Numerical Calculation of Thermal Noise

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Abstract

- **Calculation of thermal noise in a complicated system**
  - » **Comparison with previous approaches**
    1. **Modal expansion**
    2. **Levin's approach(static)**
    3. **Nakagawa's approach**
    4. **Transfer Matrix**

- ***Numerical Dynamic Approach***
  - » **Numerical solution: Finite Element Method(FEM)**
    1. **Solving equation of motion(EQM) including loss (NDA-1)**
    2. **"Dynamic" Levin's approach (NDA-2)**
  - » **The most practical, useful, and simplest way**
    - **TN calculation: no more than a structural analysis in mechanics**

Contents

1. **Introduction**
   - Fluctuation-Dissipation Theorem (FDT)
   - Overview of various approaches to calculate TN
2. **Calculation - 3 examples -**
   1. 1-D model : Elastic bar
   2. 3-D model : Mirror used in GW detector/TN measurement
   3. 3-D model : Conical mirror with flat-topped beam
3. **Summary**

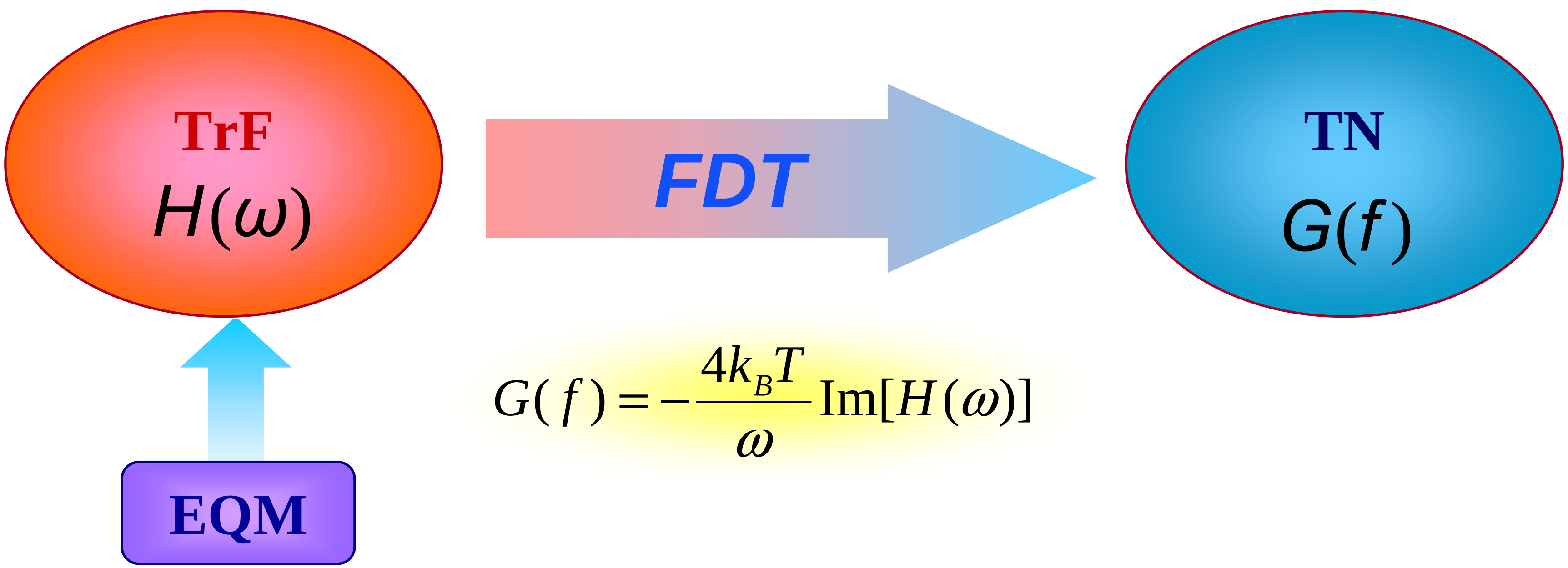
1.Introduction

- Various approaches for TN calculation
  - » For homogeneous-loss system
    - – Modal expansion
  - » For inhomogeneous-loss system
    - – Levin, Nakagawa…

- Results by now
  - » Only in a simple system
  - » Not practical
    - – These are because many people try to solve EQM *analytically*.
    - – Numerical approach can cope with this.

Fluctuation Dissipation Theorem

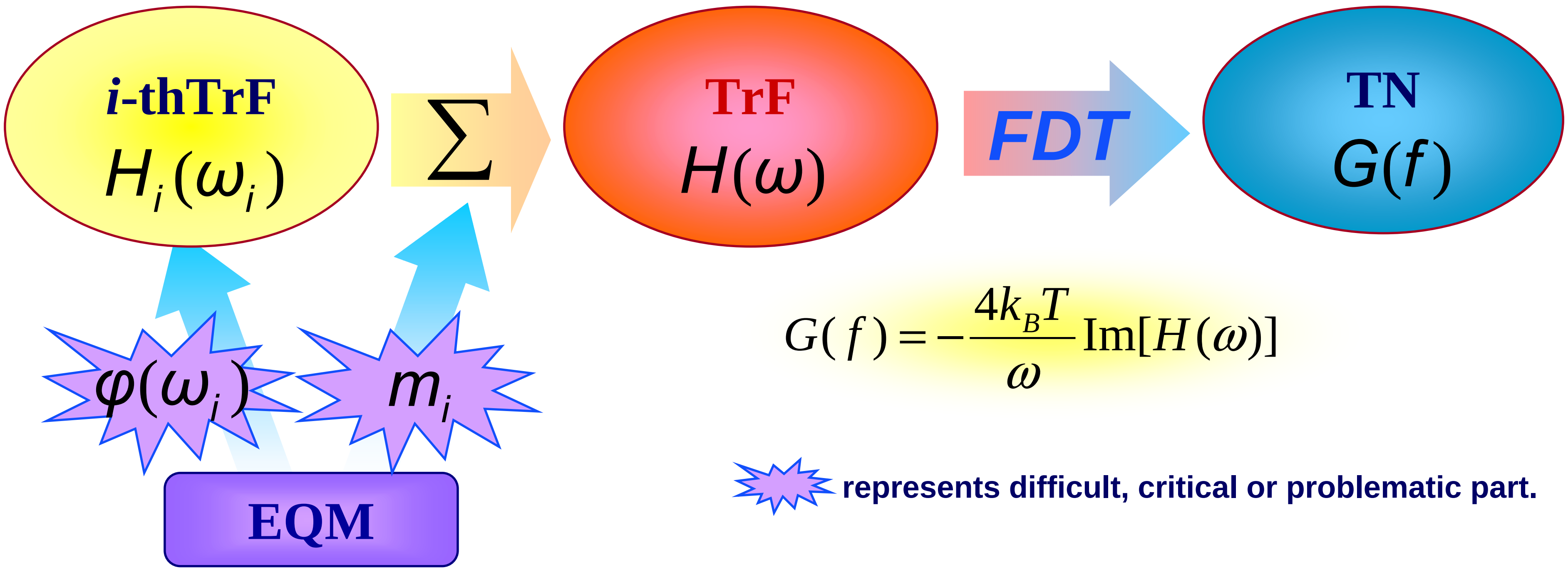
- **How can we calculate TN?**
  - » **Unique solution : FDT**
    - – **FDT represents relationship between TN and transfer function(TrF)**
    - – **Easy to apply only when TrF is analytically calculated**



$$G(f) = -\frac{4k_B T}{\omega}\mathrm{Im}[H(\omega)]$$

Mode Expansion (ME)

- **Features**
  - » **Represents total TrF by a summation of oscillator's TrFs**
    - – **Requires "effective mass" for the summation**
    - – **Valid only when normal modes are independent**
    - – **Can not be applied if the loss distribution is inhomogeneous**

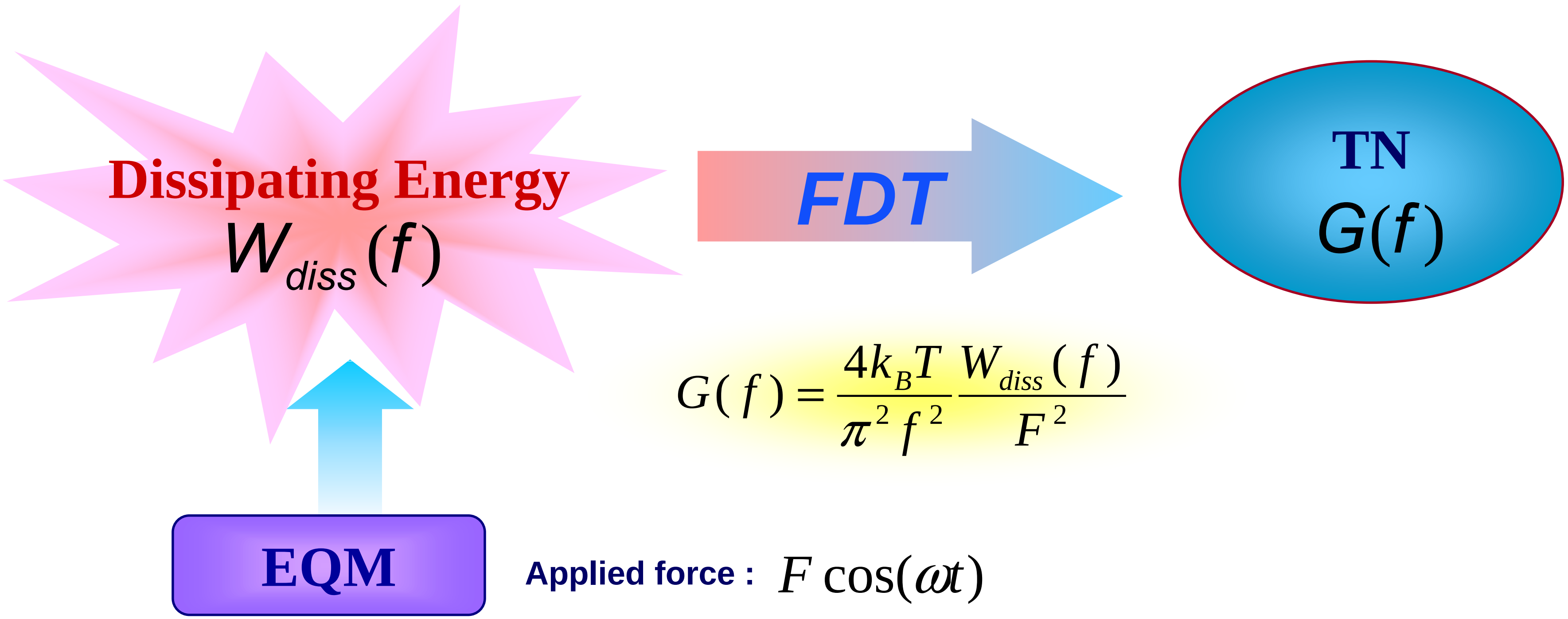


$$G(f) = -\frac{4k_B T}{\omega}\mathrm{Im}[H(\omega)]$$

represents difficult, critical or problematic part.

Levin’s Approach

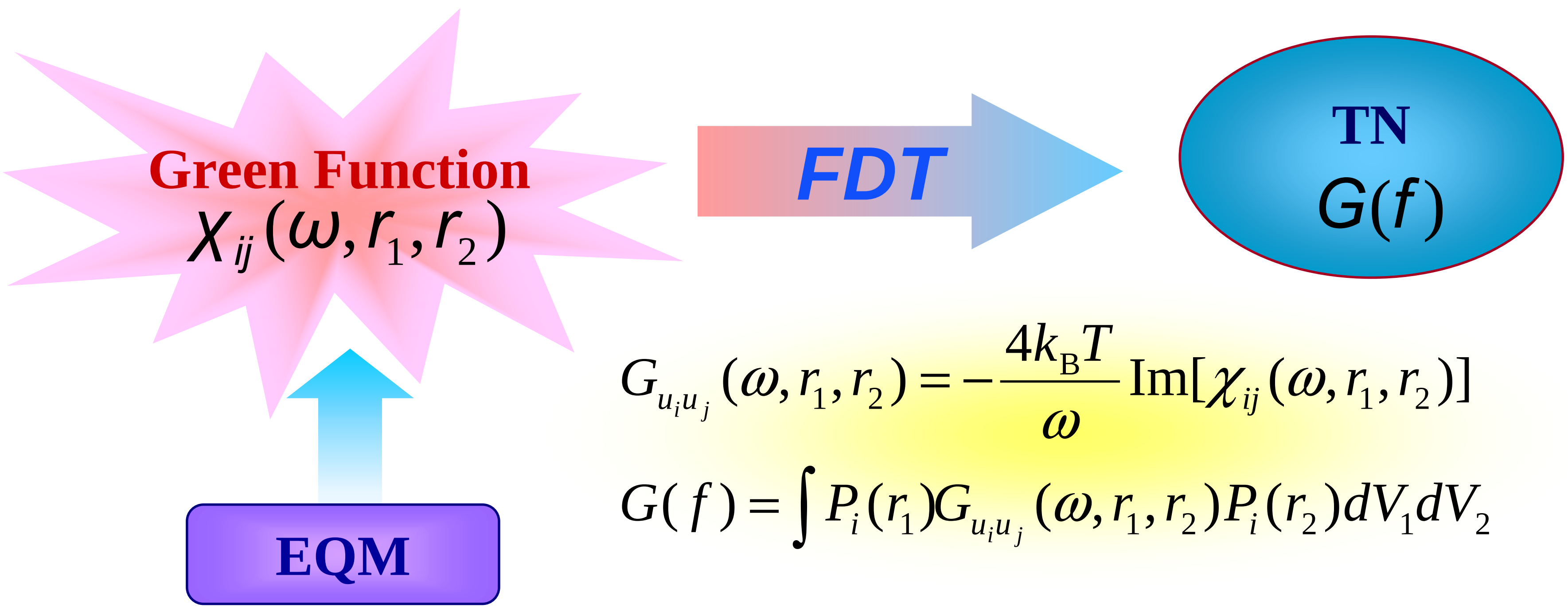
- **Features**
  - » **Represents FDT in “dissipating-energy” form**
    - – Requires to solve EQM and to know strain energy in the system
    - – Easy to apply at zero frequency (done by now)



$$G(f) = \frac{4k_B T}{\pi^2 f^2} \frac{W_{diss}(f)}{F^2}$$

Nakagawa's Approach

- **Features**
  - » **Represents FDT in "Green-function" form**
    - – Requires Green function
    - – Sometimes simple(?)
    - – Mainly applied for static calculation

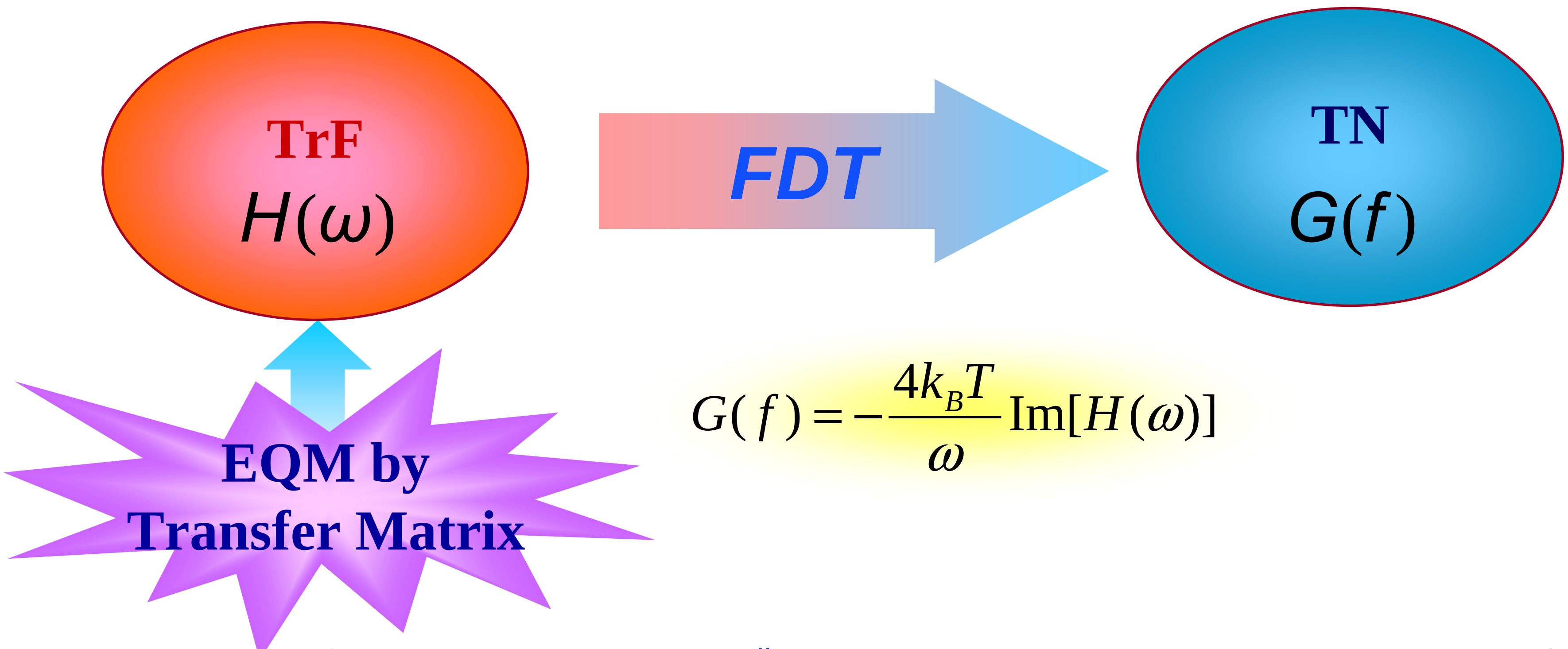


$$G_{u_i u_j}(\omega, r_1, r_2) = -\frac{4k_B T}{\omega}\,\mathrm{Im}[\chi_{ij}(\omega, r_1, r_2)]$$

$$G(f) = \int P_i(r_1) G_{u_i u_j}(\omega, r_1, r_2) P_i(r_2)\, dV_1 dV_2$$

Transfer Matrix

- **Features**
  - » **Uses transfer matrix to compute TrF [Tsubono]**
    - **Very simple, once the system is well modeled.**
    - **2-D system is its limitation (my conjecture)**



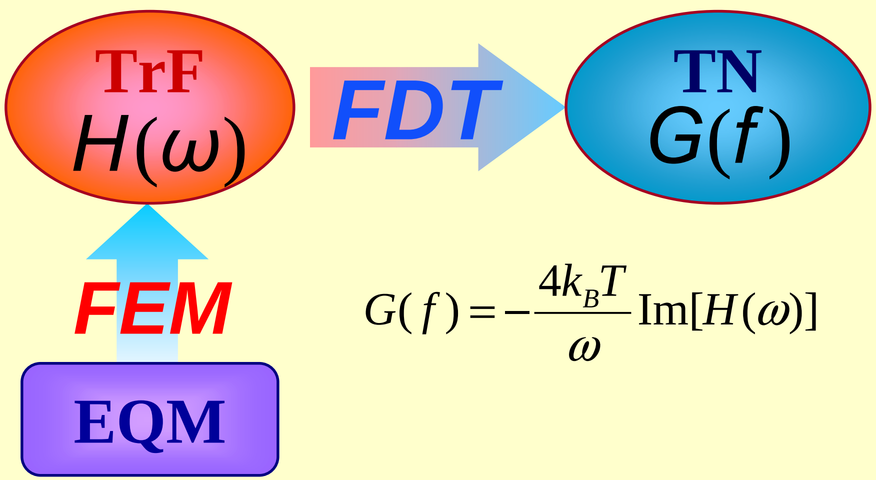
$$G(f) = -\frac{4k_B T}{\omega}\,\mathrm{Im}[H(\omega)]$$

Approaches Here

- **Numerical Dynamic Approach (NDA)**
  - » **Two possibilities**
    - – **Numerically solve EQM using Finite Element Method (FEM) (or whatever).**
    - – **Can be applied for complicated system and wide-band calculation**

NDA-1

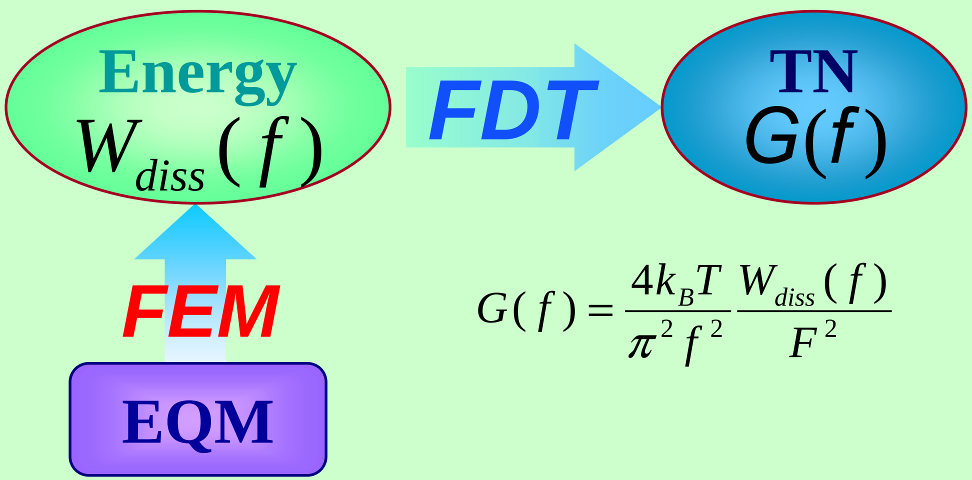
✓"Direct" approach



$$G(f) = -\frac{4k_BT}{\omega}\mathrm{Im}[H(\omega)]$$

NDA-2

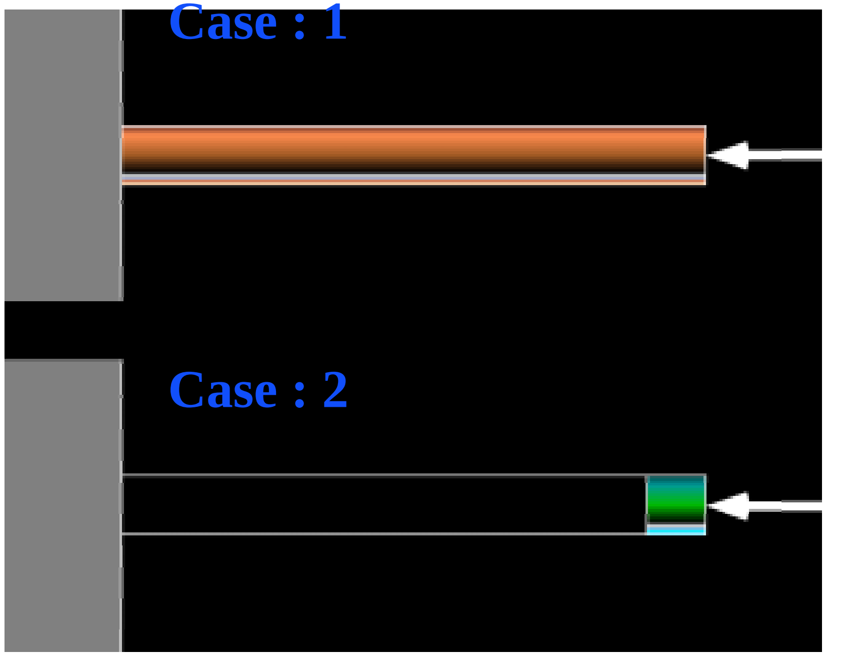
✓"Dynamic" Levin's approach



$$G(f) = \frac{4k_BT}{\pi^2 f^2}\frac{W_{diss}(f)}{F^2}$$

2.Examples

- **Example 1 : 1-D elastic bar**
  - » Elastic bar fixed at one end
  - » TN at another end point
  - » Parameters
    - – Al
    - – Diameter : 10cm
    - – Length : 1m
    - – Loss : structure/viscous
    - – Case 1 : homogeneous
      - Phi(f)=1/1000 (@1kHz)
    - – Case 2 : inhomogeneous
      - Phi(f)=30/4000 (@1kHz)
      - At 4/30 from free end
  - » Calculated using
    - – Transfer matrix
    - – NDA-1,-2
    - – ME

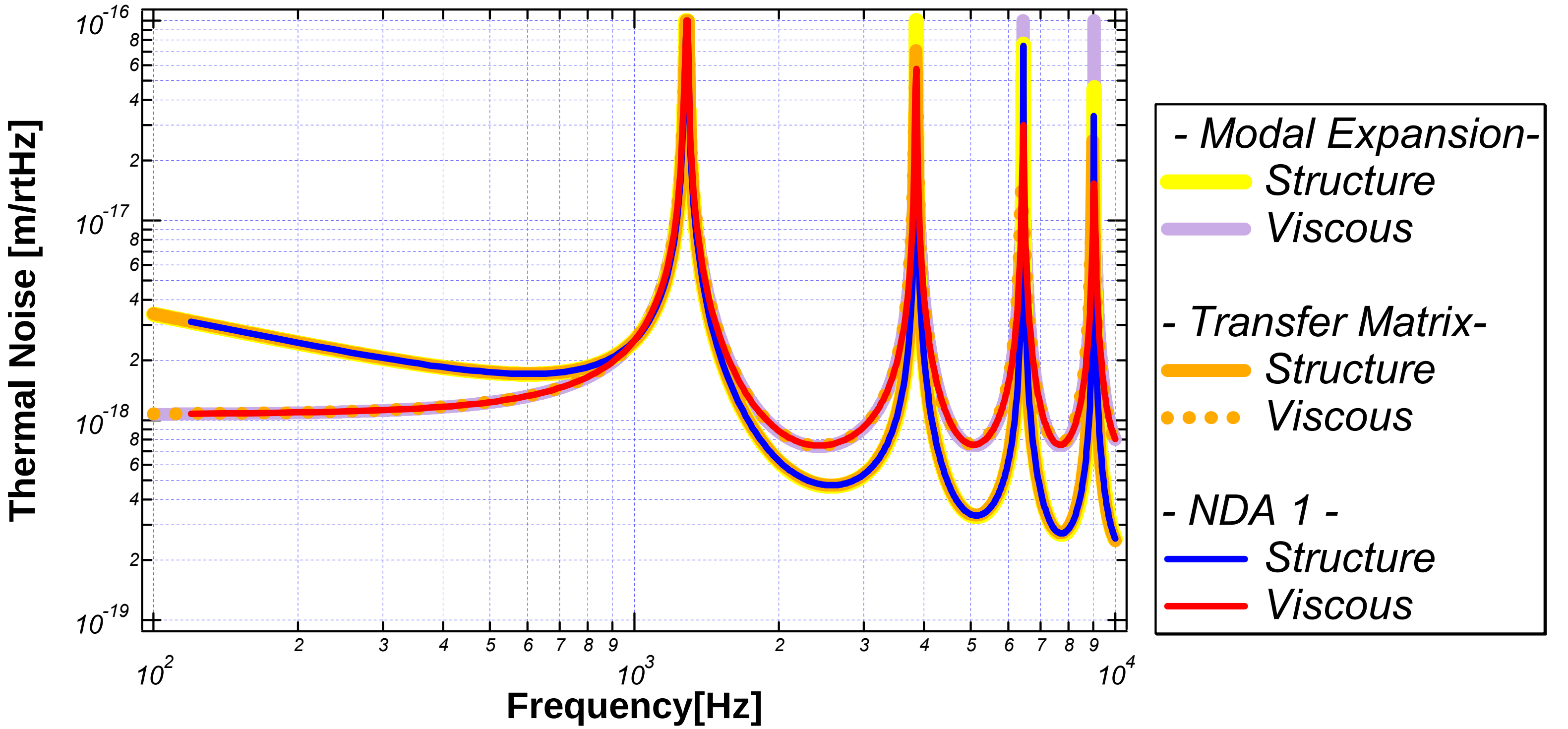


Modeling for Ex.1

- ANSYS modeling
  - » Element
    - – Beam
    - – 30 elements
  - » Done everything in ANSYS
    - – Application of FDT
  - » Calculation
    - – Done in note PC
    - – Within 1min

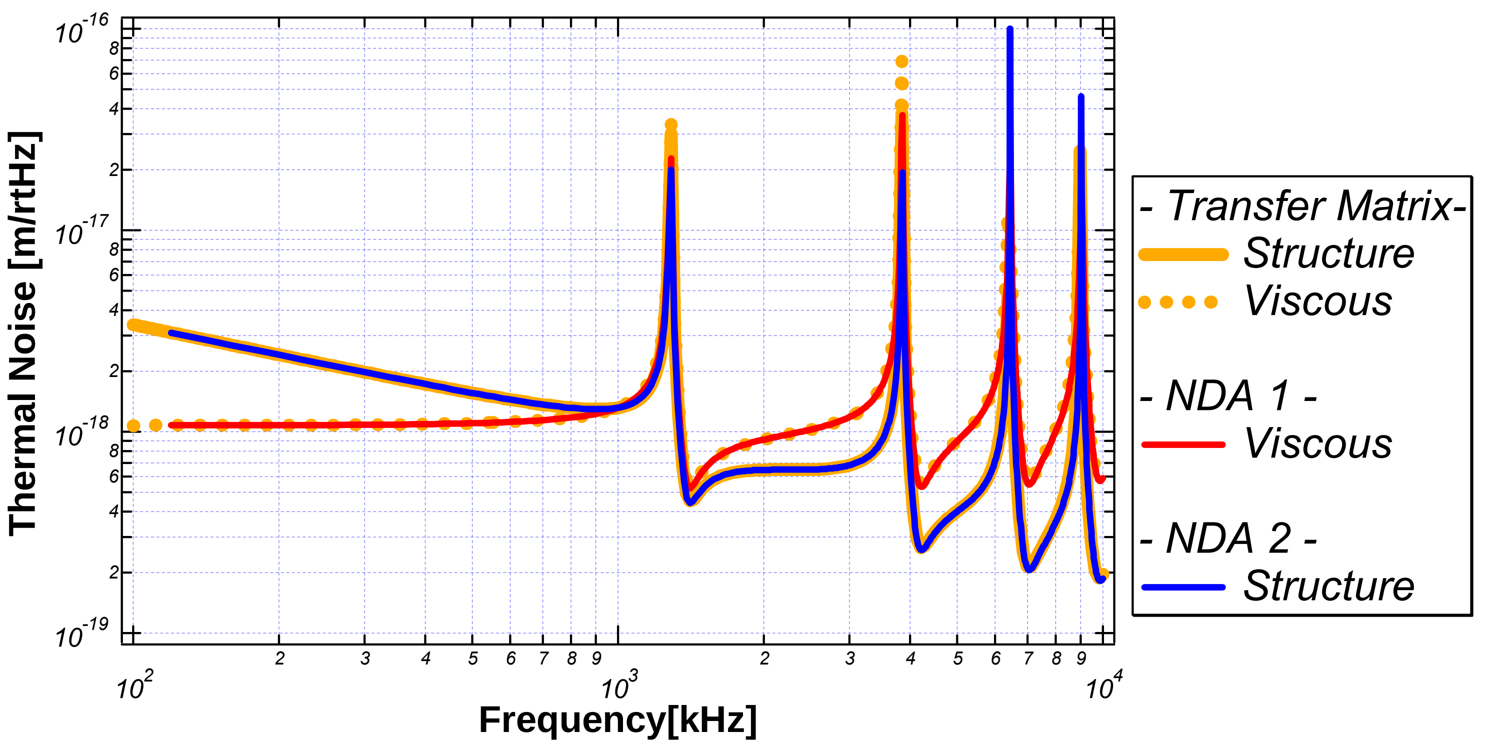
Results: Case 1

- **Homogeneous loss**
  - » **Check for ANSYS implementation**
  - » **Every method gave the same results**



Results: Case 2

- **Inhomogeneous loss**
  - » **ANSYS results with NDA-1,-2 are consistent.**
  - » **Transfer matrix (& the other methods) gave identical result.**



Example 2

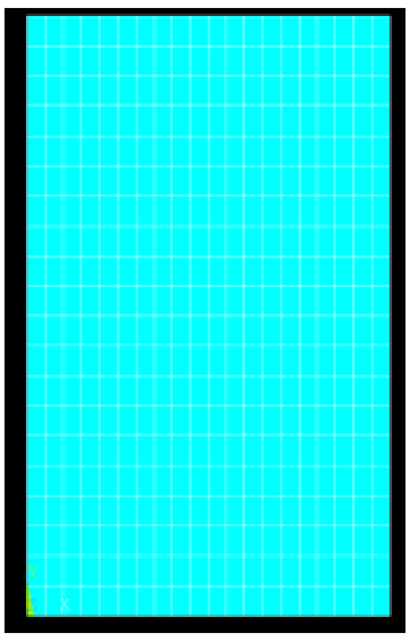
- **Example 2 : Cylindrical mirror with Gaussian Beam**
  - » **Parameters**
    - – **Case 1 : Huge bean size**
      - **W_0=10.6cm**
      - **Fused silica (phi=1/5x10^7)**
    - – **Case 2 : Small beam size (cavity)**
      - **W_0=48.9um/84.8um**
      - **BK7 (phi=1/3600)**
      - **Coating**
        - » **phi=1/10^4**
        - » **Thickness : 10um**

  - » **Calculated using**
    - – **ME (for homogeneous case)**
    - – **NDA-2**

Modeling for Ex.2

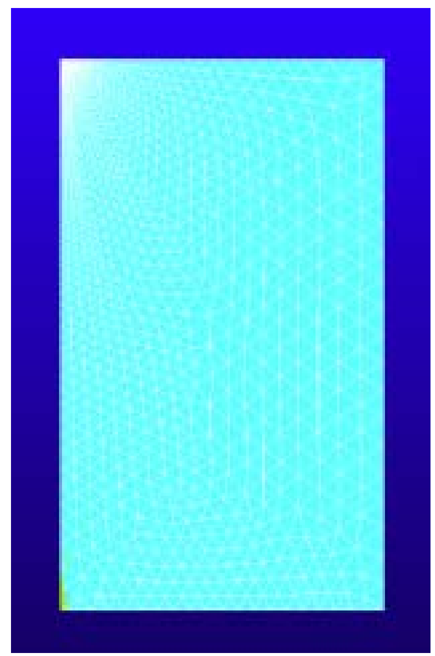
- **ANSYS modeling**
  - » **In order to avoid huge model size**
    - – **Use of axisymmetric model**

●Case 1

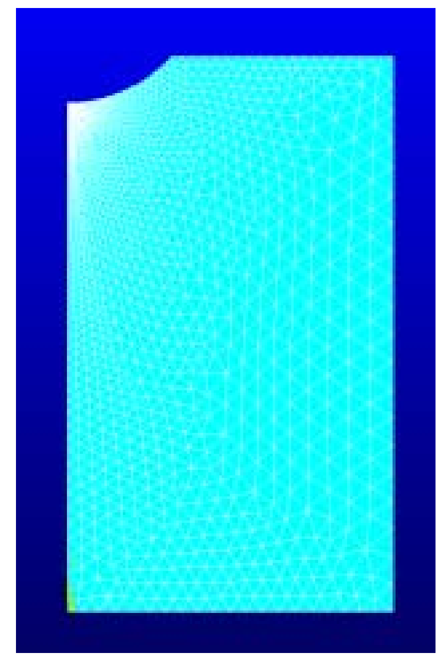


400 elements

●Case 2



5330 elements

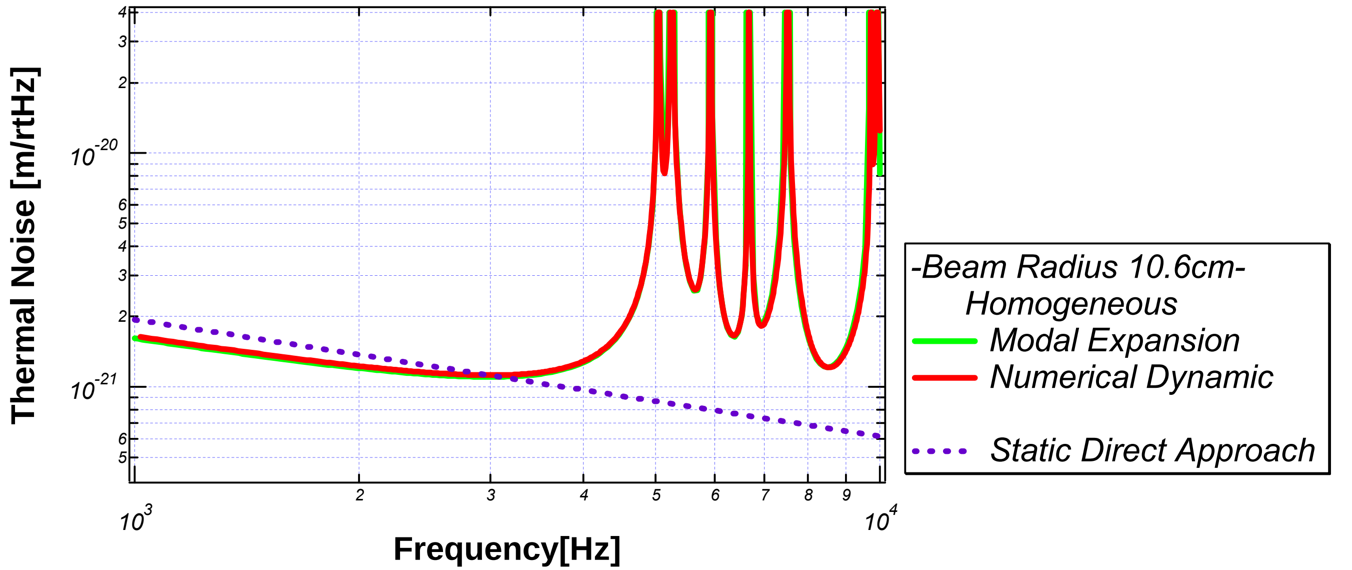


6576 elements

Results: Case 1

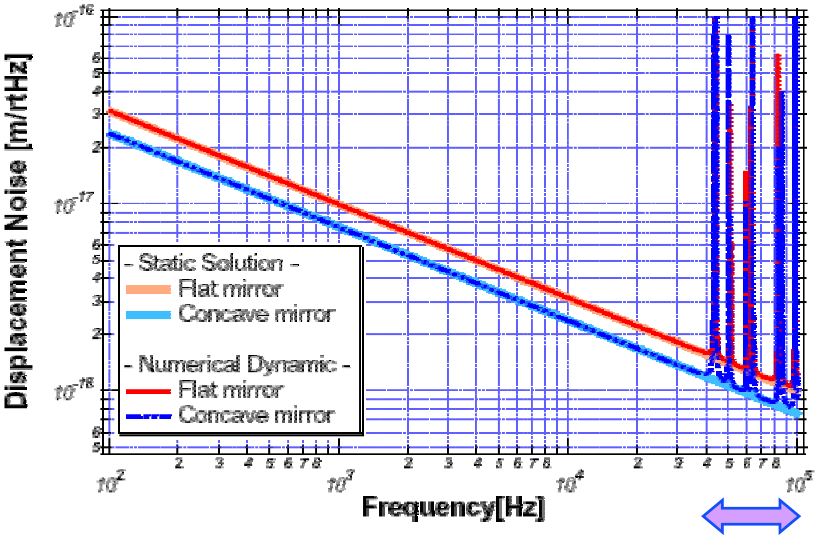
- **Comparison with ME & static solution**
  - » **Agreement with ME**
    - – **Note that ME takes much time because of numerical difficulties**
      - **NDA: ~few min / ME: ~few hours**
  - » **Static & analytic solution**

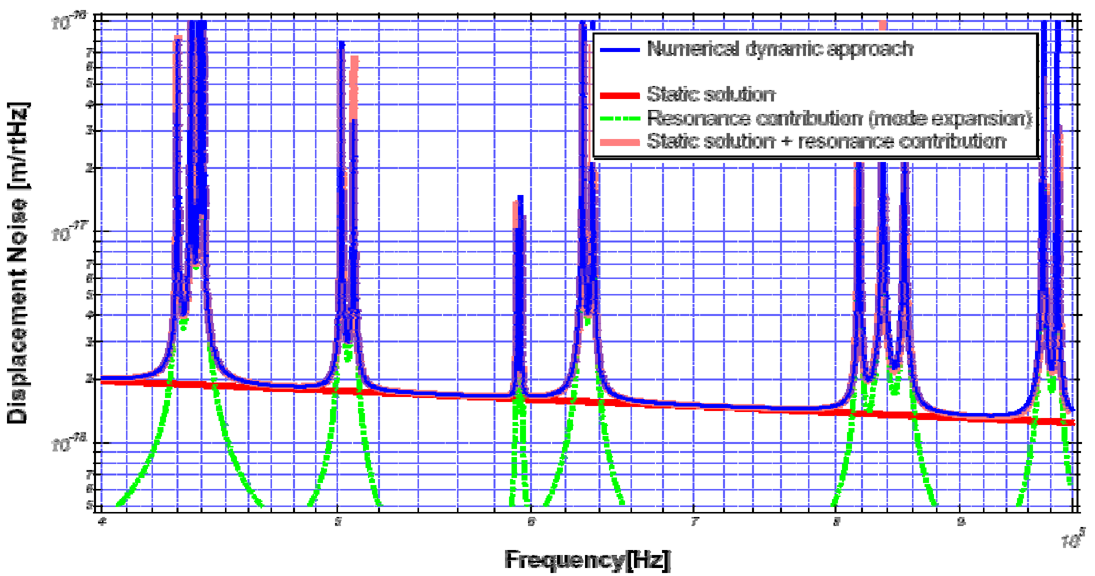
$$G(f) \approx 4k_B T \frac{(1-\sigma^2)\phi_m}{\sqrt{\pi} E w_0} \frac{1}{\omega}$$



Results: Case 2

- **Comparison with static solution**
  - » **No finite-sized effect**
    - – **Because of small beam radius**
- **Comparison with ME**
  - » **Approximated as SS+ME**
    - – **Higher order mode contribution**



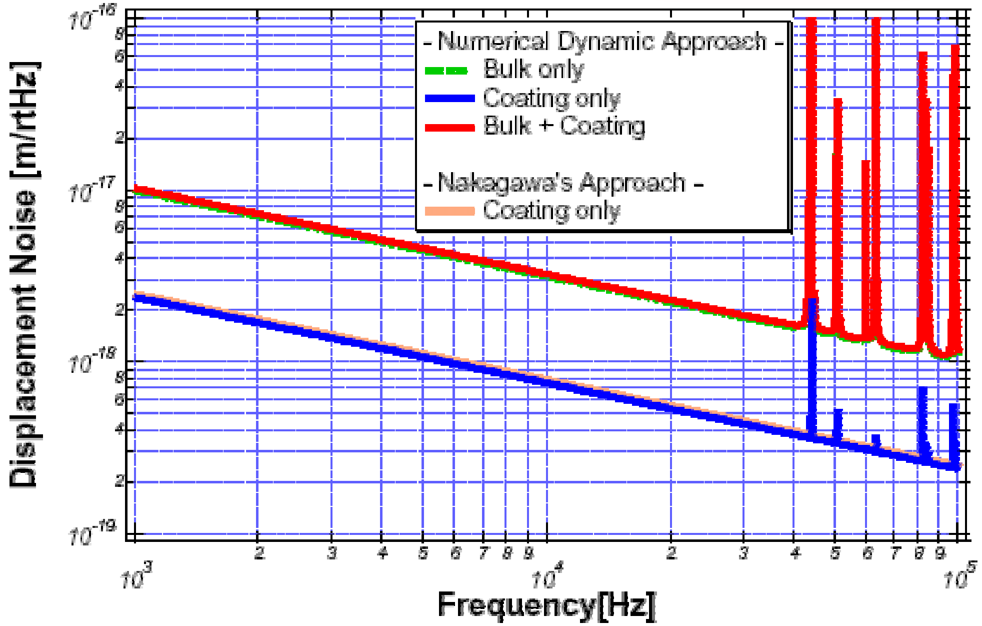


Results: Case 2

- **Inhomogeneous loss (coating)**
  - » **By Nakagawa et al.**
    - – **Static solution by Nakagawa's approach**
    - – **Infinite half space**

$$G(f) = 4k_B T \frac{(1-\sigma^2)}{2\pi^{3/2} f w_0 E} \phi_{subst} \left(1 + \frac{2}{\sqrt{\pi}} \frac{(1-2\sigma)}{(1-\sigma)} \frac{\phi_{coat}}{\phi_{subst}} \frac{d}{w_0} \right)$$

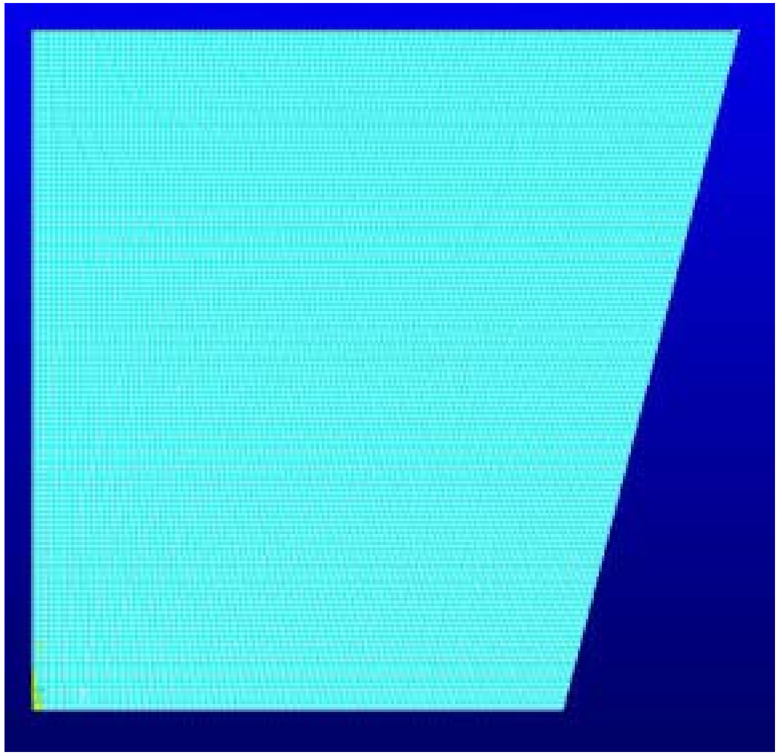
Additional term



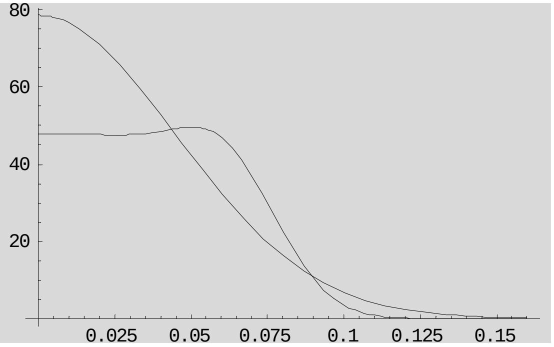
Example 3

- **Example 3: conical mirror with flat-topped(MH) beam**
  - » **Mirror**
    - – **Diameter: 0.32m & 0.24m**
    - – **Height: 0.155m**
  - » **Beam Radius**
    - – **1cm~12cm**
      - **Gaussian: w(=Sqrt(2)*r))**
      - **Flat-topped: D**
  - » **Material**
    - – **Substrate: sapphire, phi=1/10^8**
    - – **Coating: 10um, phi=1/10^4**

●**Used model (15625 elements)**



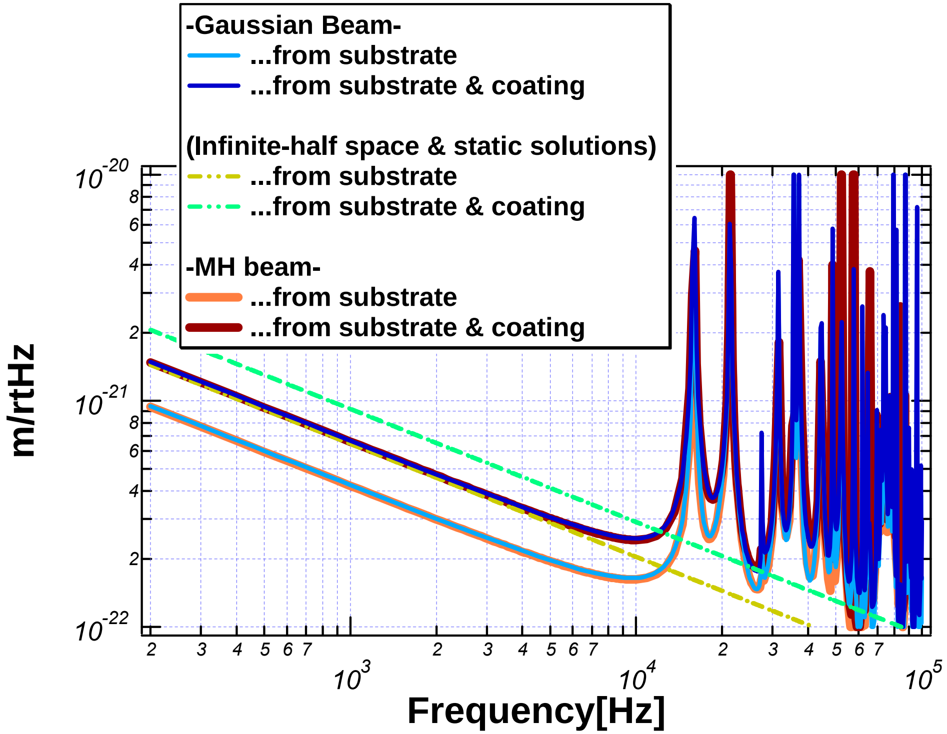
●**Power profile**



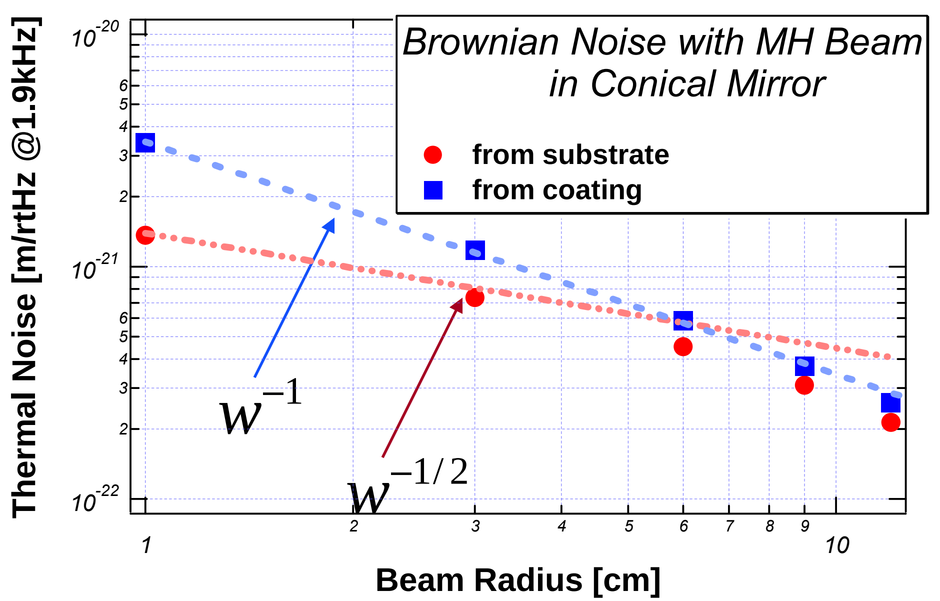
Results: Case 3

- Similar behavior with Gaussian beam

●Comparison with Gaussian beam



●Beam radius dependence



3.Summary

- Numerical Dynamic Approach
  - » Every 3-dimentional system
    - – TN in bridge, car, building, ship etc... in principle
      - Example: mirror (finite-sized)
  - » Every loss distribution
    - – At any location
    - – Every frequency dependence of loss
      - Example: coating & magnet on mirror
  - » Every frequency range
    - – TN around & between resonances
      - Analysis for inhomogeneous loss
  - » Simple but generalized
    - – Practically the most powerful method without doubt
    - – Analysis of mechanical response
      - Practical TN calculation is for mechanics designer not for physicist (?)

Future Works

- **Done calculations by now**
  - » **Related to GW detector (Not shown in this talk)**
    - – **Magnet on mirror, crystalline mirror, pendulum with elastic wire, etc…**
- **Possibilities**
  - » **More complicated system**
    - – **Pendulum with elastic mirror, friction, surface loss etc…**
  - » **Application for thermoelastic noise analysis**
    - – **If adiabatic limit, it is easy to calculate.**
    - – **Coating/resonance/anisotropic material**
    - – **Thermoelastic noise under stress**
  - » **Minimization of thermal noise**
    - – **Mirror shape, beam shape, crystal direction etc…**
    - – **Loss distribution (coating/magnet/wire etc…)**