Modeling of Thermal Noise in Mirror Coatings

LIGO-G010348-00-Z

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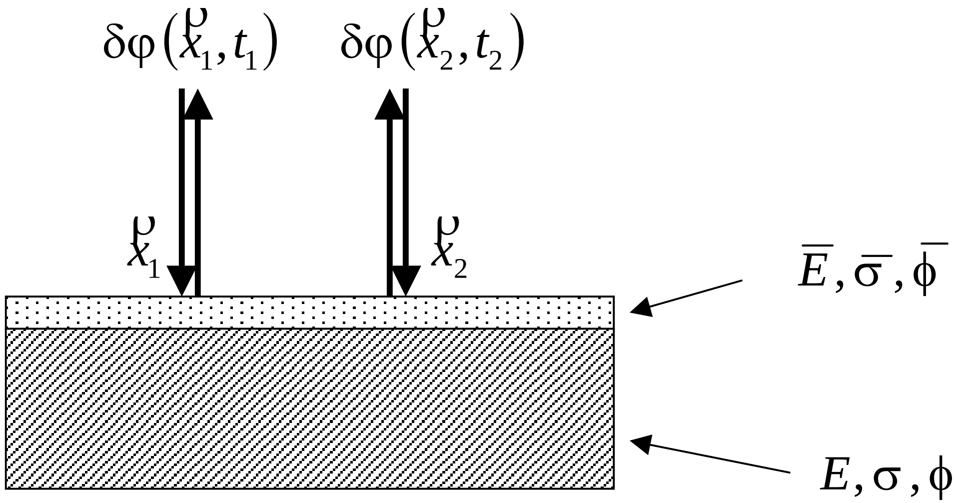
Coating Noise; Problem Statement

- Coating noise model
  - Based on half-space mirror model
    - a lossy layer (thickness *d*) on a lossy host material
  - Requirement: Compute laser phase noise correlation
  - Approach: via analytical Green’s function



$$
\begin{aligned}
&\langle \delta\varphi(\vec{x}_1, t_1)\, \delta\varphi(\vec{x}_2, t_2) \rangle \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} \langle \delta\varphi(\vec{x}_1)\, \delta\varphi(\vec{x}_2) \rangle_\omega \\
&= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} S_\varphi(\omega, \vec{x}_1 - \vec{x}_2)
\end{aligned}
$$

Coating Noise

- Phase Noise Formula

$$S_\varphi(\omega,\vec{r}_1,\vec{r}_2) = 4k^2 \iint dS' \int dS'' \psi_{00}^{w}(\vec{r}'-\vec{r}_1)\psi_{00}^{w}(\vec{r}''-\vec{r}_2)\langle u_n(\vec{r}')u_n(\vec{r}'')\rangle_\omega$$

$$\approx 4k^2 \frac{2k_B T}{\omega} \iint dS' \int dS'' \psi_{00}^{w}(\vec{r}'-\vec{r}_1)\psi_{00}^{w}(\vec{r}''-\vec{r}_2)$$

$$\times \int_V d^3x \left[\partial_k \chi_{li}^{\omega}(\vec{x},\vec{r}';c')\right] c''_{klpq}(\vec{x}) \left[\partial_p \chi_{qj}^{\omega}(\vec{x},\vec{r}'';c')\right]$$

$$c_{ijkl} = c'_{ijkl} - i c''_{ijkl} \approx \left[1 - i\phi(\omega)\right] c'_{ijkl}$$

$S_\varphi(\omega,\vec{r}_1,\vec{r}_2)$	the two-point laser-beam phase-noise power-spectrum correlation
$\langle u_i(\vec{r}')u_j(\vec{r}'')\rangle_\omega$	the displacement spectral correlation
$\vec{r}_1,\vec{r}_2$	The laser beam reflection points (the beam centers)
$\psi_{00}^{w}(\vec{r})\left(\propto e^{-2\lvert\vec{r}\rvert^2/w^2}\right)$	the Gaussian laser-beam profile function

w, k	the laser beam spot size (amplitude radius), and wave number
χ_{ij}^{ω}	elastic Green's function
c_{ijkl} [c'_{ijkl}, c''_{ijkl}]	elastic constants [dispersive and absorptive parts]
$\phi(\omega)$	loss function
k_B, T	Boltzmann constant, and temperature

Coating Noise

- Static Green's function for layer-on-substrate
  - Definition: Green's function and derivatives

$$\Phi_{ijk}(\vec{x},\vec{x}') \equiv c_{ijlm}\partial_l \chi_{mk}(\vec{x},\vec{x}')$$

  - Notation: 2D Fourier transforms in the transverse directions

$$\left[\chi_{ij}(\tilde{p},z,z')\right] \Rightarrow \begin{cases} \chi_{\theta\theta}(\tilde{p},z,z') \\ \begin{bmatrix} \chi_{pp}(\tilde{p},z,z') & \chi_{pz}(\tilde{p},z,z') \\ \chi_{zp}(\tilde{p},z,z') & \chi_{zz}(\tilde{p},z,z') \end{bmatrix} \end{cases}$$

$$\begin{cases} \vec{x} = \{x,y,z\} = \{\tilde{x},z\} \\ \vec{V}(\vec{x}) = \int \frac{d^2p}{(2\pi)^2} e^{i\tilde{p}\tilde{x}} \vec{V}(\tilde{p},z) \\ \vec{V}(\tilde{p},z) = \{V_p, V_\theta, V_z\} \end{cases}$$

$$\left[\Phi_{zij}(\tilde{p},z,z')\right] \Rightarrow \begin{cases} \Phi_{z\theta\theta}(\tilde{p},z,z') \\ \begin{bmatrix} \Phi_{zpp}(\tilde{p},z,z') & \Phi_{zpz}(\tilde{p},z,z') \\ \Phi_{zzp}(\tilde{p},z,z') & \Phi_{zzz}(\tilde{p},z,z') \end{bmatrix} \end{cases}$$

Coating Noise

- Static Green's function for layer-on-substrate (con't)

$$\left[\chi^{sos}_{\cdot\cdot}(\tilde{p},z)\right]=\frac{1}{p\bar{E}}\frac{1+\bar{\sigma}}{1-\bar{\sigma}}\left\{\begin{array}{l}\left[\left(2(1-\bar{\sigma})-pz\cdot i\tau_1\right)G^{sos}(\tilde{p},0)+\frac{1}{2}pz\tau_3\right]\cosh pz\\+\left[\left(pz\tau_3+(1-2\bar{\sigma})\tau_2\right)G^{sos}(\tilde{p},0)+\frac{3-4\bar{\sigma}}{2}-\frac{1}{2}pz(i\tau_1)\right]\sinh pz\end{array}\right\}$$

$$\left[\Phi^{sos}_{z\cdot\cdot}(\tilde{p},z)\right]=\frac{1}{2(1-\bar{\sigma})}\left[2pz\tau_3G^{sos}(\tilde{p},0)+2(1-\bar{\sigma})-pz\cdot i\tau_1\right]\cosh pz$$
$$+\frac{1}{2(1-\bar{\sigma})}\left[2\left[1-pz(i\tau_1)\right]G^{sos}(\tilde{p},0)+pz\tau_3-(1-2\bar{\sigma})\tau_2\right]\sinh pz$$

where

$$G^{sos}(\tilde{p},0)=\Delta^{-1}\left\{\begin{array}{l}\alpha(1-\sigma)\cosh 2pd+\frac{1}{4(1-\bar{\sigma})}\left\{\frac{3-4\bar{\sigma}}{2}+\alpha(1-2\sigma)(1-2\bar{\sigma})+\frac{\alpha^2}{2}(3-4\sigma)\right\}\sinh 2pd\\+\left\{\begin{array}{l}\frac{\alpha}{2}\left[(1-2\sigma)\cosh 2pd+\frac{1-\sigma}{1-\bar{\sigma}}(1-2\bar{\sigma})\sinh 2pd\right]\\+\frac{1}{8(1-\bar{\sigma})^2}\left[(1-2\bar{\sigma})(3-4\bar{\sigma})-2\alpha(1-2\sigma)(3-4\bar{\sigma})+\alpha^2(1-2\bar{\sigma})(3-4\sigma)\right]\sinh^2 pd\\-\frac{1}{8(1-\bar{\sigma})^2}(1-\alpha)\left[1+\alpha(3-4\sigma)\right](pd)^2\end{array}\right\}\tau_2\\+\frac{1}{4(1-\bar{\sigma})}pd(1-\alpha)\left[1+\alpha(3-4\sigma)\right]\tau_3\end{array}\right\}$$

$$\Delta\equiv\left\{\cosh pd-\frac{1}{2(1-\bar{\sigma})}\left[(1-2\bar{\sigma})-\alpha(3-4\sigma)\right]\sinh pd\right\}\left\{\cosh pd+\frac{1}{2(1-\bar{\sigma})}\left[(1-2\bar{\sigma})+\alpha\right]\sinh pd\right\}$$
$$+\frac{1}{4(1-\bar{\sigma})^2}(1-\alpha)\left\{1+\alpha(3-4\sigma)\right\}(pd)^2$$

$$\alpha\equiv\frac{1+\sigma}{1+\bar{\sigma}}\frac{\bar{E}}{E},\quad \tau_{1,2,3}=\text{Pauli matrices}$$

Coating Noise

- Intrinsic Thermal Phase-Noise Estimation

$$
\begin{aligned}
S_{\varphi}^{coating}(\omega,\vec{r}) = 4k^2 \cdot \frac{2k_B T}{\omega}\frac{1}{\sqrt{\pi}\,w}
\times \Bigg\{ & \phi \cdot \frac{1-\sigma^2}{E} e^{-r^2/2w^2} I_0\left(r^2/2w^2\right) \\
& + \left[\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\frac{1}{\bar{E}} + \left(\bar{\phi}-2\phi\right)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}\frac{\bar{E}}{E^2} \right. \\
& \left. \qquad - \phi\,\frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}\frac{1}{E} \right] \frac{d}{\sqrt{\pi}\,w} e^{-r^2/w^2} + O\left(d^2/w^2\right) \Bigg\}
\end{aligned}
$$

$S_{\varphi}^{coating}(\omega,\vec{r})$	The phase noise two-point correlation for a coated half-space mirror; double-sided
E, σ, ϕ	Young's modulus, Poisson ratio, and loss function of the substrate material
$\bar{E}, \bar{\sigma}, \bar{\phi}$	Those of the coating material
d	The coating thickness
$I_0(z)$	The 0-th order modified Bessel function of the first kind; $I_0(0)=1$

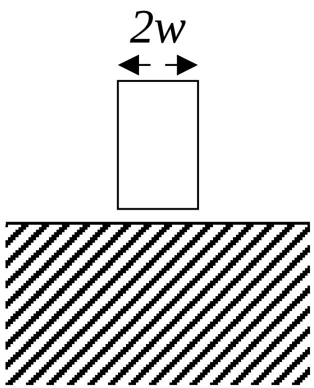
$\omega = 2\pi f$	Frequency
$\vec{r}$	a relative position vector between the two beam centers on the coating surface; $\vec{r}=0$ for a single reflection.
k, w	The laser beam wave number, and spot size (amplitude radius)
k_B, T	The Boltzmann constant and the temperature

Coating Noise

- Interpretation

$$S_\varphi^{coating}(\omega,\vec{r}) = 4k^2 \cdot \frac{2k_B T}{\omega}\frac{1}{\sqrt{\pi}\,w}$$

$$\times\left\{\underbrace{\phi\cdot\frac{1-\sigma^2}{E}e^{-r^2/2w^2}I_0\left(r^2/2w^2\right)}_{(0)}+\left[\begin{array}{l}\underbrace{\bar{\phi}\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\frac{1}{\bar{E}}}_{(1A)}+\underbrace{(\bar{\phi}-2\phi)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}\frac{\bar{E}}{E^2}}_{(1B)}\\ \qquad\qquad -\underbrace{\phi\frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}\frac{1}{\bar{E}}}_{(1C)}\end{array}\right]\frac{d}{\sqrt{\pi}\,w}e^{-r^2/w^2}\right\}$$



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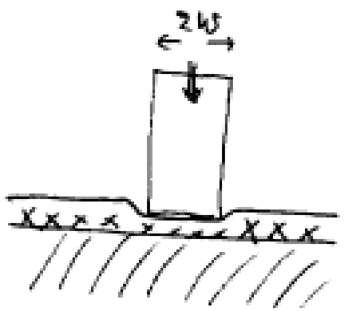
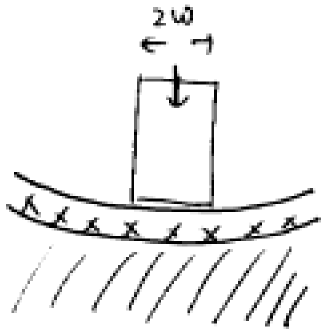
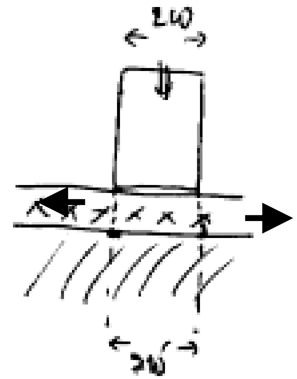


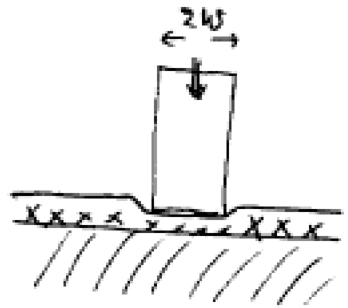




Coating Noise

- Soft coating material

$$\boxed{\varepsilon \equiv \bar{E}/E << 1}$$



$$O(1)+\frac{1}{\bar{E}}\left[\underbrace{\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}}_{(1A)}+\underbrace{(\bar{\phi}-2\phi)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}\varepsilon^2}_{(1B)}\;\underbrace{-\phi\,\frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}\varepsilon}_{(1C)}\right]\frac{d}{\sqrt{\pi}\,w}e^{-r^2/w^2}$$

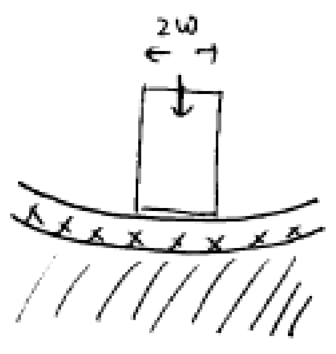
$$\cong O(1)+\underbrace{\bar{\phi}\,\frac{1}{\bar{\rho}\cdot\bar{c}_l^2}\frac{d}{\sqrt{\pi}\,w}}_{(1A)}e^{-r^2/w^2}$$

<u>(1A) = Damping of compressive deformation</u>

Coating Noise

- Stiff coating material

$$\boxed{\delta \equiv E/\bar{E} << 1}$$



$$O(1) + \frac{\bar{E}}{E^2}\left[\underbrace{\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\delta^2}_{(1A)} + \underbrace{(\bar{\phi}-2\phi)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}}_{(1B)} \underbrace{-\phi\,\frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}\delta}_{(1C)}\right]\frac{d}{\sqrt{\pi}\,w}e^{-r^2/w^2}$$

$$\cong O(1) + \underbrace{(\bar{\phi}-2\phi)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}}_{(1B)}\frac{\bar{E}}{E^2}\frac{d}{\sqrt{\pi}\,w}e^{-r^2/w^2}$$

Coating Noise

- Low shear-rigidity material

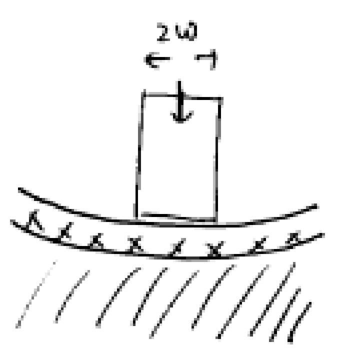
$$\boxed{\bar{\mu} \ll \bar{K}} \qquad \left[\{E,\sigma\} \to \{K,\mu\}: \quad \mu = \frac{E}{2(1+\sigma)}, \quad K = \frac{E}{3(1-2\sigma)}\right]$$

$$S_{\varphi}^{coating}(\omega, \vec{r}) = 4k^2 \cdot \frac{2k_B T}{\omega} \frac{1}{\sqrt{\pi}\, w}$$

$$\times \left\{ \phi \frac{1}{4\mu} \frac{K+4\mu/3}{K+\mu/3} e^{-r^2/2w^2} I_0\left(r^2/2w^2\right) + \left[\underbrace{\bar{\phi} \frac{1}{\bar{K}+4\bar{\mu}/3}}_{(1A)} + \underbrace{(\bar{\phi} - 2\phi) \frac{\bar{K}+\bar{\mu}/3}{\bar{K}+4\bar{\mu}/3} \left(\frac{1}{K+\mu/3} \right)^2 \bar{\mu}}_{(1B)} \underbrace{- \phi \frac{\bar{K} - 2\bar{\mu}/3}{\bar{K}+4\bar{\mu}/3} \frac{1}{K+\mu/3}}_{(1C)} \right] \frac{d}{\sqrt{\pi}\, w} e^{-r^2/w^2} \right\}$$

$$\cong 4k^2 \cdot \frac{2k_B T}{\omega} \frac{1}{\sqrt{\pi}\, w} \left\{ O(1) + \left[\underbrace{\bar{\phi} \frac{1}{\bar{K}}}_{(1A)} + \underbrace{0}_{(1B)} \underbrace{- \phi \frac{1}{K+\mu/3}}_{(1C)} \right] \frac{d}{\sqrt{\pi}\, w} e^{-r^2/w^2} \right\}$$

<u>(1B) = Damping of bending (shear) deformation</u>



(1B)

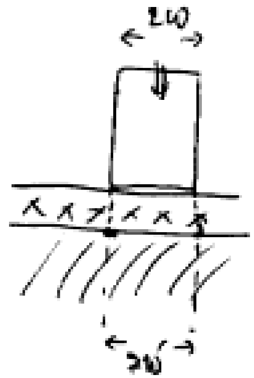
Coating Noise

- Low transversal-deformation material

$\boxed{\bar{\sigma} \approx 0}$

$$O(1) + \left[\underbrace{\bar{\phi} \frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}} \frac{1}{\bar{E}}}_{(1A)} + \underbrace{(\bar{\phi} - 2\phi) \frac{(1+\sigma)^2 (1-2\sigma)^2}{1-\bar{\sigma}^2} \frac{\bar{E}}{E^2}}_{(1B)} - \underbrace{\frac{\phi}{E} \frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}}_{(1C)} \right] \frac{d}{\sqrt{\pi}\, w} e^{-r^2/w^2}$$

$$\cong O(1) + \left[\underbrace{\bar{\phi} \frac{1}{\bar{E}}}_{(1A)} + \underbrace{(\bar{\phi} - 2\phi)(1+\sigma)^2 (1-2\sigma)^2 \frac{\bar{E}}{E^2}}_{(1B)} \underbrace{0}_{(1C)} \right] \frac{d}{\sqrt{\pi}\, w} e^{-r^2/w^2}$$



(1C)

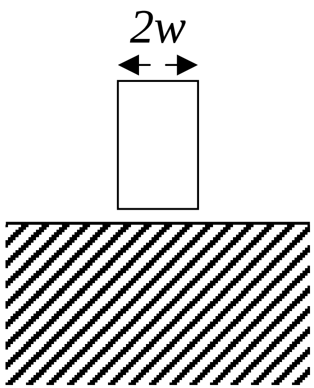
<u>(1C) = Substrate noise; force profile expansion</u>

Coating Noise

- Interpretation; Summary

$$S_{\varphi}^{coating}(\omega,\vec{r}) = 4k^2 \cdot \frac{2k_B T}{\omega}\frac{1}{\sqrt{\pi}\,w}$$

$$\times\left\{\underbrace{\phi\cdot\frac{1-\sigma^2}{E}e^{-r^2/2w^2}I_0\left(r^2/2w^2\right)}_{(0)}+\left[\begin{array}{l}\underbrace{\bar{\phi}\,\frac{(1-2\bar{\sigma})(1+\bar{\sigma})}{1-\bar{\sigma}}\frac{1}{\bar{E}}}_{(1A)}+\underbrace{(\bar{\phi}-2\phi)\frac{(1+\sigma)^2(1-2\sigma)^2}{1-\bar{\sigma}^2}\frac{\bar{E}}{E^2}}_{(1B)}\\ \qquad\qquad -\underbrace{\phi\,\frac{2(1+\sigma)(1-2\sigma)\bar{\sigma}}{1-\bar{\sigma}}\frac{1}{E}}_{(1C)}\end{array}\right]\frac{d}{\sqrt{\pi}\,w}e^{-r^2/w^2}\right\}$$



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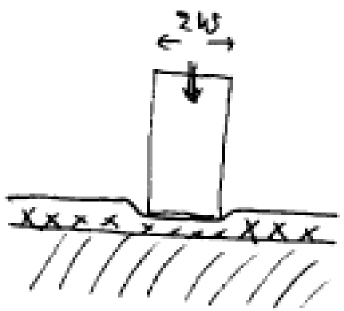
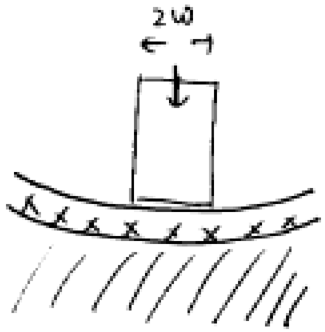
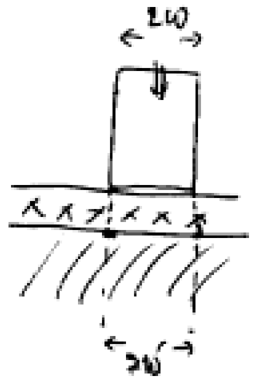


(1A)



(1B)



(1C)

Quarter-space Mirror Model

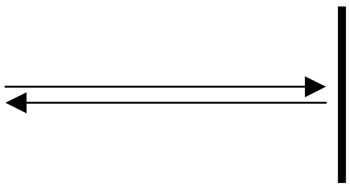
(Work in Progress)

Various Optical Configurations

- Phase noise formulas

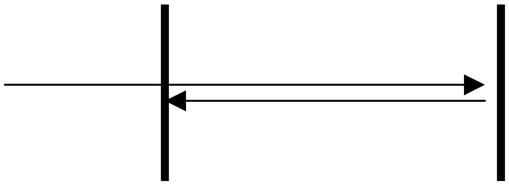
  Computed explicitly for

  – Single-reflection mirror



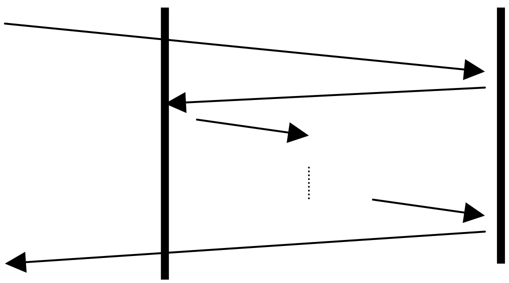
$$S_\varphi^{Single}(\omega) = S_\varphi(\omega, \vec{r}, \vec{r})$$

  – Fabry-Perot resonator



$$S_\varphi^{FP}(\omega) = \left[\frac{(1+r_I)^2}{1+r_I^2}\right]\left[1 - \frac{2r_I}{1+r_I^2}\cos 2\omega\tau\right]^{-1}\left[S_\varphi^E(\omega) + r_I^2 S_\varphi^I(\omega)\right]$$

  – Optical delay line



$$S_\varphi^{DL}(\omega) = \sum_{n=1}^{N} S_\varphi^E(\omega, \vec{r}_n, \vec{r}_n) + 2\sum_{n=2}^{N}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\cdot S_\varphi^E(\omega, \vec{r}_n, \vec{r}_q)$$

$$+\sum_{n=1}^{N-1} S_\varphi^I(\omega, \vec{\rho}_n, \vec{\rho}_n) + 2\sum_{n=2}^{N-1}\sum_{q=1}^{n-1}\cos[2(n-q)\tau\omega]\cdot S_\varphi^I(\omega, \vec{\rho}_n, \vec{\rho}_q)$$

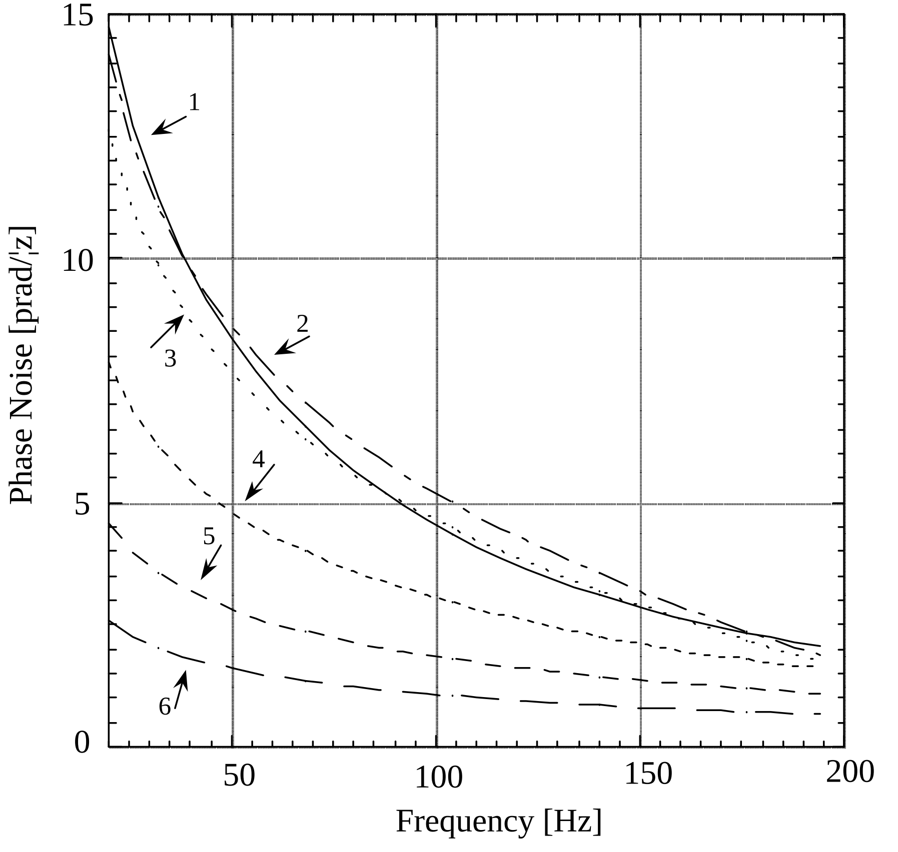
r_I	the input mirror reflection coefficient
τ	the transit time
$S_\varphi^E(\omega), \quad S_\varphi^I(\omega)$	the single-reflection phase noises of the input and end-point mirrors.
$\vec{r}_n$	the positions of the N-time reflections on the end-mirror surface
$\vec{\rho}_p$	the positions of the (N-1)-time reflections on the input-mirror surface
E, σ	Young's modulus, and Poisson ratio

If Half space:

$$S_\varphi(\omega, \vec{r}_1, \vec{r}_2) = \frac{8k_B T}{\sqrt{\pi}}\frac{\phi}{\omega}\frac{k^2}{w}\frac{1-\sigma^2}{E} e^{-(\vec{r}_1-\vec{r}_2)^2/2w^2} I_0\left((\vec{r}_1-\vec{r}_2)^2/2w^2\right)$$

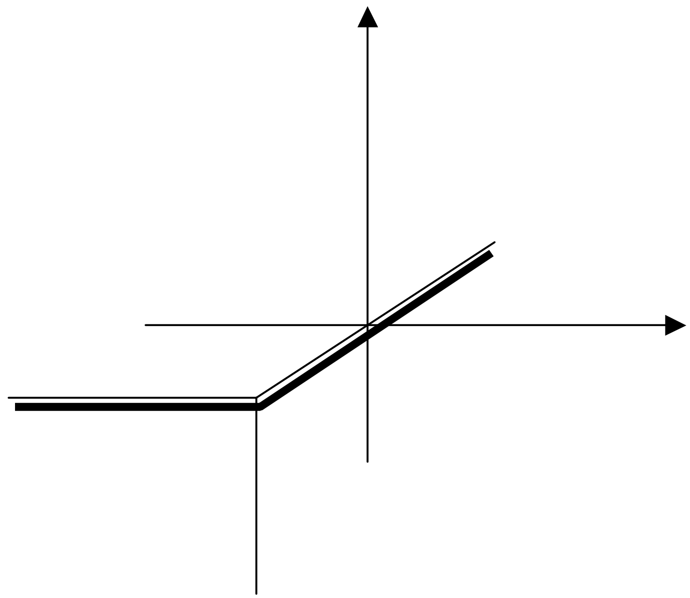
Fabry-Perot vs. Delay Line

- Fabry-Perot vs Delay lines
  - Analytical half-space mirror model
  - Fabry-Perot interferometer vs several delay lines
  - Storage time as proposed for LIGO II.
  - Delay line beam centers
    - evenly spaced
    - on a circle
- When the spots are not overlapping appreciably, the delay line is less noisy than the Fabry-Perot.
  - Noise levels are similar if
    - the spot circle radii comparable to the beam spot size
    - the spots are largely overlapping, and above several hundred Hertz.



Quarter-Space Mirror Model

Scalar model



"displacement"	$u = u(\vec{x})$
"stress"	$\vec{T} = E\vec{\nabla} u$
Field equation	$\rho\ddot{u} - \vec{\nabla}\cdot\vec{T} = 0$
"Young's modulus"	E
Loss function	ϕ

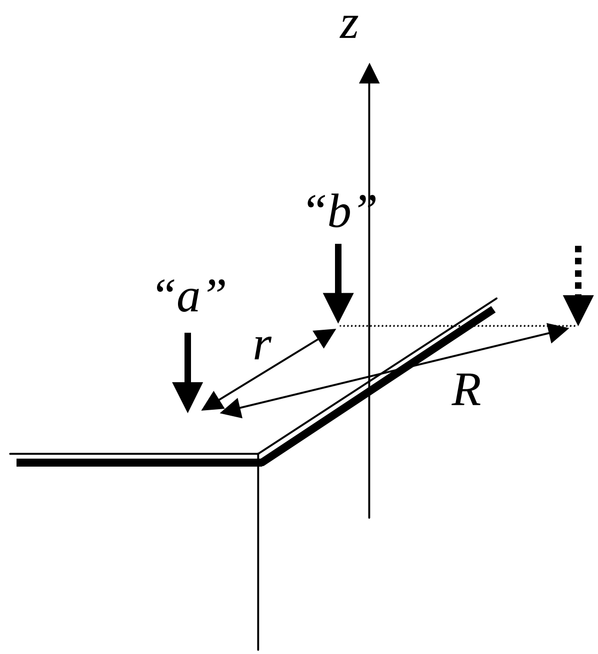
Static Green's function

$$\chi^{qs}(\vec{x},\vec{x}_o) = \frac{1}{4\pi E}\left\{ \begin{array}{l} \dfrac{1}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}} - \varepsilon(z_o)\dfrac{1}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-|z_o|)^2}} \\ -\varepsilon(y_o)\dfrac{1}{\sqrt{(x-x_o)^2+(y-|y_o|)^2+(z-z_o)^2}} + \varepsilon(y_o)\varepsilon(z_o)\dfrac{1}{\sqrt{(x-x_o)^2+(y-|y_o|)^2+(z-|z_o|)^2}} \end{array} \right\}$$

Quarter-Space Mirror Model (con't)

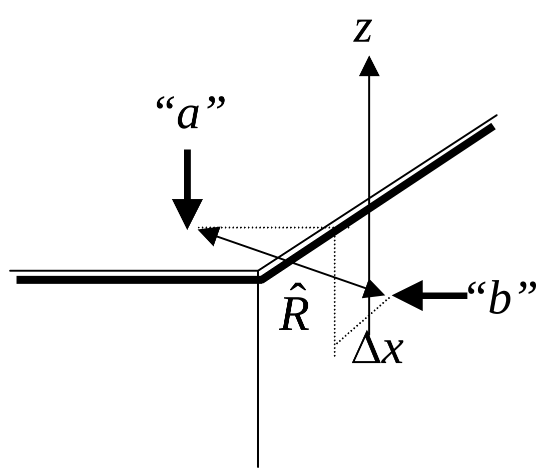
(i) Two beam spots are on the same side (e.g. the top plane)

$$S_{\varphi}^{qs}\left(\omega,\vec{x}_a,\vec{x}_b\right)=4k^2\cdot\frac{2k_BT}{\omega}\cdot\frac{1}{2\sqrt{\pi}}\frac{\phi}{E}\frac{1}{w}$$
$$\times\left[e^{-r^2/2w^2}I_0\left(r^2/2w^2\right)+e^{-R^2/2w^2}I_0\left(R^2/2w^2\right)\right]$$



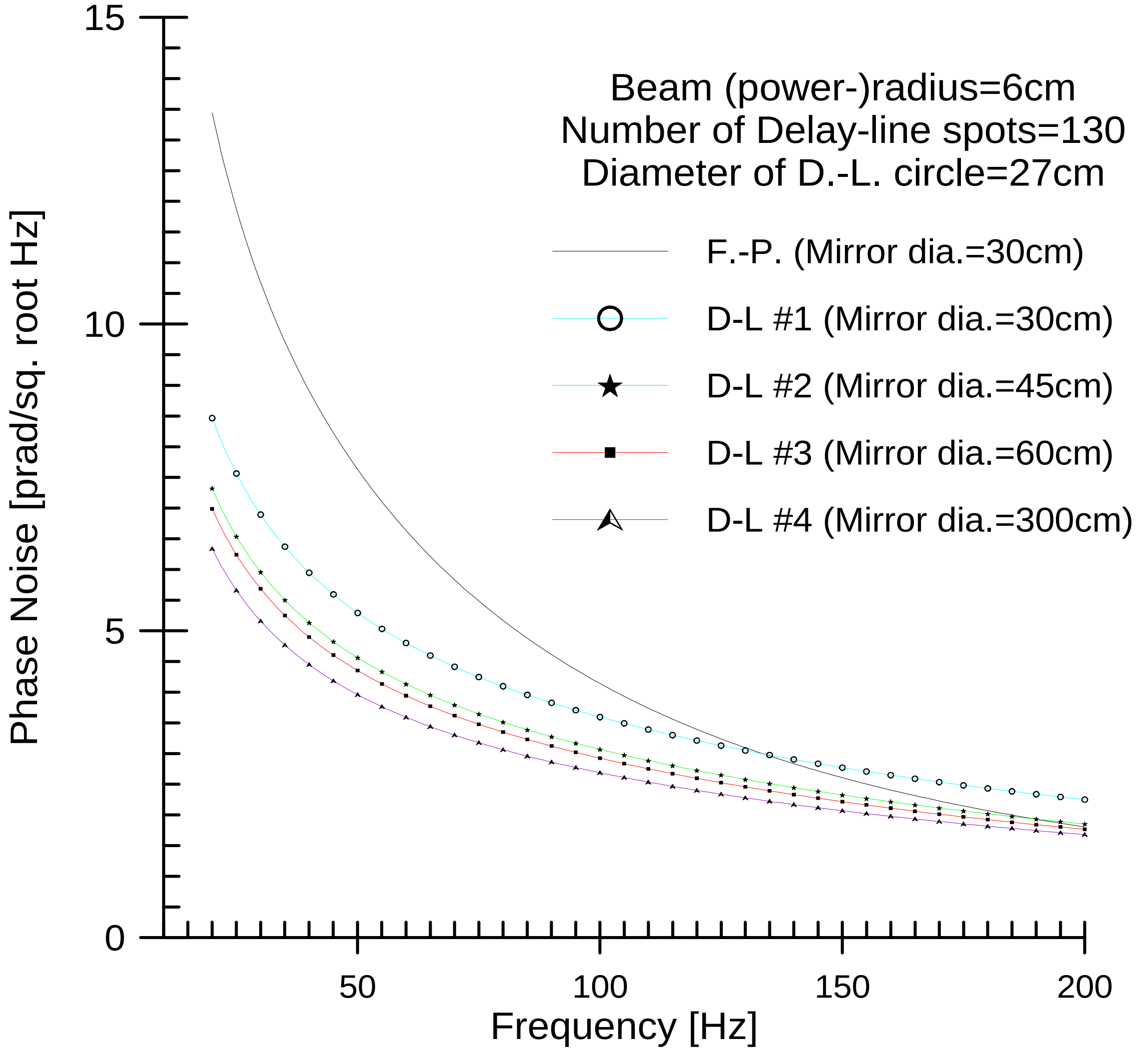
(ii) The opposite sides

$$S_{\varphi}^{qs}\left(\omega,\vec{r}_a,\vec{r}_b\right)\cong 4k^2\cdot\frac{2k_BT}{\omega}\cdot\frac{\phi}{E}\cdot\frac{1}{\sqrt{\pi}\,w}$$
$$\times\begin{cases}e^{-\hat{R}^2/2w^2}I_0\left(\hat{R}^2/2w^2\right)\\ e^{-\hat{R}^2/4w^2}I_0\left(\hat{R}^2/4w^2\right)\end{cases}\text{for}\begin{cases}R_x << R_{yz}\\ R_x >> R_{yz}\end{cases}$$



Quarter-Space Mirror Model (con't)

- Finite-mirror size effect on thermal noise (PRELIMINARY)



Summary

- Coating noise estimation
  - Intrinsic thermal noise
  - Half-space model
  - Dissimilar elastic properties
  - O(coating thickness/beam spot size)
  - Physical meanings explained.
- Finite mirror size effects on thermal noise
  - Preliminary results
  - “Scalar elasticity”
  - Quarter space model
  - Delay-line noise increases as mirror size decrease, but the rate will be tolerable