)\,\mathbf{R}_y(\pi/2-\alpha)\,\mathbf{R}_z(\beta_1), \quad (12)$$

$$\mathbf{R}_2 = \mathbf{R}_z(\psi)\,\mathbf{R}_y(\theta)\,\mathbf{R}_z(\phi)\,\mathbf{R}_y(\pi/2+\alpha)\,\mathbf{R}_z(\beta_2). \quad (13)$$

Definition of the Correlation Function

Given two signals, $V_1(t)$ and $V_2(t)$, from two gravitational wave detectors, one can construct the correlation function

$$C(\tau) = \overline{V_1\left(t + \frac{\tau}{2}\right) \; V_2\left(t - \frac{\tau}{2}\right)}. \tag{14}$$

In the explicit form,

$$\begin{aligned} C(\tau) \;\; = \;\; & F_+(\Omega_1)\, F_+(\Omega_2) \; \overline{h_+\left(t + \frac{\tau_s + \tau}{2}\right) \; h_+\left(t - \frac{\tau_s + \tau}{2}\right)} + \\ & F_+(\Omega_1)\, F_\times(\Omega_2) \; \overline{h_+\left(t + \frac{\tau_s + \tau}{2}\right) \; h_\times\left(t - \frac{\tau_s + \tau}{2}\right)} + \\ & F_\times(\Omega_1)\, F_+(\Omega_2) \; \overline{h_\times\left(t + \frac{\tau_s + \tau}{2}\right) \; h_+\left(t - \frac{\tau_s + \tau}{2}\right)} + \\ & F_\times(\Omega_1)\, F_\times(\Omega_2) \; \overline{h_\times\left(t + \frac{\tau_s + \tau}{2}\right) \; h_\times\left(t - \frac{\tau_s + \tau}{2}\right)}. \end{aligned} \tag{15}$$

In these notations the coincidence corresponds to $\tau = -\tau_s$.

Definition of the Correlation Function (2)

Introduce two matrices: $\mathbf{M}$ and $\mathbf{N}$ and write the correlation function in a compact (matrix) form:

$$C(\tau) = \mathrm{Tr}\left\{\mathbf{M}^T\,\mathbf{N}(\tau)\right\}. \tag{16}$$

The first matrix represents angular dependence of the correlation function

$$M_{ij} = F_i(\Omega_1)\,F_j(\Omega_2). \tag{17}$$

The second matrix represents temporal behavior of the correlation function

$$N_{ij}(\tau) = \overline{h_i\left(t + \frac{\tau_s + \tau}{2}\right)\,h_j\left(t - \frac{\tau_s + \tau}{2}\right)}. \tag{18}$$

The Fourier transform of the matrix N_{ij} is

$$\tilde{N}_{ij}(\omega) = e^{i\omega\tau_s}\,\tilde{h}_i(\omega)\,\tilde{h}_j(\omega)^*. \tag{19}$$

Correspondingly, the correlation function takes the form:

$$\tilde{C}(\omega) = \mathrm{Tr}\left\{\mathbf{M}^T\,\tilde{\mathbf{N}}(\omega)\right\}. \tag{20}$$

Averaging Over ψ, ϕ, and θ

The simplest would be to average over all angles:

$$\mathbf{M} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}. \tag{21}$$

For Hanford-Livingston detectors: $a = -0.1413$. In this case,

$$\tilde{C}(\omega) = a\, e^{i\omega\tau_s}\mathcal{E}(\omega), \tag{22}$$

which is similar to Wiener-Khinchin theorem. Indeed, $\mathcal{E}$ is the energy density of the gravitational wave:

$$\mathcal{E}(\omega) = |\tilde{h}_+(\omega)|^2 + |\tilde{h}_\times(\omega)|^2. \tag{23}$$

Such an averaging procedure is not self-consistent because the arrival time of the gravitational wave is associated with angle θ:

$$\cos\theta = \frac{\tau_s}{\tau_0}. \tag{24}$$

Averaging Over ψ, ϕ

Consider fixed θ and average over ψ and ϕ. The result is

$$\mathbf{M}(\theta) = \begin{pmatrix} a(\theta) & b(\theta) \\ -b(\theta) & a(\theta) \end{pmatrix}. \tag{25}$$

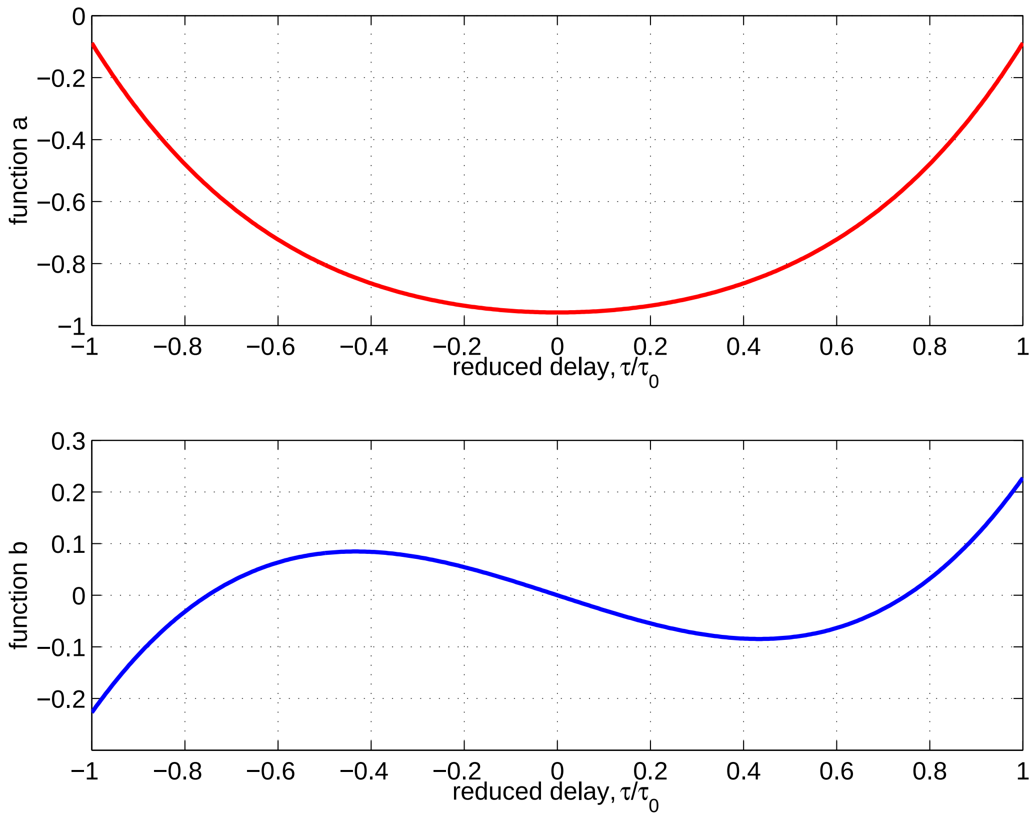
where a and b are trigonometric functions of θ. For any two detectors their functional form is

$$\begin{aligned} a(\theta) &= a_0 + a_2 \cos^2\theta + a_4 \cos^4\theta, & (26) \\ b(\theta) &= b_1 \cos\theta + b_3 \cos^3\theta. & (27) \end{aligned}$$

For example, the directional overlap of Hanford-Livingston pair is

$$\begin{aligned} a(\tau_s) &= -0.239 + 0.133 \left(\frac{\tau_s}{\tau_0}\right)^2 + 0.084 \left(\frac{\tau_s}{\tau_0}\right)^4, & (28) \\ b(\tau_s) &= 0.073 \left(\frac{\tau_s}{\tau_0}\right) - 0.13 \left(\frac{\tau_s}{\tau_0}\right)^3. & (29) \end{aligned}$$

Averaging Over ψ, ϕ (continued)



The maximum value of $b/a = 2.57$.

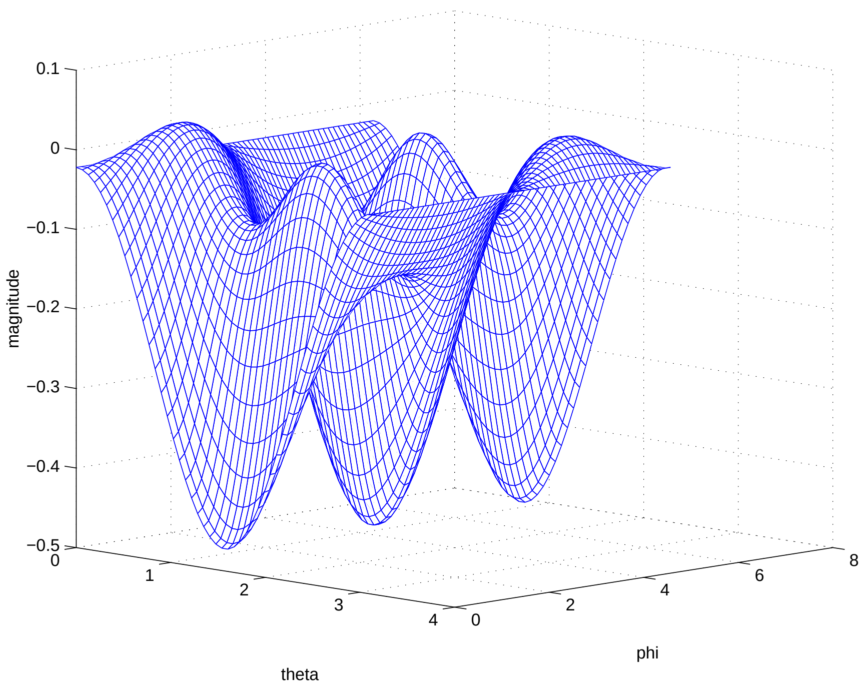
Averaging Over ψ

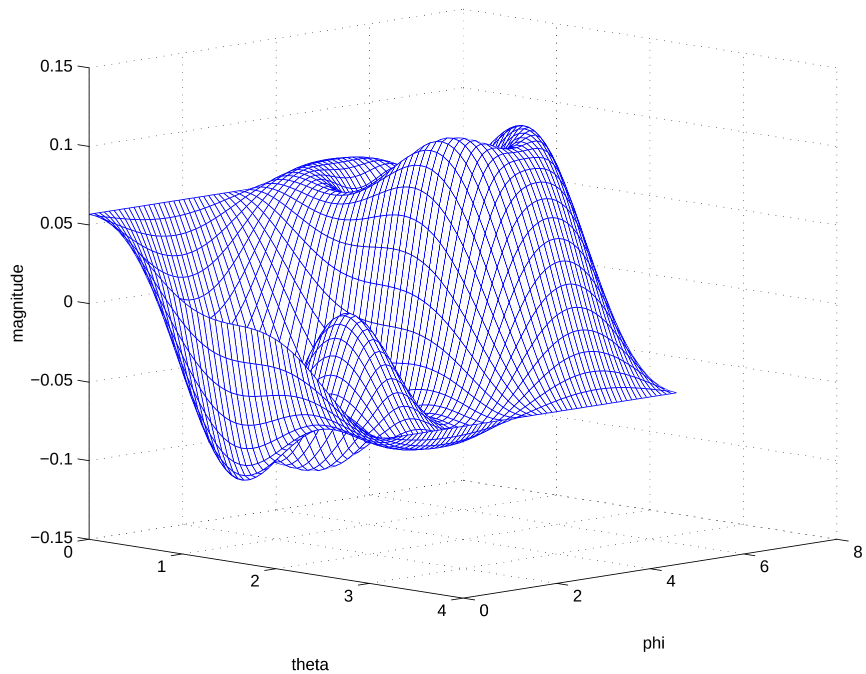
$$
\begin{aligned}
a \;=\;& -0.02168 - 0.1909\cos^2\theta + 0.1905\cos^4\theta \\
& -0.06095\cos^4\theta\;\cos^4\phi - 0.1668\cos^4\theta\;\cos^2\phi \\
& -0.06095\cos^4\phi - 0.3423\cdot 10^{-3}\cos\phi\;\sin\phi \\
& +0.5566\cos^2\theta\;\cos^2\phi - 0.3898\cos^2\phi \\
& +0.1219\cos^2\theta\;\cos^4\phi - 0.1731\cos\theta\sin\theta\;\sin\phi \\
& -0.2100\cos^3\theta\sin\theta\;\sin\phi\;\cos^2\phi - 0.1300\cdot 10^{-3}\sin\phi\;\cos^3\phi \\
& +0.3355\cdot 10^{-3}\cos^2\theta\;\cos\phi\;\sin\phi + 0.4323\cos^3\theta\sin\theta\;\sin\phi \\
& +0.2100\cos\theta\sin\theta\;\sin\phi\;\cos^2\phi - 0.1300\cdot 10^{-3}\cos^4\theta\;\cos^3\phi\;\sin\phi \\
& +0.6822\cdot 10^{-5}\cos^4\theta\;\cos\phi\;\sin\phi + 0.2601\cdot 10^{-3}\cos^2\theta\;\cos^3\phi\;\sin\phi,
\end{aligned}
$$

$$
\begin{aligned}
b \;=\;& -0.2064\cos^3\theta - 0.8454\cdot 10^{-4}\cos\phi\;\sin\theta\;\cos^2\theta^2 \\
& +0.09418\sin\phi\sin\theta\;\cos^2\theta + 0.5934\cdot 10^{-3}\cos^2\theta\;\cos^3\phi\;\sin\theta \\
& +0.1965\sin\phi\cos^2\phi\;\sin\theta - 0.3962\cdot 10^{-3}\sin\theta\;\cos\phi \\
& +0.03033\sin\theta\;\sin\phi - 0.5934\cdot 10^{-3}\cos^3\phi\;\sin\theta \\
& +0.1525\cos^3\theta\;\cos^2\phi - 0.1525\cos\theta\cos^2\phi \\
& -0.1965\sin\theta\cos^2\theta\;\sin\phi\cos^2\phi + 0.1496\cos\theta.
\end{aligned}
$$

M_{++} and $M_{+\times}$ in Cartesian Coordinates

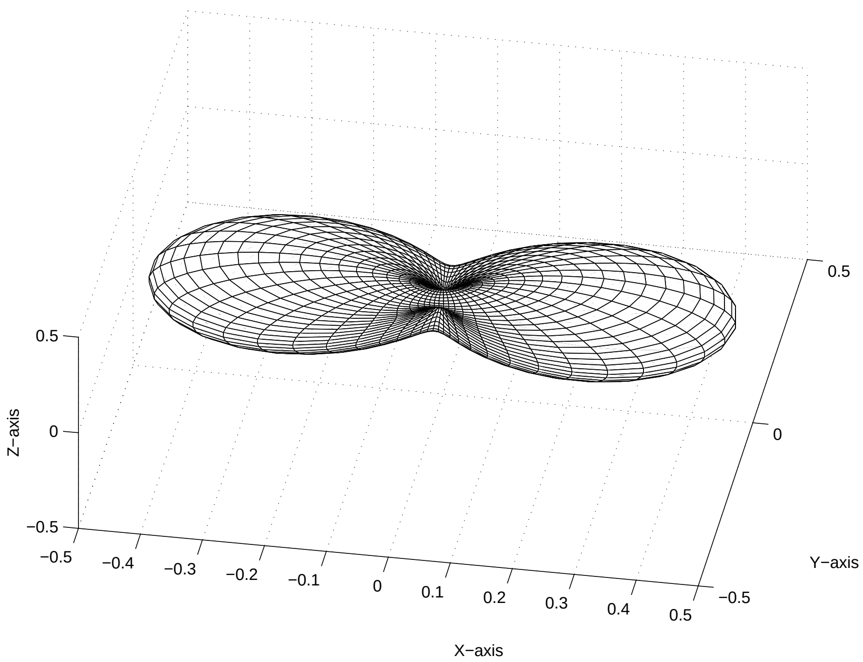
a (same as M_{++}) and b (same as $M_{+\times}$) as functions of θ, ϕ:

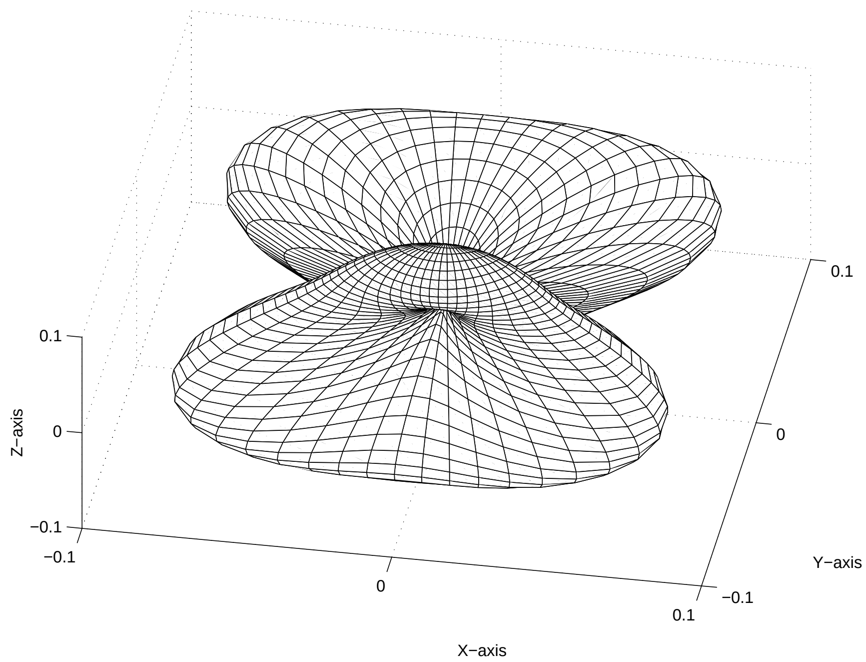




M_{++} and $M_{+\times}$ in Spherical Coordinates

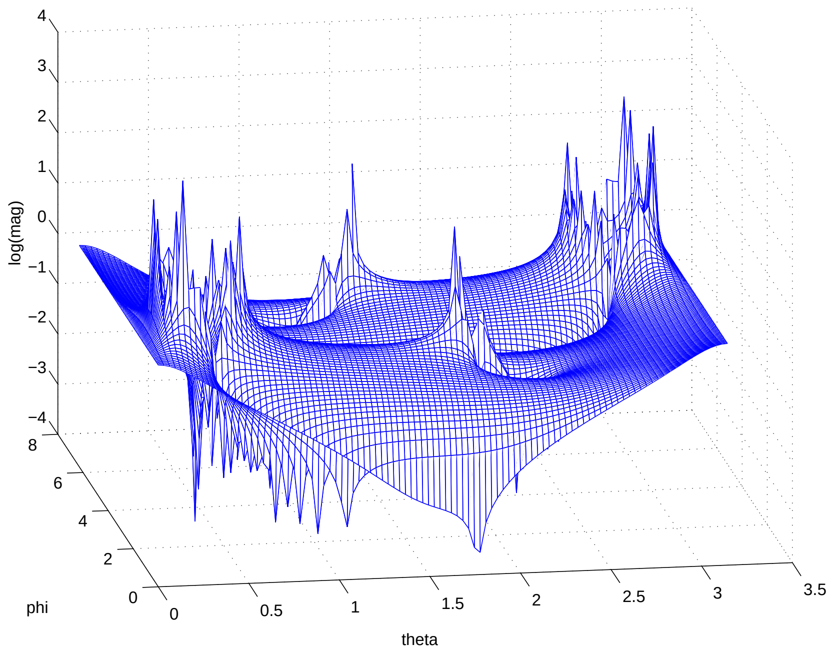
a (same as M_{++}) and b (same as $M_{+\times}$) as functions of θ, ϕ:



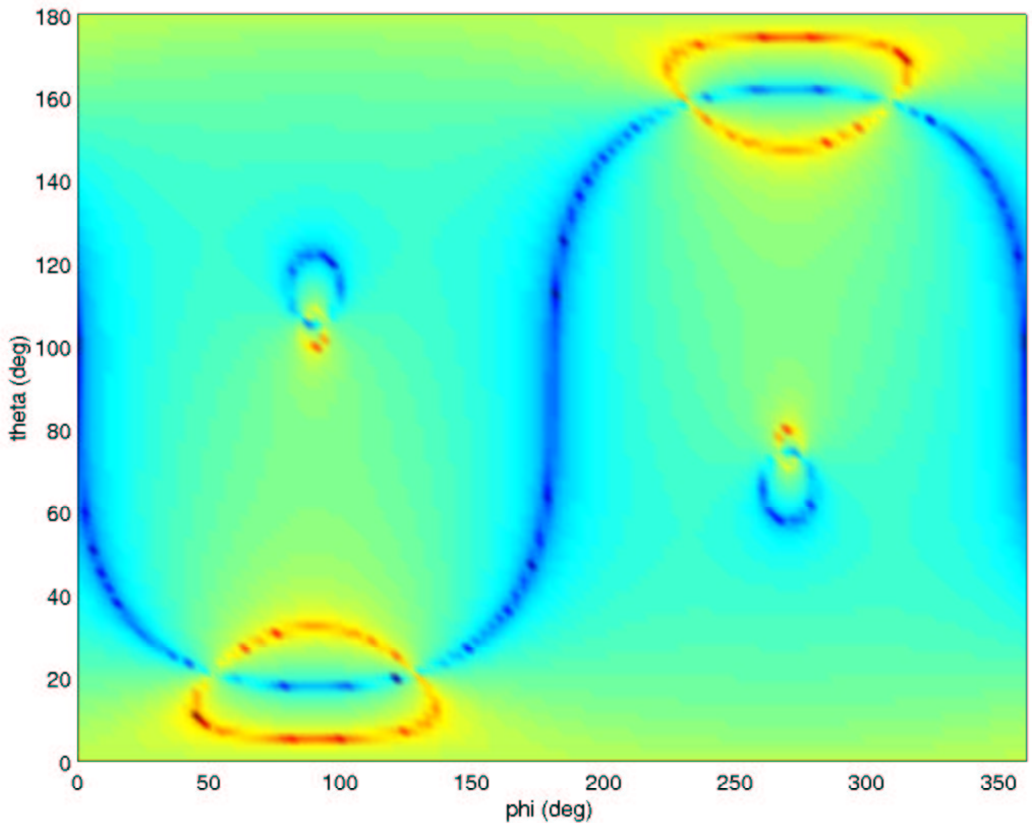


Relative Magnitude of M_{++} and $M_{+\times}$

$$\mu(\theta, \phi) = \frac{b(\theta, \phi)}{a(\theta, \phi)}. \tag{30}$$



Variation of μ on the Sky



Properties of the Correlation Function

For gravitational wave with unknown polarizations (average over ψ), the angular correlation matrix takes the form

$$\mathbf{M}(\theta,\phi) = \begin{pmatrix} a(\theta,\phi) & b(\theta,\phi) \\ -b(\theta,\phi) & a(\theta,\phi) \end{pmatrix}. \tag{31}$$

Then the correlation function is

$$\tilde{C}(\omega) = e^{i\omega\tau_s}\left[a\,\mathcal{E}(\omega) + ib\,\mathcal{P}(\omega)\right], \tag{32}$$

where $\mathcal{E}$ is symmetric and $\mathcal{P}$ is antisymmetric function of frequency. The symmetric component is

$$\mathcal{E}(\omega) = |\tilde{h}_+(\omega)|^2 + |\tilde{h}_\times(\omega)|^2, \tag{33}$$

and has the meaning of the energy density of the gravitational wave. The antisymmetric component is

$$\mathcal{P}(\omega) = \frac{1}{i}\left[\,\tilde{h}_+(\omega)\,\tilde{h}^*_\times(\omega) - \tilde{h}_\times(\omega)\,\tilde{h}^*_+(\omega)\,\right], \tag{34}$$

and has the meaning of the spin density.

Both $\mathcal{E}$ and $\mathcal{P}$ are real.

Spin-2 Representation of SO(3)

For gravitational waves, the states with particular helicity are

$$u(\omega) \;=\; [\tilde{h}_+(\omega) + i\tilde{h}_\times(\omega)]/\sqrt{2}, \tag{35}$$
$$d(\omega) \;=\; [\tilde{h}_+(\omega) - i\tilde{h}_\times(\omega)]/\sqrt{2}. \tag{36}$$

For these states, rotation by angle α along z-axis generates

$$u(\omega) \;\rightarrow\; e^{+2i\alpha}\, u(\omega), \tag{37}$$
$$d(\omega) \;\rightarrow\; e^{-2i\alpha}\, d(\omega). \tag{38}$$

The invariant quantities are the densities of positive and negative helicity:

$$N_\uparrow(\omega) \;=\; u(\omega)\, u(\omega)^*, \tag{39}$$
$$N_\downarrow(\omega) \;=\; d(\omega)\, d(\omega)^*. \tag{40}$$

Their sum and difference,

$$\mathcal{E}(\omega) \;=\; N_\uparrow(\omega) + N_\downarrow(\omega), \tag{41}$$
$$\mathcal{P}(\omega) \;=\; N_\uparrow(\omega) - N_\downarrow(\omega), \tag{42}$$

represent the total number of eigenstates and the net spin in the wave.

Observable Spin Density

An excess of states with a particular helicity will give rise to nonzero values for the spin density $\mathcal{P}(\omega)$. Such an excess of spin states in the gravitational wave can be characterized by

$$r(\omega) = \frac{\mathcal{P}(\omega)}{\mathcal{E}(\omega)}, \tag{43}$$

which is bounded by $|r(\omega)| \leq 1$. Nonzero values for this quantity would indicate the presence of spin in the gravitational wave and thus confirm the prediction of general relativity.

However, the ratio $r(\omega)$ cannot be directly observed. The observed ratio of the spin density to the energy density is

$$r_{\text{obs}}(\omega) = \mu(\theta, \phi)\, r(\omega). \tag{44}$$

This relation shows that the ratio $r(\omega)$ becomes suppressed (by a factor of μ) if the detectors are well aligned with respect to each other. Furthermore, even if the detectors are not aligned, there are regions of the sky for which the sources would not allow detectable spin density.

Magnitude and Phase

Complex representation of the correlation function:

$$\tilde{C}(\omega) = \mathcal{R}(\omega)\, e^{i\chi(\omega)}. \tag{45}$$

The magnitude and phase of the correlation function are

$$\mathcal{R}(\omega) = \sqrt{a^2\,\mathcal{E}(\omega)^2 + b^2\,\mathcal{P}(\omega)^2}, \tag{46}$$

$$\chi(\omega) = \omega\tau_s + \arctan\{r_{\mathsf{obs}}(\omega)\}. \tag{47}$$

The phase is directly related to the time delay τ_s. Indeed, the slope of the phase at the peak frequency is

$$\left.\frac{d\chi}{d\omega}\right|_{\omega=\omega_p} = \tau_s + \left[\frac{1}{1 + r^2_{\mathsf{obs}}(\omega_p)}\right] \left.\frac{dr_{\mathsf{obs}}}{d\omega}\right|_{\omega=\omega_p}. \tag{48}$$

Any deviation of the phase from the linear dependence would indicate the presence of the spin density:

$$\delta\chi(\omega) = \chi(\omega) - \left.\frac{d\chi}{d\omega}\right|_{\omega=\omega_p} (\omega - \omega_p). \tag{49}$$

This quantity can be measured in the experiment.