LIGO-G030057 17-20 March, 2003

Correlation Function in the Coincidence Analysis for Laser Gravitational-Wave Detectors

M. Rakhmanov and S. Klimenko

University of Florida, Gainesville

LIGO Scientific Collaboration Meeting
LIGO Livingston Observatory, Livingston, LA

Overview

Objectives:

- determine the location of g.w. source on the sky
- find the angle of g.w. polarization axes.
- restore the waveforms

Related material:

- S. Klimenko and I. Yakushin, *Burst data analysis in wavelet domain for two interferometers*, LSC meeting
- Y. Gürsel and M. Tinto, *Near optimal solution to the inverse problem for gravitational-wave bursts*, Phys. Rev. D 40 (1989) 3884
- B.F. Schutz, *Data processing, analysis, and storage for interferometric antennas*, in Detection of gravitational waves. ed. D. Blair
- W.E. Althouse et al., *Precision alignment of the LIGO 4 km arms using the dual-frequency differential global positioning system*, Rev. Sci. Instr. 72 (2001) 3086

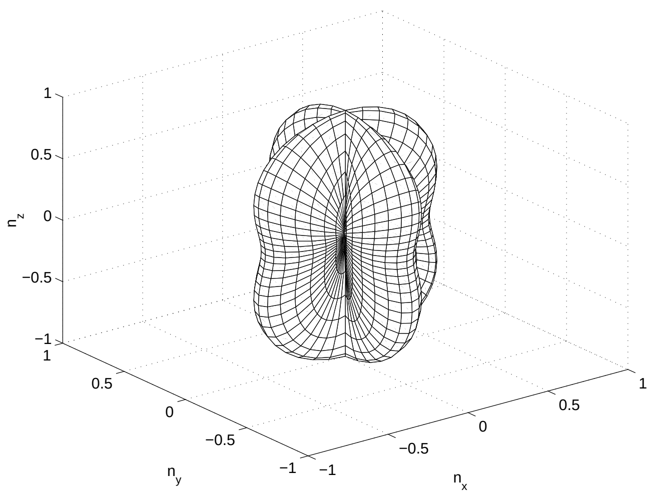
Gravitational-Wave Signal

Gravitational-wave signal:

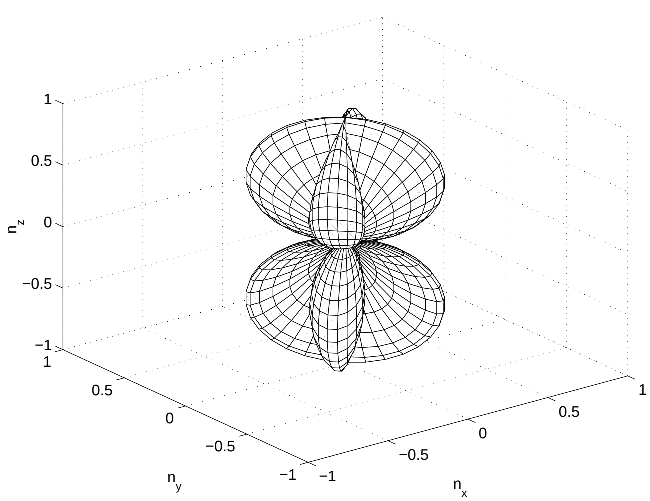
$$V(t) = F_+(\Omega)\, h_+(t,\vec{r}) + F_\times(\Omega)\, h_\times(t,\vec{r}),$$

$\Omega = (\phi, \theta, \psi)$, $F_i(\Omega)$ antenna patterns.

$$F_+(\Omega) = 2^{-1}\, \cos 2\phi\, (1 + \cos^2\theta)\, \cos 2\psi - \sin 2\phi\, \cos\theta\, \sin 2\psi.$$



$$F_\times(\Omega) = 2^{-1}\, \cos 2\phi\, (1 + \cos^2\theta)\, \sin 2\psi - \sin 2\phi\, \cos\theta\, \cos 2\psi.$$



Coincidence Analysis

1 detector:

- one response: $V(t)$
- cannot tell directional information: θ, ϕ and ψ
- cannot separate $h_+(t)$ and $h_\times(t)$

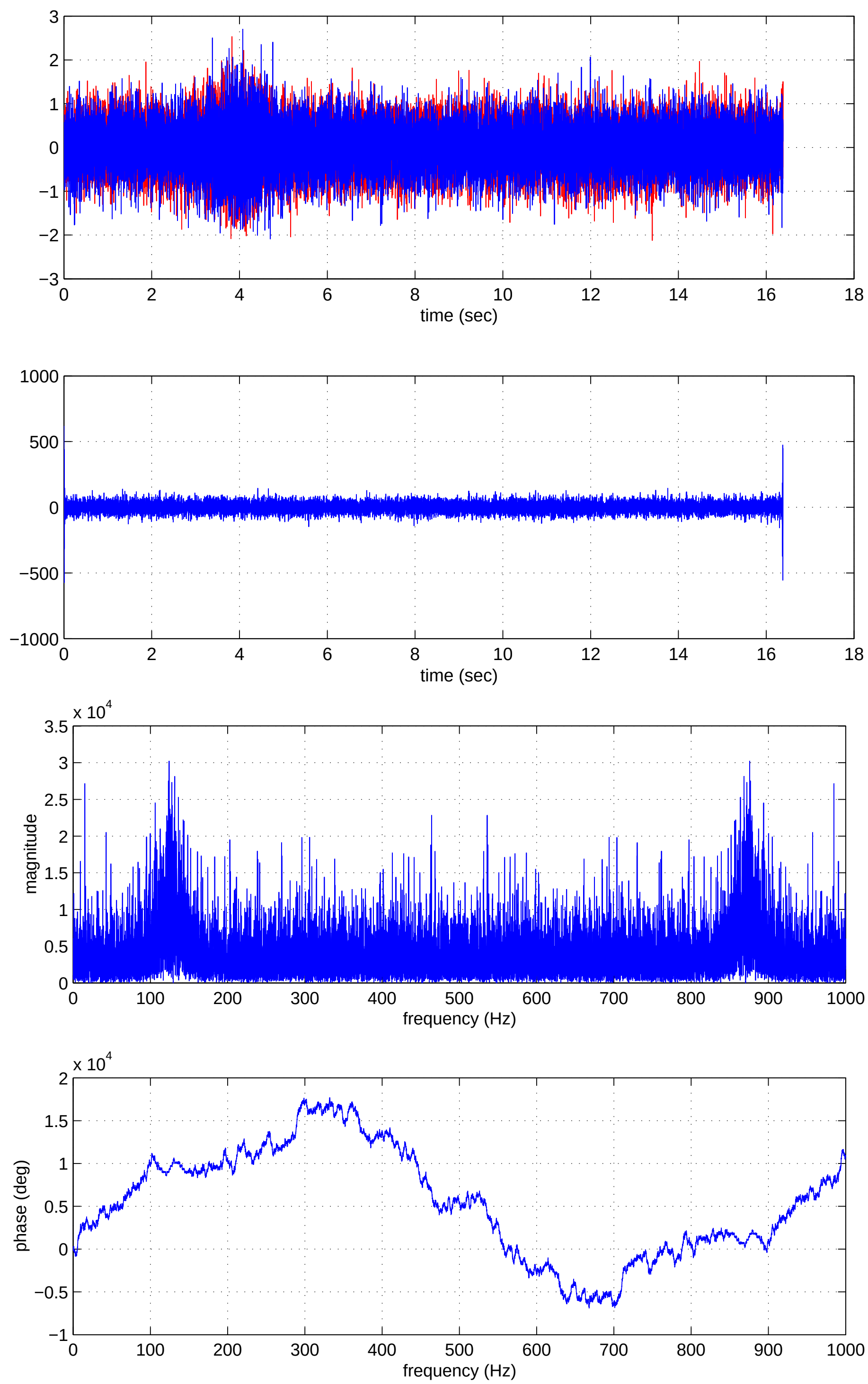
2 detectors:

- two responses: $V_1(t)$ and $V_2(t)$ but one corr.function
- can tell the delay τ_s and therefore θ
- cannot tell ϕ and ψ

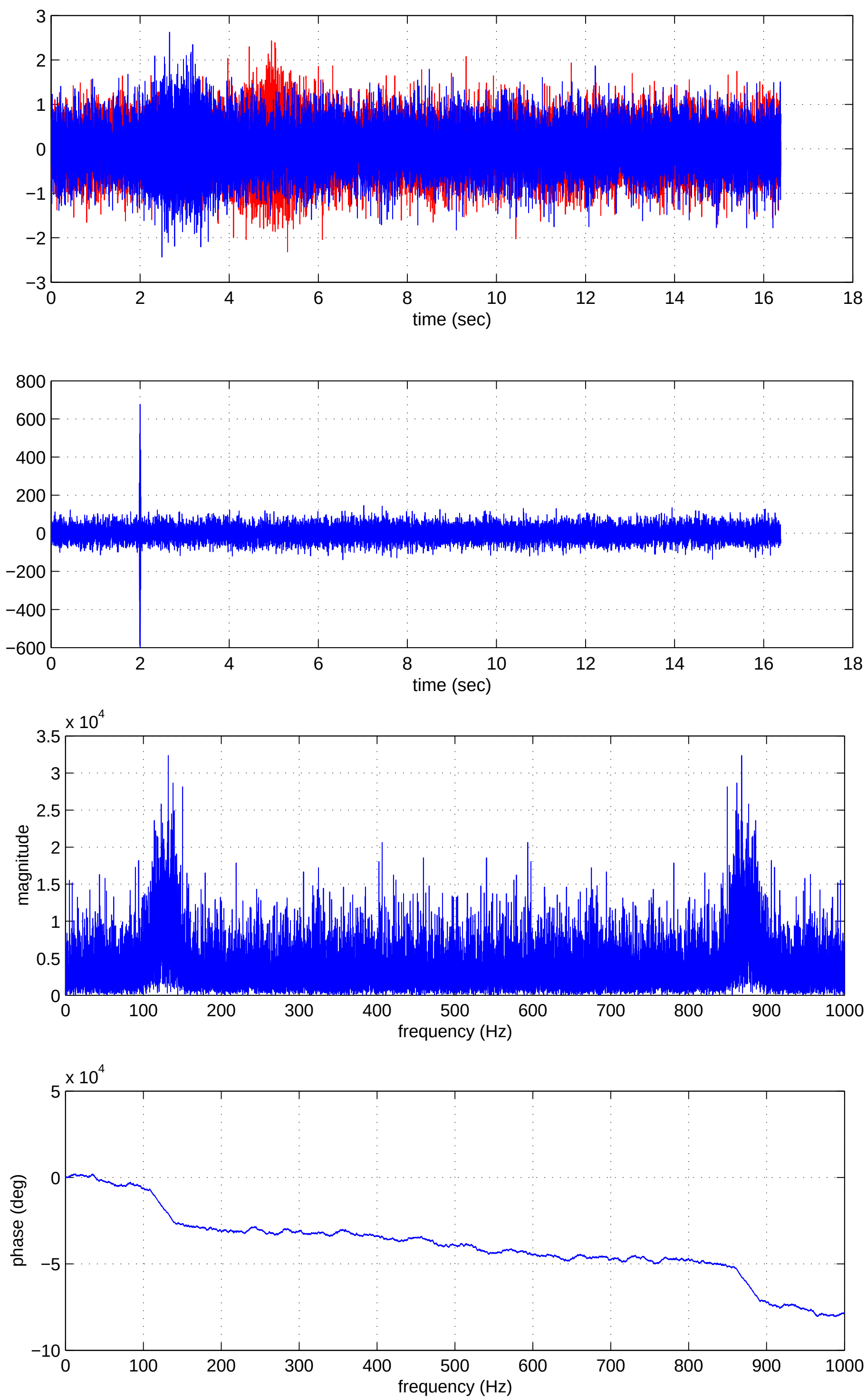
3 detectors:

- three responses: $V_1(t)$, $V_2(t)$, and $V_3(t)$
- three correlation functions
- full separation of $h_+(t)$ and $h_\times(t)$
- redundancy allows to test general relativity

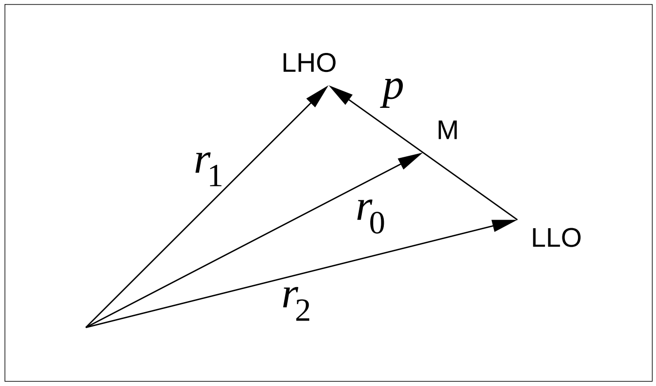
Model Correlation ($\tau_s = 0$)



Model Correlation ($\tau_s > 0$)



Mid-point Parametrization



Vector $\vec{p}$ connects the two detectors:

$$\vec{p} = \vec{r}_1 - \vec{r}_2,$$

Midpoint for two detectors (M):

$$\begin{aligned} \vec{r}_1 &= \vec{r}_0 + \vec{p}/2, \\ \vec{r}_2 &= \vec{r}_0 - \vec{p}/2. \end{aligned}$$

The delay is equally split between the two detectors:

$$\begin{aligned} h(t, \vec{r}_1) &= h(t + \tau_s/2), \\ h(t, \vec{r}_2) &= h(t - \tau_s/2), \end{aligned}$$

where τ_s is the gravitational-wave arrival delay

$$\tau_s = \frac{\vec{n} \cdot \vec{p}}{c}.$$

G.W. signals from two detectors:

$$\begin{aligned} V_1(t) &= F_+(\Omega_1)\, h_+(t + \tau_s/2) + F_\times(\Omega_1)\, h_\times(t + \tau_s/2), \\ V_2(t) &= F_+(\Omega_2)\, h_+(t - \tau_s/2) + F_\times(\Omega_2)\, h_\times(t - \tau_s/2). \end{aligned}$$

Correlation of Signals

Introduce artificial delay: τ (coincidence: $\tau = -\tau_s$)

Correlation function for detectors #1 and #2:

$$C(\tau) = \overline{V_1(t - \tau/2) V_2(t + \tau/2)}.$$

Expanded form:

$$\begin{aligned}
&F_+(\Omega_1)\,F_+(\Omega_2)\ \overline{h_+\left(t + \frac{\tau_s + \tau}{2}\right)\,h_+\left(t - \frac{\tau_s + \tau}{2}\right)} + \\
&F_+(\Omega_1)\,F_\times(\Omega_2)\ \overline{h_+\left(t + \frac{\tau_s + \tau}{2}\right)\,h_\times\left(t - \frac{\tau_s + \tau}{2}\right)} + \\
&F_\times(\Omega_1)\,F_+(\Omega_2)\ \overline{h_\times\left(t + \frac{\tau_s + \tau}{2}\right)\,h_+\left(t - \frac{\tau_s + \tau}{2}\right)} + \\
&F_\times(\Omega_1)\,F_\times(\Omega_2)\ \overline{h_\times\left(t + \frac{\tau_s + \tau}{2}\right)\,h_\times\left(t - \frac{\tau_s + \tau}{2}\right)}.
\end{aligned}$$

Compact form:

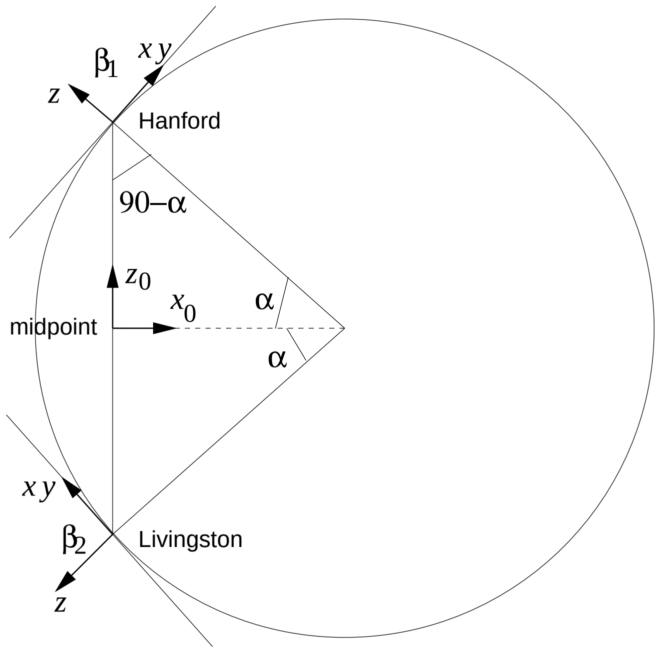
$$C(\tau) = \mathrm{Tr}\left\{\, \mathbf{M}^T\, \mathbf{N}(\tau) \,\right\},$$

where

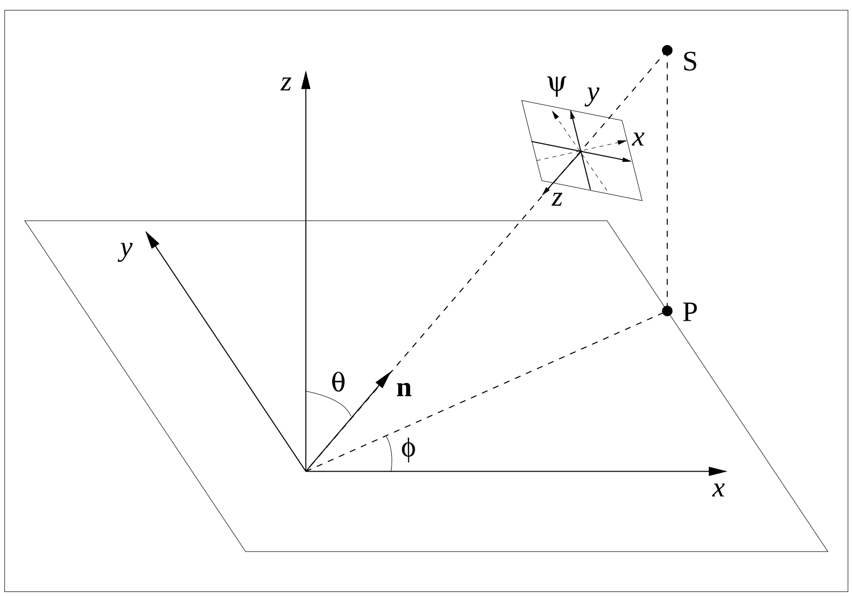
$$\begin{aligned}
\mathbf{M}_{ij} &= F_i(\Omega_1)\,F_j(\Omega_2), \\
\mathbf{N}_{ij} &= \overline{h_i\left(t + \frac{\tau_s + \tau}{2}\right)\,h_j\left(t - \frac{\tau_s + \tau}{2}\right)}.
\end{aligned}$$

Rotational Transformations

Orientation of the midpoint and the detector frames on earth.



Orientation of the GW coordinate system with respect to the midpoint coordinate system.



Induced Metric Transformations

For each detector

$$\mathbf{q}_s = \mathbf{R}\,\mathbf{q}.$$

where (symbolic form)

$$\mathbf{R} = \mathbf{RRRRR}.$$

induced transformation of the metric:

$$\mathbf{H} = \mathbf{R}^T\,\mathbf{H}_s\,\mathbf{R}.$$

the strain tensor in g.w. frame

$$\mathbf{H}_s = h_+\,\mathbf{E}_+ + h_\times\,\mathbf{E}_\times.$$

basis tensors are

$$\mathbf{E}_+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{E}_\times = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For Hanford,

$$\begin{aligned} \mathbf{H}_1^+ &= \mathbf{R}_1^T\,\mathbf{E}_+\,\mathbf{R}_1, \\ \mathbf{H}_1^\times &= \mathbf{R}_1^T\,\mathbf{E}_\times\,\mathbf{R}_1. \end{aligned}$$

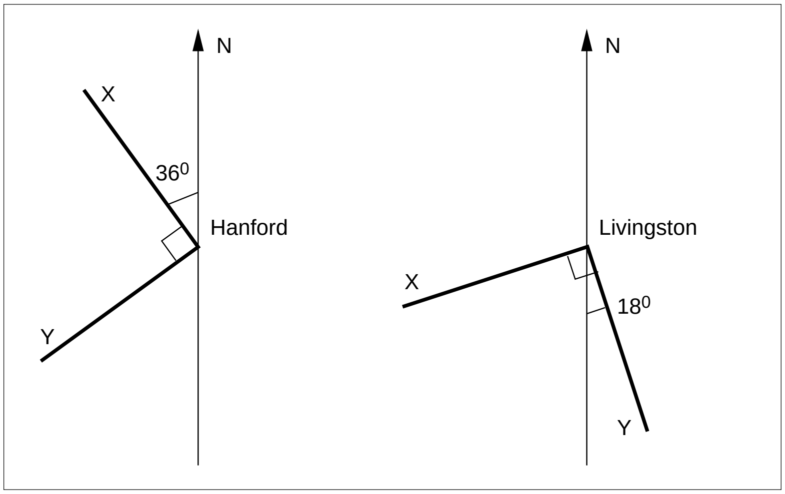
For Livinngston $1 \rightarrow 2$.

The sensitivity function is

$$F_+(\Omega_1) = \mathbf{H}_1^+(1,1) - \mathbf{H}_1^+(2,2),$$

and so on.

Correlation of Antenna Patterns



Average over ϕ and ψ but not θ.

Correlation of directional sensitivities

$$\mathbf{M}_{ij}(\tau_s) = \langle F_i(\Omega_1)\, F_j(\Omega_2)\rangle_{\phi,\psi}.$$

The result is

$$\mathbf{M}(\tau_s) = \begin{bmatrix} a(\tau_s) & b(\tau_s) \\ -b(\tau_s) & a(\tau_s) \end{bmatrix},$$

where $a(\tau_s)$ and $b(\tau_s)$ are polynomials (order ≤ 4).

For any detector locations and orientation angles:

$$\begin{aligned} a(\tau_s) &= a_0 + a_2 \left(\frac{\tau_s}{\tau_0}\right)^2 + a_4 \left(\frac{\tau_s}{\tau_0}\right)^4, \\ b(\tau_s) &= b_1 \left(\frac{\tau_s}{\tau_0}\right) + b_3 \left(\frac{\tau_s}{\tau_0}\right)^3. \end{aligned}$$

τ_0 is maximum delay $\tau_0 = \frac{|\vec{p}|}{c}$

H1-L1 Directional Overlap

Hanford – Livingston correlation:

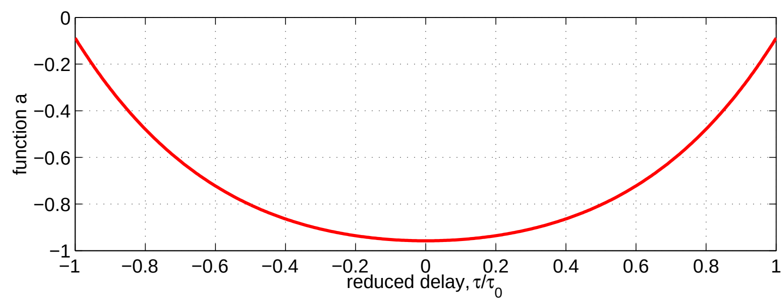
α - angle between Z axes of the detectors,

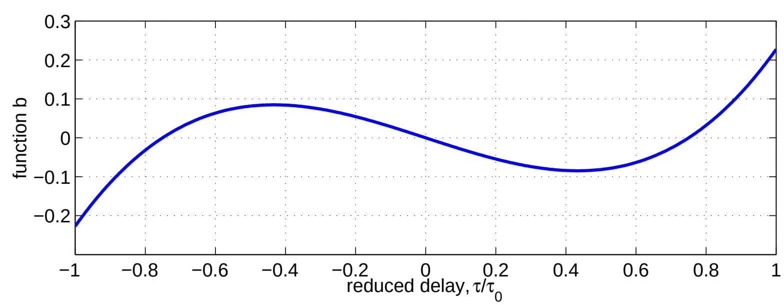
β_1 and β_2 - angles of X axes (arms) with $\vec{p}$:

$$\begin{aligned} \alpha &= 13.61^0, \\ \beta_1 &= 28.42^0, \\ \beta_2 &= -61.52^0. \end{aligned}$$

Numerical calculations yield

$$\begin{aligned} a(\tau_s) &= -0.9577 + 0.5325\left(\frac{\tau_s}{\tau_0}\right)^2 + 0.3369\left(\frac{\tau_s}{\tau_0}\right)^4, \\ b(\tau_s) &= -0.2931\left(\frac{\tau_s}{\tau_0}\right) - 0.5204\left(\frac{\tau_s}{\tau_0}\right)^3. \end{aligned}$$





a and b as functions of τ_s.

Correlation Function

Correlation wave-forms:

$$N_{ij}(\tau) = \overline{h_i\left(t + \frac{\tau_s + \tau}{2}\right)\, h_j\left(t - \frac{\tau_s + \tau}{2}\right)}.$$

Correlation function in time domain:

$$\begin{aligned} C(\tau) \quad = \quad & a(\tau_s)\,[N_{++}(\tau) + N_{\times\times}(\tau)] + \\ & b(\tau_s)\,[N_{+\times}(\tau) \; - \; N_{\times +}(\tau)]\,. \end{aligned}$$

Fourier-domain representation:

$$\tilde{h}_j(\omega) = \int_{-T/2}^{T/2} e^{i\omega t}\, h_j(t)\, dt.$$

Correlation function in Fourier domain:

$$\tilde{C}(\omega) = [a(\tau_s)\,\mathcal{E}(\omega) + ib(\tau_s)\,\mathcal{P}(\omega)]\, e^{i\omega\tau_s}.$$

$\mathcal{E}$ is symmetric component

$$\mathcal{E}(\omega) = |\tilde{h}_+(\omega)|^2 + |\tilde{h}_\times(\omega)|^2,$$

$\mathcal{P}$ is antisymmetric component

$$\mathcal{P}(\omega) = \frac{1}{i}\left[\tilde{h}_+(\omega)\,\tilde{h}^*_\times(\omega) - \tilde{h}_\times(\omega)\,\tilde{h}^*_+(\omega)\right].$$

G.W. Eigenstates of Rotations

Spin - quantum number that defines how the wave transforms under rotations

$$J_x,\ J_y,\ J_z$$

For gravitational waves spin is parallel to the direction of propagation (z-axis).

States with spin-$\uparrow$ and spin-$\downarrow$:

$$\begin{aligned} u(\omega) &= \frac{1}{\sqrt{2}}\,[\tilde{h}_+(\omega) + i\tilde{h}_\times(\omega)], \\ d(\omega) &= \frac{1}{\sqrt{2}}\,[\tilde{h}_+(\omega) - i\tilde{h}_\times(\omega)]. \end{aligned}$$

These are eigen-states of J_z (where z is g.w.-propagation).

Rotation by α along z-axis:

$$\begin{aligned} u(\omega) &\rightarrow e^{+2i\alpha}\,u(\omega), \\ d(\omega) &\rightarrow e^{-2i\alpha}\,d(\omega). \end{aligned}$$

$N_\uparrow + N_\downarrow$ is energy density:

$$\mathcal{E}(\omega) = u(\omega)u(\omega)^* + d(\omega)d(\omega)^*,$$

$N_\uparrow - N_\downarrow$ is spin density:

$$\mathcal{P}(\omega) = u(\omega)u(\omega)^* - d(\omega)d(\omega)^*.$$

Phase of the Correlation Function

Complex representation

$$\tilde{C}(\omega) = \mathcal{R}(\omega)\; e^{i\chi(\omega)}$$

Magnitude of the correlation function

$$\mathcal{R}(\omega) = \sqrt{a(\tau_s)^2\; \mathcal{E}(\omega)^2 + b(\tau_s)^2\; \mathcal{P}(\omega)^2}.$$

Ratio of detected spin density to energy density:

$$r(\omega) = \frac{b(\tau_s)\; \mathcal{P}(\omega)}{a(\tau_s)\; \mathcal{E}(\omega)}.$$

Phase of the correlation function

$$\chi(\omega) = \omega\tau_s + \arctan\left\{r(\omega)\right\}.$$

Slope of χ:

$$\tau_s + \left.\frac{dr}{d\omega}\right|_{\omega=0}.$$

Deviations from the linear trajectory: $r(\omega) \neq 0$

Conclusions

Basic algorithm:

- determine rough delay τ_s from $C(\tau)$
- calculate $a(\tau_s)$ and $b(\tau_s)$
- estimate the slope of $\chi(\omega)$
- apply non-linear fit to estimate $r(\omega)$
- reconstruct $\mathcal{E}(\omega)$ and $\mathcal{P}(\omega)$

Conclusions:

- Linear phase of $C(\omega)$ indicates delay τ_s
- Nonlinear phase of $C(\omega)$ indicates spin density
- Allows to tell conclusively whether g.w. have spin
- Two detectors fix θ for G.W. source
- Three detectors fix θ and ϕ