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Mechanical Simulation Engine : Physics

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1 Potential energy of the elastic line

The motion equation of the elastic line $y(z)$ in tension T is

$$EIy'''' - Ty'' = 0 \tag{1}$$

where E is the Young modulus and I the inertia of the beam[1].

The coordinate system for the motion in the plane y, z is displayed in Fig. 1. With the choice of z oriented upward, the x axis exits from the plane.

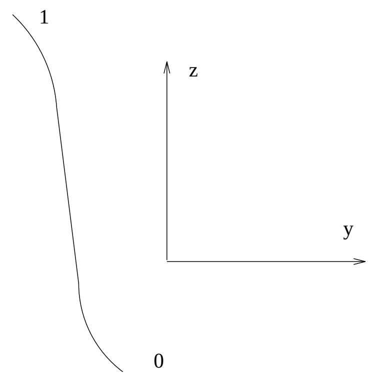

Figure 1: The chosen coordinate system

This equation can be easily derived [1] by a variational principle, introducing the potential energy

$$U = \frac{1}{2}\int_0^L \left[EI\left(y''\right)^2 + Ty'^2\right] dz \tag{2}$$

It is convenient to redefine the unit length introducing

$$z = \zeta L \tag{3}$$

which gives

$$\begin{aligned} U &= \frac{EI}{2L^3}\int_0^1 \left[y''^2 + k^2 y'^2\right] \\ &= \frac{TL}{2}\frac{1}{k^2L^2}\int_0^1 \left[y''^2 + k^2 y'^2\right] \end{aligned} \tag{4}$$

where

$$k^2 \equiv \frac{TL^2}{EI} \ . \tag{5}$$

[1]For a cilyndrical beam of radius r it is $I = (\pi r^4)/4$

1.1 Clamped boundary conditions

The most general solution for the elastic line equation,

$$y'''' - k^2 y'' = 0 \tag{6}$$

satisfying the boundary conditions of clamps with a certain position and orientation

$$\mathbf{y}^T = (y(0),\, y'(0),\, y(1),\, y'(1)) \tag{7}$$

is

$$y(\zeta) = \mathbf{c}^T \cdot \begin{pmatrix} \sinh k\zeta \\ \cosh k\zeta \\ k\zeta \\ 1 \end{pmatrix} \tag{8}$$

where $\mathbf{c}^T = (c_1,\, c_2,\, c_3,\, c_4)$ is a vector of coefficients.
The potential energy can be evaluated as a function of $\mathbf{c}$ in matrix form, by noting that

$$\begin{aligned} y''(\zeta)^2 &= k^4 \mathbf{c}^T \cdot \left[\begin{pmatrix} \sinh k\zeta \\ \cosh k\zeta \\ 0 \\ 0 \end{pmatrix} \otimes (\sinh k\zeta,\, \cosh k\zeta,\, 0,\, 0) \right] \cdot \mathbf{c} \\ &= k^4 \mathbf{c}^T \cdot \mathbf{Y}_2\,(\zeta) \cdot \mathbf{c} \end{aligned} \tag{9}$$

where the non zero entries of the matrix $\mathbf{Y}_2$ are

$$\mathbf{Y}_2[1:2, 1:2] \equiv \begin{bmatrix} \sinh(k\zeta)^2 & \sinh k\zeta \cosh k\zeta \\ \sinh k\zeta \cosh k\zeta & \cosh(k\zeta)^2 \end{bmatrix}. \tag{10}$$

Analogously one finds

$$k^2\,(y')^2 = k^4 \mathbf{c}^T \cdot \mathbf{Y}_1 \cdot \mathbf{c} \tag{11}$$

where

$$\mathbf{Y}_1[1:3, 1:3] = \begin{bmatrix} \cosh(k\,z)^2 & \cosh(k\,z)\,\sinh(k\,z) & \cosh(k\,z) \\ \cosh(k\,z)\,\sinh(k\,z) & \sinh(k\,z)^2 & \sinh(k\,z) \\ \cosh(k\,z) & \sinh(k\,z) & 1 \end{bmatrix} \tag{12}$$

Integrating in ζ one easily finds

$$U = \frac{T L}{2} \frac{k^2}{L^2} \mathbf{c} \cdot \mathbf{A} \cdot \mathbf{c} \tag{13}$$

where

$$\mathbf{A}[1:3, 1:3] = \begin{bmatrix} \frac{\sinh(2\,k)}{2\,k} & \frac{\sinh(k)^2}{k} & \frac{\sinh(k)}{k} \\ \frac{\sinh(k)^2}{k} & \frac{\sinh(2\,k)}{2\,k} & \frac{2\sinh(\frac{k}{2})^2}{k} \\ \frac{\sinh(k)}{k} & \frac{2\sinh(\frac{k}{2})^2}{k} & 1 \end{bmatrix}. \tag{14}$$

To obtain the potential energy as a function of the boundary conditions $\mathbf{x}$, we impose the boundary relations

$$\mathbf{B} \cdot \mathbf{c} = \mathbf{x} \tag{15}$$

where the matrix $\mathbf{B}$ is

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ k & 0 & k & 0 \\ \sinh(k) & \cosh(k) & k & 1 \\ k\cosh(k) & k\sinh(k) & k & 0 \end{bmatrix} . \tag{16}$$

It follows that

$$\begin{aligned} U &= \frac{TL}{2}\frac{k^2}{L^2}\mathbf{x}^T \cdot \left[\left(\mathbf{B}^{-1}\right)^T \cdot \mathbf{A} \cdot \mathbf{B}^{-1}\right] \cdot \mathbf{x} \\ &\equiv \frac{TL}{2}\mathbf{x}^T \cdot \mathbf{W} \cdot \mathbf{x} \end{aligned} \tag{17}$$

where the matrix $\mathbf{W}$ is given by

$$\begin{aligned} \mathbf{W} &= \frac{1}{L^2\left[k\cosh(k/2) - 2\sinh(k/2)\right]} \times \\ & \begin{bmatrix} k\cosh(\frac{k}{2}) & \sinh(\frac{k}{2}) & -\left(k\cosh(\frac{k}{2})\right) & \sinh(\frac{k}{2}) \\ \sinh(\frac{k}{2}) & \frac{(k\cosh(k)-\sinh(k))}{2k\sinh(\frac{k}{2})} & -\sinh(\frac{k}{2}) & \frac{(-k+\sinh(k))}{2k\sinh(\frac{k}{2})} \\ -\left(k\cosh(\frac{k}{2})\right) & -\sinh(\frac{k}{2}) & k\cosh(\frac{k}{2}) & -\sinh(\frac{k}{2}) \\ \sinh(\frac{k}{2}) & \frac{(-k+\sinh(k))}{2k\sinh(\frac{k}{2})} & -\sinh(\frac{k}{2}) & \frac{(k\cosh(k)-\sinh(k))}{2k\sinh(\frac{k}{2})} \end{bmatrix} . \end{aligned} \tag{18}$$

An alternate form, useful for asymptotic expansions, is

$$\begin{aligned} \mathbf{W} &= \frac{k}{L^2\left[k - 2\tanh(\frac{k}{2})\right]} \times \\ & \begin{bmatrix} 1 & \frac{\tanh(k/2)}{k} & -1 & \frac{\tanh(k/2)}{k} \\ \frac{\tanh(k/2)}{k} & \frac{1}{k}\left(\frac{1}{\tanh(k)} - \frac{1}{k}\right) & -\frac{\tanh(k/2)}{k} & \frac{1}{k}\left(\frac{\tanh(k)}{k} - \frac{1}{\cosh(k)}\right) \\ -1 & -\frac{\tanh(k/2)}{k} & 1 & -\frac{\tanh(k/2)}{k} \\ \frac{\tanh(k/2)}{k} & \frac{1}{k}\left(\frac{\tanh(k/2)}{k} - \frac{1}{\cosh k}\right) & -\frac{\tanh(k/2)}{k} & \frac{1}{k}\left(\frac{1}{\tanh k} - \frac{1}{k}\right) \end{bmatrix} \end{aligned} \tag{19}$$

A similar formula hold for the x coordinate: taking into account that the variable z had been rescaled to $z = L\zeta$, and referring to Fig. 1, the correspondence between derivative x', y' and rotation angles around axes $\mathbf{y}, \mathbf{x}$ are

$$\begin{aligned} x'(0,1) &= +L\theta_y(0,1) \\ y'(0,1) &= -L\theta_x(0,1) \end{aligned} \tag{20}$$

Hence the total potential energy of the elastic line is given by

$$
\begin{aligned}
U &= \frac{TL}{2}\left[\mathbf{y}^T \cdot \mathbf{W} \cdot \mathbf{y} + \mathbf{x}^T \cdot \mathbf{W} \cdot \mathbf{x}\right] \\
\mathbf{x} &= (x(0),\, L\theta_y(0),\, x(1), L\theta_y(1)) \\
\mathbf{y} &= (y(0),\, -L\theta_x(0),\, y(1), -L\theta_x(1))
\end{aligned} \tag{21}
$$

1.1.1 Approximations of W

The parameter k is in general rather large. Let us first consider the lowest filter, and assume a steel with approximately $E = 2 \times 10^{11}, \mathrm{GPa}$; we consider for VIRGO a beam with

$$L = .66\mathrm{m},\; r = 0.855 \times 10^{-3} \tag{22}$$

and a suspended mass of about 300 Kg.

$$I = \frac{\pi r^4}{4} \simeq 4.2 \times 10^{-13}\mathrm{m}^4, \quad T \simeq 3 \times 10^3\mathrm{N} \tag{23}$$

and

$$k^2 \equiv \frac{TL^2}{EI} \simeq 1.6 \times 10^4 \;. \tag{24}$$

Considering instead the upper filter, we have

$$I \simeq 7.4 \times 10^{-12}, \quad T =\simeq 10^4\mathrm{N}, \quad L = 1\mathrm{m} \tag{25}$$

and we get

$$k^2 \simeq 6.7 \times 10^3 \;. \tag{26}$$

In general therefore $k \geq 80$, and this allows to expand the hyperbolic functions. We can set $\tanh(k/2) \simeq 1$ and $\cosh(k) \simeq \infty$, obtaining with very good approximation

$$\mathbf{W} = \frac{k}{L^2\,(k-2)} \begin{bmatrix} 1 & \frac{1}{k} & -1 & \frac{1}{k} \\ \frac{1}{k} & \frac{k-1}{k^2} & -\frac{1}{k} & \frac{1}{k^2} \\ -1 & -\frac{1}{k} & 1 & -\frac{1}{k} \\ \frac{1}{k} & \frac{1}{k^2} & -\frac{1}{k} & \frac{k-1}{k^2} \end{bmatrix} \;; \tag{27}$$

this expression can be further expanded in $1/k$.

1.1.2 Large tension limit

In this case we let $k \to \infty$, obtaining

$$\mathbf{W}_0 = \frac{1}{L^2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

that is the potential energy is simply

$$U \simeq U_0 \equiv \frac{TL}{2}\left\{\left[\frac{y(0)-y(1)}{L}\right]^2+\left[\frac{x(0)-x(1)}{L}\right]^2\right\} \tag{29}$$

which is the same formula that would result by considering the elastic line as a rigid constraint of length L. This term corresponds to the gravitational contribution to the horizontal recalling force introduced by Del Fabbro in [2].

The first correction to this formula comes by retaining in the formula 27 the $1/k$ term. We get

$$\mathbf{W} \simeq \mathbf{W}_0 + \mathbf{W}_1 + O(1/k^2) \tag{30}$$

where

$$\mathbf{W}_1 = \frac{1}{k\,L^2}\begin{bmatrix} 2 & 1 & -2 & 1 \\ 1 & 1 & -1 & 0 \\ -2 & -1 & 2 & -1 \\ 1 & 0 & -1 & 1 \end{bmatrix} . \tag{31}$$

Hence the potential is given by

$$U \simeq U_0 + U_1 \tag{32}$$

where

$$\begin{aligned} U_1 = \frac{TL}{2}\frac{1}{k}\Bigg\{&\left[\frac{x(0)-x(1)}{L}+\theta_y(0)\right]^2+\left[\frac{x(0)-x(1)}{L}+\theta_y(1)\right]^2 \\ &+\left[\frac{y(0)-y(1)}{L}-\theta_x(0)\right]^2+\left[\frac{y(0)-y(1)}{L}-\theta_x(1)\right]^2\Bigg\} . \end{aligned} \tag{33}$$

This term corresponds to the one in Eq. 9 of the Del Fabbro work [3], apart a correction in the overall factor: indeed the correct flexural constant H is given by

$$H = \frac{TL}{k} = \sqrt{TEI} \tag{34}$$

while in [3] it is smaller by a factor $1/2$. This error can be traced back to the work by Saulson. Also the signs of the angles differ from [3]; this may be due to a different convention in the orientation of the z axis.

For completeness we report the second correction in $1/k$ to the matrix $\mathbf{W}$,

$$\mathbf{W}_2 = \frac{1}{k^2L^2}\begin{bmatrix} 4 & 2 & -4 & 2 \\ 2 & 1 & -2 & 1 \\ -4 & -2 & 4 & -2 \\ 2 & 1 & -2 & 1 \end{bmatrix} ; \tag{35}$$

we obtain for the potential energy

$$U \simeq U_0 + U_1 + U_2 \tag{36}$$

where

$$U_2 = \frac{TL}{2}\frac{4}{k^2}\left\{\left[\frac{x(0)-x(1)}{L}+\frac{\theta_y(0)+\theta_y(1)}{2}\right]^2+\left[\frac{y(0)-y(1)}{L}-\frac{\theta_x(0)+\theta_x(1)}{2}\right]^2\right\} . \tag{37}$$

Higher order terms go to zero as the tension T increases and will be neglected.

1.2 Mixed boundary: hinged-clamped

A different boundary condition results by having the upper end of the beam *hinged*, that is freely swinging. It corresponds to the boundary conditions

$$\mathbf{y}^T = (y(0), y'(0), y(1)) \tag{38}$$

supplemented by the null momentum condition in the upper suspension, $y''(1) = 0$. The most general solution for this equation is

$$y(\zeta) = \mathbf{c}^T \cdot \begin{pmatrix} \sinh(k\zeta) - \tanh(k)\cos(k\zeta) \\ k\zeta \\ 1 \end{pmatrix} \tag{39}$$

As before, the potential energy is written in terms of the vector $\mathbf{c}$ as

$$U = \frac{TL}{2}\frac{k^2}{L^2}\mathbf{c}^T \cdot \mathbf{A} \cdot \mathbf{c} \tag{40}$$

where

$$\mathbf{A}[1:2,1:2] \equiv \begin{bmatrix} \frac{\tanh(k)}{k} & \frac{\tanh(k)}{k} \\ \frac{\tanh(k)}{k} & 1 \end{bmatrix} . \tag{41}$$

In terms of the vector $\mathbf{y}$, with the usual transformations, we get

$$U = \frac{TL}{2}\mathbf{y}^T \cdot \mathbf{W} \cdot \mathbf{y} \tag{42}$$

where

$$\mathbf{W} = \frac{1}{L^2\left[1-\tanh(k)/k\right]}\begin{bmatrix} 1 & \frac{\tanh(k)}{k} & -1 \\ \frac{\tanh(k)}{k} & \frac{\tanh(k)}{k} & -\frac{\tanh(k)}{k} \\ -1 & -\frac{\tanh(k)}{k} & 1 \end{bmatrix} \tag{43}$$

2 Violin modes: both ends clamped

According to the Landau textbook, we can write the motion equation of the thin wire as

$$\rho A \frac{\partial^2 y(z,t)}{\partial t^2} = -EI\frac{\partial^4 y(z,t)}{\partial z^4} + T\frac{\partial^2 y(z,t)}{\partial z^2} \tag{44}$$

where A is the wire section, hence ρA is the linear mass density. This equation can be derived from the lagrangian

$$\mathcal{L} = \frac{1}{2}\int_0^L \rho A \left(\frac{\partial y(z,t)}{\partial t}\right)^2 dz - \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial^2 y(z,t)}{\partial z^2}\right)^2 + T\left(\frac{\partial y(z,t)}{\partial z}\right)^2\right] dz \ . \tag{45}$$

We look for an oscillating solution at frequency ω, that is

$$EIy'''' - Ty'' = \omega^2 \rho A y \tag{46}$$

is the equation to be solved, with the prescribed boundary conditions.

2.1 Analytic solution

We make the common ansatz for the solution

$$y(z) = \sum_{i=1}^{4} c_i e^{\lambda_i z} \tag{47}$$

which gives the algebraic equation, for each root

$$EI\lambda_i^4 - T\lambda_i^2 - \omega^2 \rho A = 0 \tag{48}$$

whose general solution is

$$\begin{aligned} \lambda_{1,2} &= \pm\sqrt{\frac{T}{2EI}}\sqrt{1+\sqrt{1+\frac{4\omega^2\rho A EI}{T^2}}} \\ i\lambda_{3,4} &= \pm i\sqrt{\frac{T}{2EI}}\sqrt{\sqrt{1+\frac{4\omega^2\rho A EI}{T^2}}-1} \ ; \end{aligned} \tag{49}$$

the solution is therefore of the form

$$y(z) = A\cos\lambda_3 z + B\sin\lambda_3 z + C\cosh\lambda_1 z + D\sinh\lambda_1 z \ ; \tag{50}$$

they are also useful the identities

$$\begin{aligned} \frac{T}{EI} &= \lambda_1^2 - \lambda_3^2 \\ \omega^2\frac{\rho A}{EI} &= \lambda_1^2\lambda_3^2 \ . \end{aligned} \tag{51}$$

One has to further impose the boundary conditions: on the 0 site one sets

$$y(0) = y'(0) \tag{52}$$

which means that

$$y(z) = a\left[\cos\lambda_3 z - \cosh\lambda_1 z\right] + b\left[\lambda_1\sin\lambda_3 z - \lambda_3\sinh\lambda_1 z\right] \ . \tag{53}$$

We further impose $y(L) = 0$, obtaining

$$
\begin{aligned}
y(z) &= a\{[\lambda_1 \sin\lambda_3 L - \lambda_3 \sinh\lambda_1 L][\cos\lambda_3 z - \cosh\lambda_1 z] \\
&\quad - [\cos\lambda_3 L - \cosh\lambda_1 L][\lambda_1 \sin\lambda_3 z - \lambda_3 \sinh\lambda_1 z]\} \qquad (54)
\end{aligned}
$$

and we finally obtain the eigenvalue equation by setting $y'(L) = 0$, which gives the eigenvalue equation

$$
[\lambda_3 \sin\lambda_3 L + \lambda_1 \sinh\lambda_1 L][\lambda_3 \sinh\lambda_1 L - \lambda_1 \sin\lambda_3 L] = \lambda_1\lambda_3 [\cos\lambda_3 L - \cosh\lambda_1 L]^2 \ . \qquad (55)
$$

This equation depends on ω through $\lambda_{1,3}$ and its solutions give the eigenfrequencies of the system.

2.2 Lattice solution

We can introduce a lattice for discretizing the left-hand side over the $z = [0,\, L]$ interval and reduce the problem to a standard eigenvalue problem. We write

$$
\begin{aligned}
y'' &= \frac{1}{a^2}[y(i+a) - 2y(i) + y(i-a)] \ , \\
y'''' &= \frac{1}{a^4}[y(i+2a) - 4y(i+a) + 6y(i) - 4y(i-a) + y(i-2a)] \qquad (56)
\end{aligned}
$$

and we further impose null clamped boundary conditions, for simplicity with zero deviation and angle.

We show in Fig. 2 an example of calculation of the lowest 4 violin frequencies, for a wire with the following characteristics:

$$
\begin{aligned}
T &= 10^4 N \\
r &= 3.5 \times 10^{-3} m \\
E &= 2.0 \times 10^{11} Pascal \\
\rho &= 7.0 \times 10^3 Kg/m^3 \\
L &= 1m \qquad (57)
\end{aligned}
$$

using lattices of different sizes: the full lines are derived from the numerical solution of the eigenvalue equation, while the dotted lines come from the lattice simulation. The lattice solution has been used as a starting point for finding the root in the eigenvalue equation, this is why the two curves follow each other: the meaning is that the accuracy of $1.0e-12$ used in finding the roots is not enough, from a certain lattice on, to distinguish the two solutions.

We see that lowest frequencies stabilise faster.

2.3 Violin modes: finite element approach

In order to evaluate violin mode frequencies, we may attempt a description in terms of finite elements.

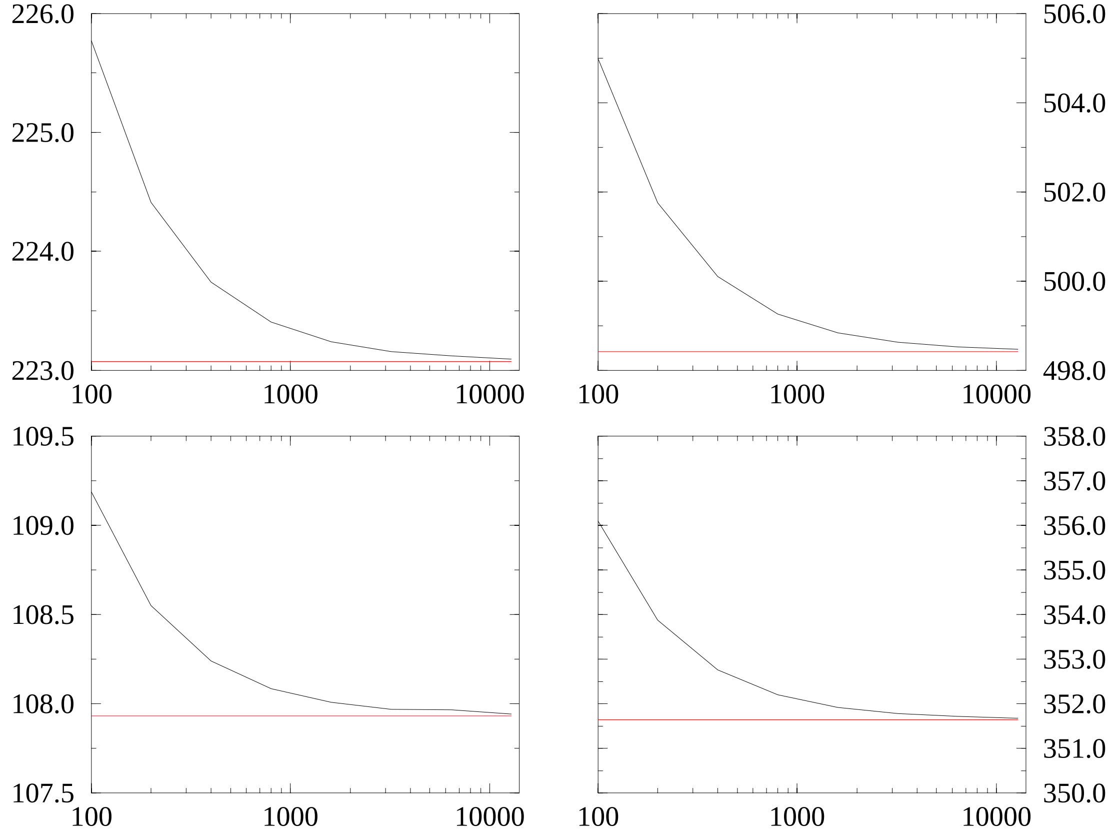

Figure 2: An example calculation of violin frequencies

The idea is to discretize the problem introducing a finite element of length δL, and attempting for each finite element a solution in the form

$$ \tag{58} $$

we can attempt a variable separation, writing the motion in terms of the general solution of the static problem, with time-dependent coefficients.

3 Violins and pendola

We are first interested in the motion equation for a simple pendulum made of a beam with a mass M suspended at the end.

3.1 Lagrangian derivation

Let us write down the Lagrangian for the trasversal motion of a pair of masses connected by a beam under tension: we have

$$\mathcal{L} = \frac{1}{2}M_1\left[\dot{y}(0)\right]^2 + \frac{1}{2}M_2\left[\dot{y}(L)\right]^2 + \frac{1}{2}J_1\left[\dot{y}'(0)\right]^2 + \frac{1}{2}J_2\left[\dot{y}'(L)\right]^2 + \frac{1}{2}\int_0^L \rho A\left[\dot{y}(z)\right]^2 dz - \frac{1}{2}\int_0^L EI\left[y''(z)\right]^2 + T\left[y'(z)\right]^2 dz \; ; \tag{59}$$

the motion equations are obtained by varying $y(z)$ and considering the variations δy, $\delta y'$ as independent.

We obtain, apart total derivatives with respect to the time,

$$\begin{aligned}\delta\mathcal{L} &= \left[-M_1\ddot{y}(0) - EIy'''(0) + Ty'(0)\right]\delta y(0) + \left[-M_2\ddot{y}(L) + EIy'''(L) - Ty'(L)\right]\delta y(L) \\ &+ \left[-J_1\ddot{y}'(0) + EIy''(0)\right]\delta y'(0) + \left[-J_2\ddot{y}'(L) - EIy''(L)\right]\delta y'(L) \\ &- \int_0^L \left[\rho A\ddot{y}(z) + EIy''''(z) - Ty''(z)\right]\delta y(z) dz \; .\end{aligned} \tag{60}$$

Let us start limiting to an infinite mass and inertia at the suspension point, that is a clamped wire: the motion equation along the beam is as before

$$EIy'''' - Ty'' = -\rho A\ddot{y} \tag{61}$$

with the boundary conditions

$$\begin{aligned} y(0) &= y'(0) = 0 \qquad \text{clamped upper end} \\ y''(L) &= 0 \qquad \text{no momentum in the lower end} \\ EI\,y'''(L) - Ty'(L) &= M\ddot{y}(L) \qquad \text{suspended mass inertia ;}\end{aligned} \tag{62}$$

it is assumed that the tension is $T = M\,g$, that is it is given by the weight of the mass.

3.2 Analytical solution

We have already found the general solution for Eq. 61, imposing the boundary conditions in 0, that is

$$y(z) = a\left[\cos\lambda_3 z - \cosh\lambda_1 z\right] + b\left[\lambda_1\sin\lambda_3 z - \lambda_3\sinh\lambda_1 z\right] \; , \tag{63}$$

with λ_1, λ_3 given in Eq. 49. The conditions $y''(L) = 0$ gives

$$\begin{aligned} y(z) &= \lambda_1\lambda_3\left[\lambda_3\sin\lambda_3 L + \lambda_1\sinh\lambda_1 L\right]\left[\cos\lambda_3 z - \cosh\lambda_1 z\right] \\ &- \left[\lambda_3^2\cos\lambda_3 L + \lambda_1^2\cosh\lambda_1 L\right]\left[\lambda_1\sin\lambda_3 z - \lambda_3\sinh\lambda_1 z\right] \; .\end{aligned} \tag{64}$$

We finally impose the condition coming from the inertia of the suspended mass, that is

$$EIy'''(L) - Ty'(L) + \omega^2 My(L) = 0 \tag{65}$$

which needs to be solved numerically to find the eigenfrequencies.

3.3 Lattice approach

As before we may attempt also a lattice formulation: we assume that the lattice contains N sites, numbered $y(-1)$, $y(0)$, $y(1), ...y(N)$, $y(N+1)$; the simulation lattice goes from $y(1)$ to $y(N)$, and we use the extra sites to impose the boundary conditions

$$\begin{aligned} y(-1) = y(0) &= 0 \\ y(N+1) - 2y(N) + y(N-1) &= 0 \\ \frac{EI}{a^3}\left[y(N+1) - 3y(N) + 3y(N-1) - y(N-2)\right] - \frac{T}{a}\left[y(N) - y(N-1)\right] &= -\omega^2 M y(N); \end{aligned} \tag{66}$$

we can therefore eliminate $y(N+1)$ in the third equation, obtaining

$$\frac{EI}{a^4}\left[y(N-2) - 2y(N-1) + y(N)\right] + \frac{T}{a^2}\left[y(N) - y(N-1)\right] = \omega^2 \frac{M}{a} y(N) \ ; \tag{67}$$

while in the bulk we have the equation

$$\begin{aligned} \frac{EI}{a^4} \quad & \left[y(i-2) - 4y(i-1) + 6y(i) - 4y(i+1) + y(i+2)\right] \\ & -\frac{T}{a^2}\left[y(i-1) - 2y(i) + y(i+1)\right] = \omega^2 \rho A y(i) \end{aligned} \tag{68}$$

for $i \in [1,\, N-2]$, and for $i = N-1$

$$\begin{aligned} \frac{EI}{a^4} \quad & \left[y(N-3) - 4y(N-2) + 5y(N-1) - 2y(N)\right] \\ & -\frac{T}{a^2}\left[y(N-2) - 2y(N-1) + y(N)\right] = \omega^2 \rho A y(N-1) \end{aligned} \tag{69}$$

We have to find the eigenvalues of a problem in the form

$$\mathbf{A} \cdot \mathbf{y} = \omega^2 \mathbf{B} \cdot \mathbf{y} \tag{70}$$

where the indices of the matrices range in $[1:N,\, 1:N]$; explicitly

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \cdots \\ & \vdots & & 1 & 0 \\ 0 & & & 0 & \frac{M}{a\rho A} \end{bmatrix} ,$$

$$\mathbf{A} = \begin{bmatrix} 6\kappa_1 + 2\kappa_2 & -4\kappa_1 - \kappa_2 & \kappa_1 & 0 & \cdots & & \\ -4\kappa_1 - \kappa_2 & 6\kappa_1 + 2\kappa_2 & -4\kappa_1 - \kappa_2 & \kappa_1 & 0 & \cdots & \\ \kappa_1 & -4\kappa_1 - \kappa_2 & 6\kappa_1 + 2\kappa_2 & -4\kappa_1 - \kappa_2 & \kappa_1 & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \kappa_1 & -4\kappa_1 - \kappa_2 & 5\kappa_1 + 2\kappa_2 & -2\kappa_1 - \kappa_2 \\ 0 & \cdots & & 0 & \kappa_1 & -2\kappa_1 - \kappa_2 & \kappa_1 + \kappa_2 \end{bmatrix}$$

where

$$\kappa_1 = \frac{EI}{a^4 \rho A}, \qquad \kappa_2 = \frac{T}{a^2 \rho A} \; ; \tag{71}$$

note that **A** is symmetric, **B** is symmetric and positive definite. Then in general the problem has only real eigenvalues, as it should be.

In the lattice approach it is easy to account for a tension varying along the beam by effect of the mass of the beam itself. In the matrix **A** one inserts a κ_2 depending on the site i. It is necessary to ensure that the matrix remains symmetric! A simple way is to take the average of the two secondary diagonals.

In Fig. 3 we show the first three lowest resonances for the wire suspending the seventh filter in the VIRGO Super Attenuator: the simulation is performed taking into account the weight of the beam, and the diagrams show the convergence of the method varying the lattice size.

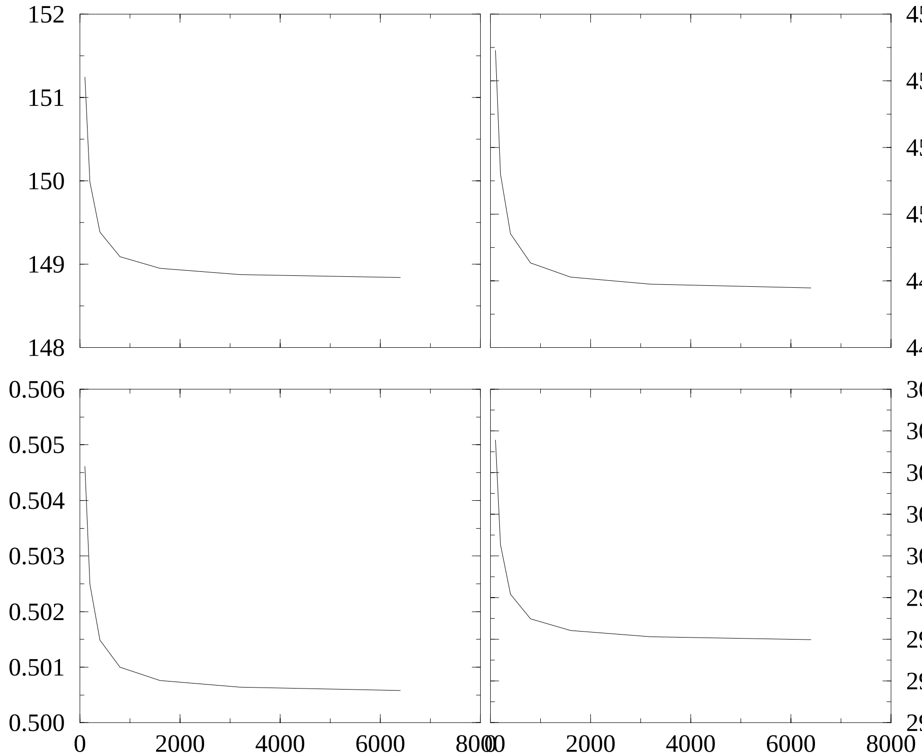

Figure 3: Convergence of eigenfrequencies: filter 7

The parameters chosen are

$$\begin{aligned} M &= 223 Kg \\ r &= 2.0 \times 10^{-3} m \\ E &= 2.0 \times 10^{11} Pascal \end{aligned}$$

$$\rho = 8.0 \times 10^3 Kg/m^3$$
$$L = 1m \ . \tag{72}$$

In Fig. 4 we show the spectrum of the lowest resonances in dependence on the filter index; indices $1-7$ refer to filters $1-7$, index 8 to the marionetta

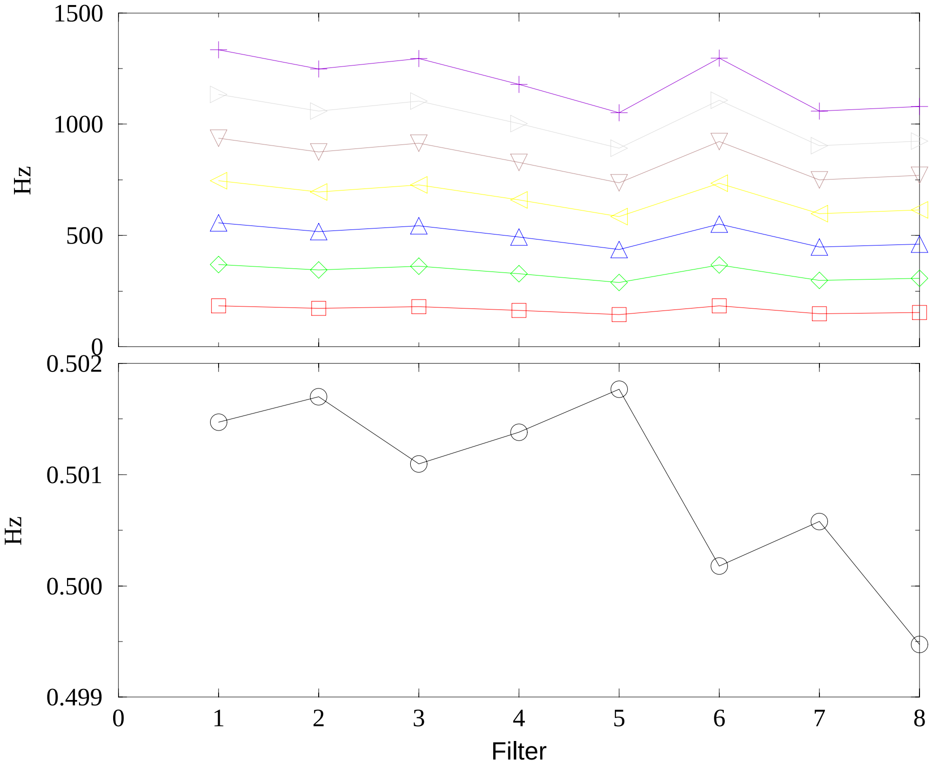

Figure 4: Spectrum in dependence of the filter

We note that going to lower filters the tension lowers and correspondingly the eigenfrequencies.

3.4 Transfer function

The computation of the eigenfrequencies allows to obtain the transfer function of the system, according to the discussion in App. A.

The analytical approach is simple: we go back to the resolution of the system and we impose a sinusoidal motion

$$y(0,t) = \sin \omega t, \qquad y'(0) = 0 \ : \tag{73}$$

a solution satisfying the motion equation in the bulk and the boundary condition given above is

$$y(z,t) = \sin \omega t \left\{ [a \cos \lambda_3 z + (1-a) \cosh \lambda_1 z] + b \left[\lambda_1 \sin \lambda_3 z - \lambda_3 \sinh \lambda_1 z \right] \right\} \tag{74}$$

to set the variables a, b we impose the boundary conditions on the motion of the suspended mass:

$$y''(L,t) = 0 \tag{75}$$

and, using the identities in Eq. 51

$$y'''(L,t) - \left(\lambda_1^2 - \lambda_3^2\right) y'(L,t) + \frac{M}{\rho A}\lambda_1^2\lambda_3^2 y(L,t) = 0 \ . \tag{76}$$

We obtain

$$y(L) = -\frac{\left(\lambda_1^2+\lambda_3^2\right)\left(\lambda_3^2\cos(\lambda_3 L)+\lambda_1^2\cosh(\lambda_1 L)\right)}{\left\{\begin{array}{c} 2\lambda_1^2\lambda_3^2 + \left(\lambda_1^4+\lambda_3^4\right)\cos(\lambda_3 L)\cosh(\lambda_1 L) + \lambda_1\lambda_3\left(\lambda_1^2-\lambda_3^2\right)\sin(\lambda_3 L)\sinh(\lambda_1 L) + \\ +\frac{M}{\rho A}\lambda_1\lambda_3\left(\lambda_1^2+\lambda_3^2\right)\left[\lambda_3\cos(\lambda_3 L)\sinh(\lambda_1 L) - 1\sin(\lambda_3 L)\cosh(\lambda_1 L)\right]\end{array}\right\}} \ . \tag{77}$$

This result should be compared with the transfer function for a pendulum made of a wire with $E = 0$:

$$y(L) = \frac{1}{\cos\kappa L - \frac{M\omega}{\sqrt{\rho A T}}\sin\kappa L} \tag{78}$$

where $\kappa = \omega\sqrt{\frac{\rho A}{T}}$; we show in Fig. 5 an example of computation of the violin modes for the beam holding the first filter in the Pisa Super Attenuator: the relevant parameters are $L = 1$ m, $M = 1040$ Kg and $r = 1.75$ mm;

we see that the effect of the flexural term in the motion equation for the beam is very mild: the tension dominates and at least for the frequency range interesting in VIRGO the transfer function is well approximated by Eq. 78.

We can compare this result with the marionetta stage, having the following characteristics: $L = 1$ m, $M = 60$ Kg, $r = 0.5$ mm: we see in Fig 6 that also in this case the effect of the flexural term is almost irrelevant.

3.4.1 Inclusion of rocking modes

A non zero momentum of inertia J for the suspended mass changes the boundary condition at the suspended mass: in the frequency domain

$$EIy''(L) = \omega^2 J y'(L) \tag{79}$$

is the equation of motion for the rotational degree of freedom.

The transfer function results

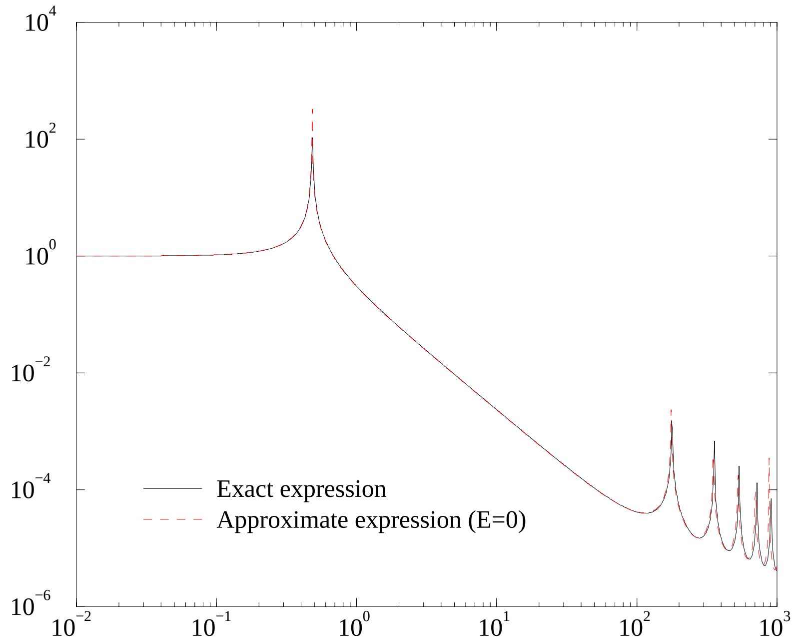

Figure 5: First filter transfer function

$$y(L) = -\frac{\left(\lambda_1^2+\lambda_3^2\right)\left[\lambda_3^2\,\cos(\lambda_3 L)+\lambda_1^2\,\cosh(\lambda_1 L)-\frac{J}{\rho A}\,\lambda_1^2\,\lambda_3^2\left(\lambda_3\sin(\lambda_3 L)+\lambda_1\,\sinh(\lambda_1 L)\right)\right]}{\left\{\begin{array}{c} 2\lambda_1^2\lambda_3^2+\left(\lambda_1^4+\lambda_3^4\right)\cos(\lambda_3 L)\cosh(\lambda_1 L)+\lambda_1\lambda_3\left(\lambda_1^2-\lambda_3^2\right)\sin(\lambda_3 L)\sinh(\lambda_1 L)+ \\ +\frac{M}{\rho A}\lambda_1\lambda_3\left(\lambda_1^2+\lambda_3^2\right)\left[\lambda_3\cos(\lambda_3 L)\sinh(\lambda_1 L)-1\sin(\lambda_3 L)\cosh(\lambda_1 L)\right]+ \\ +\frac{J}{\rho A}\lambda_1\lambda_3\left\{\lambda_1\left(\lambda_1^2\cosh(\lambda_1 L)\sin(\lambda_3 L)+\lambda_3^2\cosh(\lambda_1 L)\sin(\lambda_3 L)\right)+\right. \\ \left.+\lambda_3\left(\lambda_1^2\sinh(\lambda_1 L)\cos(\lambda_3 L)-\lambda_3^2\sinh(\lambda_1 L)\cos(\lambda_3 L)\right)+\right. \\ \left.+\frac{M}{\rho A}\left[\lambda_1^3\lambda_3^3\left(\cos(\lambda_3 L)\cosh(\lambda_1 L)-1\right)+\lambda_1^2\lambda_3^2\left(\lambda_1^2+\lambda_3^2\right)\sin(\lambda_3 L)\sinh(\lambda_1 L)\right]\right\} \end{array}\right\}}. \tag{80}$$

This very complicated expression is plotted in Fig. 7, for the case of a filter of $M = 1040$ Kg and a momentum of inertia of 8.63 Kg m^2, and a suspending beam with $r = 1.75$ mm. We show only, on a linear frequency scale, the ranges of frequencies around the principal resonance and the first violin mode: note the appearance of a new resonance corresponding to the rocking mode, as well as the shift in the position of violin modes.

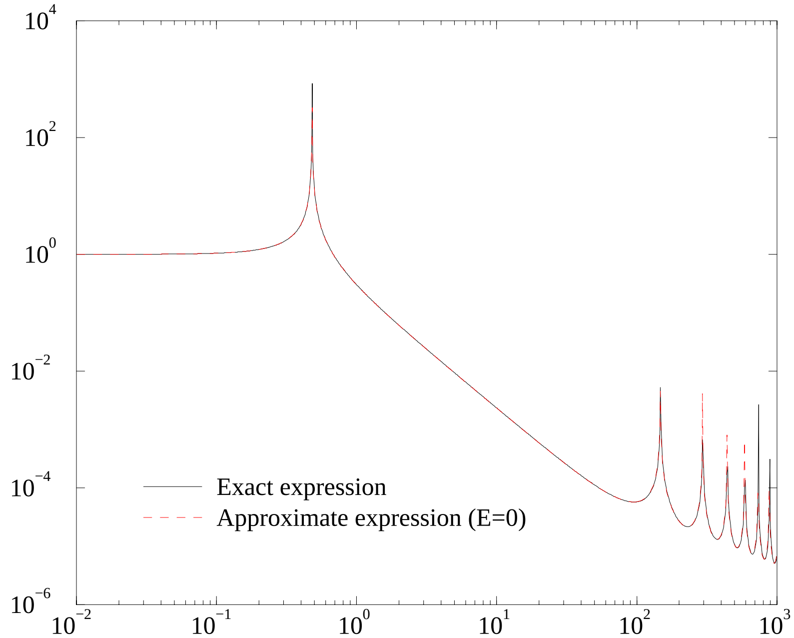

Figure 6: "Marionetta" transfer function

3.5 Double pendulum

The next exercise is to consider a double pendulum, made of a pair of subsequent pendola of lengths L_1, L_2 and masses M_1, M_2.

Again we are interested in computing the resonances and the transfer functions. We set for simplicity $E = 0$, then the equations for the first wire are

$$\begin{aligned} T_1 y_1''(z,t) &= \rho_1 A_1 \ddot{y}_1(z,t) \\ y_1(0) &= 0 \\ T_2 y_2'(0,t) - T_1 y_1'(L_1,t) &= M_1 \ddot{y}_1(L_1,t) \end{aligned} \tag{81}$$

while for the second wire (we choose the origin of z at the upper end of each wire separately)

$$\begin{aligned} T_2 y_2''(z,t) &= \rho_2 A_2 \ddot{y}_2(z,t) \\ y_2(0,t) &= y_1(L_1,t) \\ T_2 y_2'(L_2,t) &= -M_2 \ddot{y}_2(L_2,t) \end{aligned} \tag{82}$$

and

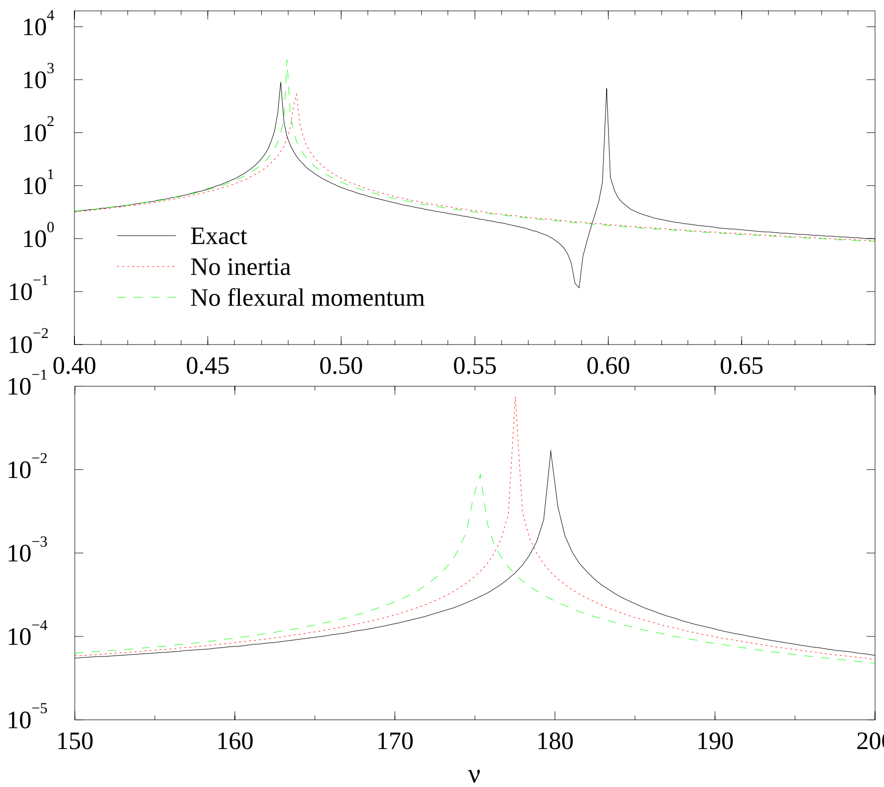

Figure 7: First filter transfer function, with rocking

$$T_1 = g\,(M_1 + M_2)\,, \qquad T_2 = gM_2\ . \tag{83}$$

We define

$$\kappa_{1,\,2} = \omega\sqrt{\frac{\rho_{1,2}A_{1,2}}{T_{1,2}}} \tag{84}$$

and rewrite the equations as

$$\begin{aligned}
y_1''(z) &= -\kappa_1^2 y_1(z)\\
y_1(0) &= 0\\
\mu y_2'(0) - (1+\mu)\,y_1'(L_1) &= -\frac{\omega^2}{g}y_1(L_1)\\
y_2''(z) &= -\kappa_2^2 y_2(z)\\
y_2(0) &= y_1(L_1)\\
y_2'(L_2) &= \frac{\omega^2}{g}y_2(L_2)\ ;
\end{aligned} \tag{85}$$

where $\mu = M_2/M_1$; note that there is no reason to assume that the line formed by the two wires is differentiable in M_1.

The general solution is, apart an overall factor

$$
\begin{aligned}
y_1(z) &= \alpha \sin \kappa_1 z \\
y_2(z) &= y_1(L_1) \cos \kappa_2 z + \beta \sin \kappa_2 z \ .
\end{aligned}
\tag{86}
$$

with

$$
\begin{aligned}
\alpha &= \kappa_2 \cos(\kappa_2 L_2) - \frac{\omega^2}{g} \sin(\kappa_2 L_2) \\
\beta &= \frac{\omega^2}{g} \cos(\kappa_2 L_2) \sin(\kappa_1 L_1) + \kappa_2 \sin(\kappa_1 L_1) \sin(\kappa_2 L_2)
\end{aligned}
\tag{87}
$$

as imposed by the motion equation for the mass 2.
Imposing further the equation of motion for the mass 1 we obtain the eigenvalue equation

$$
\begin{aligned}
&\left\{g^2 \left[\kappa_2^2 \, \mu - \kappa_1 \, \kappa_2 \ (1+\mu)\right] - \omega^4\right\} \cos(\kappa_1 \, L_1 - \kappa_2 \, L_2) + \\
&\quad - \left\{g^2 \left[\kappa_2^2 \, \mu + \kappa_1 \, \kappa_2 \ (1+\mu)\right] - \omega^4\right\} \cos(\kappa_1 \, L_1 + \kappa_2 \, L_2) + \\
&\quad - g \ (1+\mu) \ \omega^2 \ \left[(\kappa_1 - \kappa_2) \ \sin(\kappa_1 \, L_1 - \kappa_2 \, L_2) - (\kappa_1 + \kappa_2) \ \sin(\kappa_1 \, L_1 + \kappa_2 \, L_2)\right] = 0
\end{aligned}
\tag{88}
$$

for instance, considering the values valid for the two upper filters, $M_1 = 142$, $M_2 = 898$ Kg, $L_{1,2} = 1$ m, $r_{1,2} = 1.75$ mm, one obtains, up to 1 kHz, the resonances listed in the table.

Res. #	ν (Hz)	Res. #	ν (Hz)
0	0.35882	7	546.023
1	1.87314	8	676.503
2	169.136	9	728.028
3	182.016	10	910.034
4	338.256	11	1014.75
5	364.018	12	1183.88
6	507.379	13	1274.05

3.6 Double pendulum TF

In order to compute the transfer function we simply set $y_1(0) = 1$, omitting the sinuisoidal time dependence.

The general solution in the bulk is

$$
\begin{aligned}
y_1(z) &= \cos(\kappa_1 z) + \beta_1 \sin(\kappa_1 z) \\
y_2(z) &= [\cos(\kappa_1 L_1) + \beta_1 \sin(\kappa_1 L_1)] \cos(\kappa_2 z) + \beta_2 \sin(\kappa_2 z) \ ;
\end{aligned}
\tag{89}
$$

the equations to impose are

$$
\begin{aligned}
y_2'(L_2) &= \frac{\omega^2}{g} y_2(L_2) \\
\mu y_2'(0) - (1+\mu) \, y_1'(L_1) &= -\frac{\omega^2}{g} y_1(L_1)
\end{aligned}
\tag{90}
$$

which can be solved for $\beta_{1,2}$ and substituted to obtain

$$y_2(L_2) = \frac{2\, g^2\, \kappa_1\, \kappa_2\, (1+\mu)}{\left\{ \begin{array}{c} [g^2\, \kappa_1\, \kappa_2\, (1+\mu) - g^2\, \kappa_2^2\, \mu + \omega^4] \cos(\kappa_1\, L_1 - \kappa_2\, L_2) + \\ + [g^2\, \kappa_1\, \kappa_2\, (1+\mu) + g^2\, \kappa_2^2\, \mu - \omega^4] \cos(\kappa_1\, L_1 + \kappa_2\, L_2) + \\ + g\omega^2\, (1+\mu)\, [(\kappa_1 - \kappa_2) \sin(\kappa_1\, L_1 - \kappa_2\, L_2) - (\kappa_1 + \kappa_2) \sin(\kappa_1\, L_1 + \kappa_2\, L_2)] \end{array} \right\}} \tag{91}$$

We show in Fig. 3.6 the transfer function from the suspension to the mass 1 for the pair of filters 1 and 2 in the VIRGO SuperAttenuator (same parameters as in the previous section); we also show the transfer function from the suspension to the first mass, and to the middle point of the second wire.

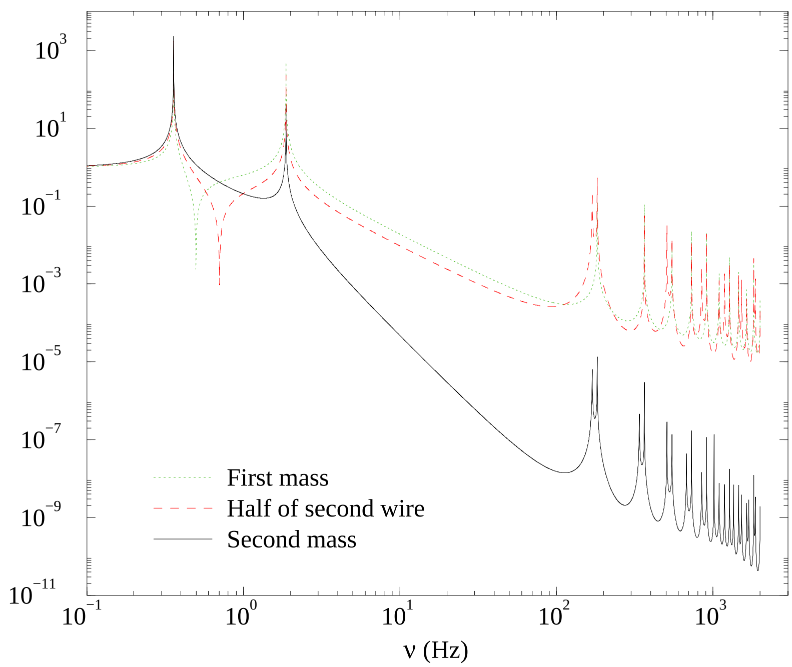

Figure 8: Transfer functions for a double pendulum

A few comments:

- The motion of the middle point of the second wire is of the same order as the motion of the first filter, which is the source of violin excitation.
- Some of the violin resonances do not contribute to the motion of the middle point of the second wire, because it is with a good approximation a nodal point for some of the eigenmodes.
- The motion of second filter is attenuated as expected by a factorisation of the transfer functions due to pendola resonances at low frequencies.

3.7 Multiple pendola: the 7 VIRGO filters

In order to solve the same problem for n filters in the VIRGO Super Attenuator, we equip ourselves with a bit of mathematics.

Each wire suspending a filter has the general solution

$$y_i(z) = \alpha_i \sin(\kappa_i z) + \beta_i \cos(\kappa_1 z) \tag{92}$$

with the boundary conditions

$$\begin{aligned}
&y_1(0) = 1 \\
&y_i(L_i) = y_{i+1}(0), \quad i \in [1,\, n-1] \\
&\mu_i y'_{i+1}(0) - (1+\mu_i)\, y'_i(L_i) = -\frac{\omega^2}{g} y_i(L_i), \quad i \in [1,\, n-1] \\
&y'_n(L_n) = \frac{\omega^2}{g} y_n(L_n)\ ;
\end{aligned} \tag{93}$$

where

$$\mu_i = \frac{T_{i+1}}{M_i g} = \frac{1}{M_i} \sum_{j=i+1}^{n} M_j\ . \tag{94}$$

These system of $2n$ equations in $2n$ unknowns, can be solved for α_i, β_i to get the solution.

We show in Fig. 3.7 the resulting transfer function for a 7 stage VIRGO Super-Attenuator, without marionetta and mirror: the parameters used are taken from [4].

stage #	M (Kg)	L (m)	$d = 2r$ (mm)
1	142	1	3.5
2	164	1	3.5
3	135	1	3
4	132	1	3
5	128	1	3
6	116	1	2
7	163	1	2
8	30	1	1
9	30	.7	.2

The first violin resonance appears at about XX Hz, however in the figure the Q value is set to zero and therefore the height of the peak is incorrect.

4 Matrix method for transfer functions

It is possible to greatly simplify the computation of transfer functions by exploiting in the frequency domain the MDLE approach.

To be definite, let us consider a chain of pendola connected by concentrated masses, and assume for simplicity a second order system: define for each wire i the state vector

$$\psi_i(z) = \begin{pmatrix} y_i(z) \\ y'_i(z) \end{pmatrix} \tag{95}$$

and note that the motion equation

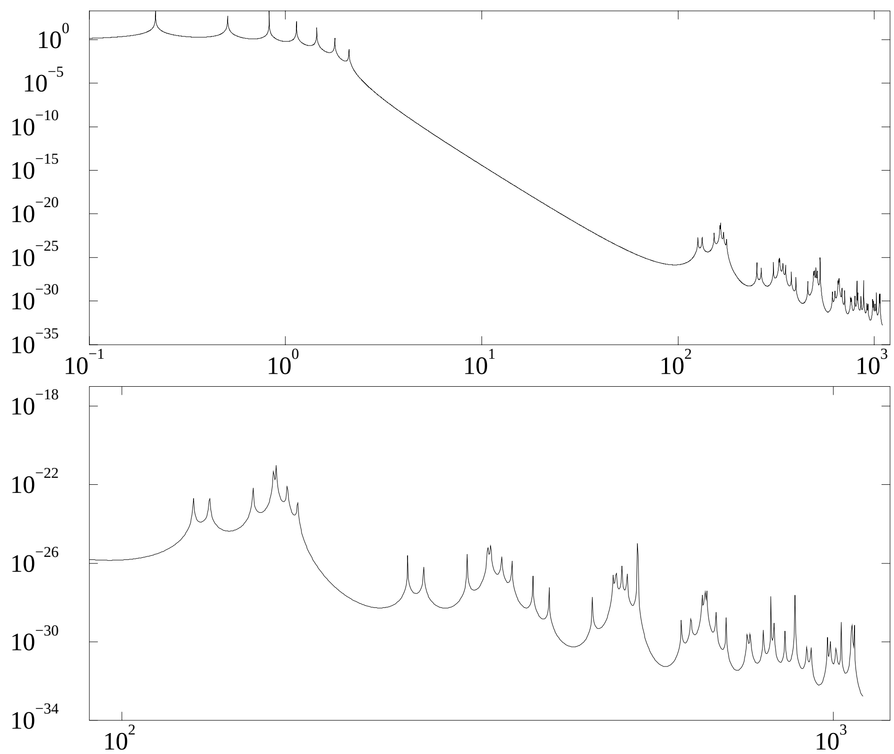

Figure 9: Transfer functions for a 7 stages pendulum

$$Ty''(z) = -\omega^2 \rho A y(z) \tag{96}$$

can be rewritten in first order form as

$$\frac{d}{dz}\psi_i(z) = \mathbf{H} \cdot \psi_i(z) \tag{97}$$

where

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -\frac{\omega^2 \rho A}{T} & 0 \end{bmatrix} . \tag{98}$$

This first order equation can be easily integrated giving

$$\psi_i(L_i) = \mathbf{U}_i(L_i) \cdot \psi_i(0) \tag{99}$$

where

$$\mathbf{U}_i(L_i) = e^{\mathbf{H}L_i} = \begin{bmatrix} \cos\kappa_i L_i & \frac{1}{\kappa_i}\sin\kappa L_i \\ -\kappa\sin\kappa_i L_i & \cos\kappa_i L_i \end{bmatrix} \tag{100}$$

and as before $\kappa_i = \omega\sqrt{\rho_i A_i / T_i}$.
The boundary condition introduced by each filter

$$
\begin{aligned}
y_{i+1}(0) &= y_i(L_i) \\
T_{i+1}y'_{i+1}(0) - T_i y'_i(L_i) &= -\omega^2 M_i y_i(L_i)
\end{aligned}
\tag{101}
$$

can be in turn written in matrix form as

$$
\begin{aligned}
\psi_{i+1}(0) &= \mathbf{Z}_i \cdot \psi_i(L_i) \\
\mathbf{Z}_i &= \begin{bmatrix} 1 & 0 \\ -\omega^2 \frac{M_i}{T_{i+1}} & \frac{T_i}{T_{i+1}} \end{bmatrix} .
\end{aligned}
\tag{102}
$$

These equations are closed by the motion equation for the last mass,

$$
T_n y'_n(L_n) = \omega^2 M_n y_n(L_n) \tag{103}
$$

that is

$$
\begin{pmatrix} y_n(L_n) \\ 0 \end{pmatrix} = \mathbf{Z}_n \cdot \mathbf{\Phi}_{n,1} \cdot \begin{pmatrix} \psi_1(0) \\ \psi'_1(0) \end{pmatrix} \tag{104}
$$

where

$$
\begin{aligned}
\mathbf{\Phi}_{n,1} &= \mathbf{U}_n(L_n) \cdot \mathbf{Z}_{n-1} \cdots \mathbf{U}_1(L_1) \\
\mathbf{Z}_n &= \begin{bmatrix} 1 & 0 \\ -\frac{\omega^2}{g} & 1 \end{bmatrix}
\end{aligned}
\tag{105}
$$

leading to

$$
y_n(L_n) = \frac{\det \mathbf{\Phi}_{n,1}}{(\mathbf{Z}_n \cdot \mathbf{\Phi}_{n,1})_{2,2}} . \tag{106}
$$

Now

$$
\det \mathbf{\Phi}_{n,1} = \prod_{i=2}^{n} \det \mathbf{U}_i(L_n) \det \mathbf{Z}_{i \leftarrow i-1} \det \mathbf{U}_1(L_1) \tag{107}
$$

and the U matrices are by construction with unit determinant, hence

$$
\det \mathbf{\Phi}_{n,1} = \prod_{i=2}^{n} \det \mathbf{Z}_{i-1} = \frac{T_1}{T_n} . \tag{108}
$$

Hence finally

$$
y_n(L_n) = \frac{T_1}{T_n} \frac{1}{(\mathbf{Z}_n \cdot \mathbf{\Phi}_{n,1})_{2,2}} \tag{109}
$$

which is numerically stable.

Having a stable expression for $y_n(L_n)$, we can immediately solve also for an intermediate element m, by writing

$$
\begin{aligned}
\begin{pmatrix} y_n(L_n) \\ 0 \end{pmatrix} &= \mathbf{Z}_n \cdot \mathbf{\Phi}_{n,m} \cdot \begin{pmatrix} \psi_m(0) \\ \psi'_m(0) \end{pmatrix} \qquad \text{where} \\
\mathbf{\Phi}_{n,m} &= \mathbf{U}_n(L_n) \cdot \mathbf{Z}_{n-1} \cdots \mathbf{U}_m(L_m)
\end{aligned} \tag{110}
$$

and solving for $\psi_m(0)$

$$
\begin{aligned}
\psi_m(0) &= \frac{(\mathbf{Z}_n \cdot \mathbf{\Phi}_{n,m})_{2,2}}{\det \mathbf{\Phi}_{n,m}} y_n(L_n) \\
&= \frac{T_1}{T_m} \frac{(\mathbf{Z}_n \cdot \mathbf{\Phi}_{n,m})_{2,2}}{(\mathbf{Z}_n \cdot \mathbf{\Phi}_{n,1})_{2,2}}
\end{aligned} \tag{111}
$$

These algorithm is very fast and allows to compute at once the transfer functions to all the stages.

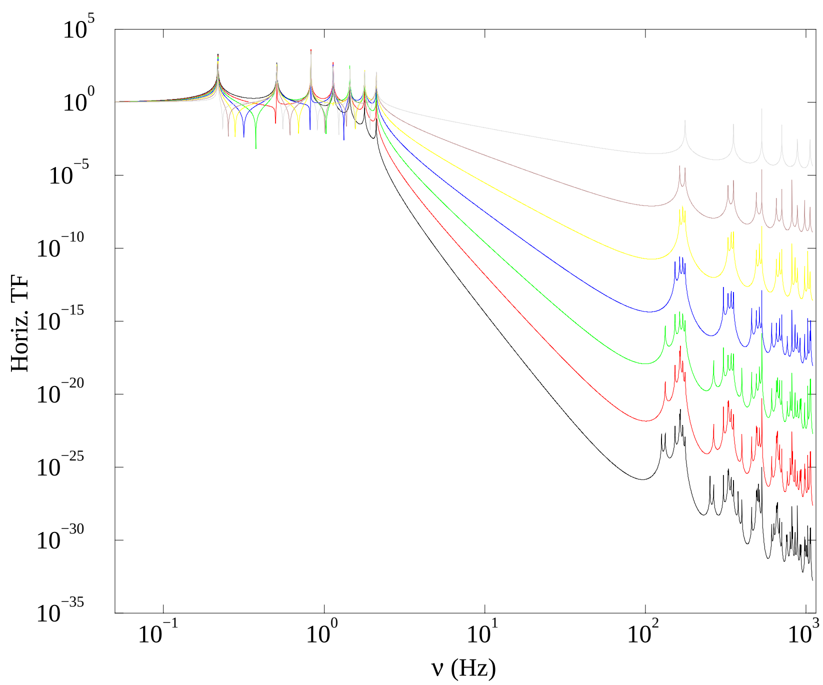

Figure 10: Transfer functions for 7 filters

As an example we show in Fig. **??** the 7 transfer functions between the horizontal motion of the suspension and the 7 filters.

4.1 Higher order linear equations

The generalization to higher order linear equations is straightforward: for simplicity we deal directly with the inclusion of the flexural momentum contribution.

The equation of motion is

$$EIy''''(z) - Ty''(z) = \rho A \omega^2 y(z) \tag{112}$$

which is of fourth order: we introduce

$$\psi(z) = \begin{pmatrix} y(z) \\ y'(z) \\ y''(z) \\ y'''(z) \end{pmatrix} \tag{113}$$

and rewrite the equation as

$$\begin{aligned} \frac{d}{d\,z}\psi(z) &= \mathbf{H} \cdot \psi(z) \qquad \text{where} \\ \mathbf{H} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \lambda_1^2 \lambda_3^2 & 0 & \lambda_1^2 - \lambda_3^2 & 0 \end{bmatrix} , \end{aligned} \tag{114}$$

where $\lambda_{1,3}$ have been defined in Eq. 49.
As before we define

$$\mathbf{U}(L) = \exp\left(\mathbf{H} L\right) \ ; \tag{115}$$

we can write the $\mathbf{H}$ matrix in form

$$\mathbf{H} = \mathbf{V} \cdot \mathbf{H}_d \cdot \mathbf{V}^{-1} \tag{116}$$

where

$$\mathbf{V} = \begin{bmatrix} -\lambda_1^{-3} & \lambda_1^{-3} & \frac{-i}{\lambda_3^3} & \frac{i}{\lambda_3^3} \\ \lambda_1^{-2} & \lambda_1^{-2} & -\lambda_3^{-2} & -\lambda_3^{-2} \\ -\frac{1}{\lambda_1} & \frac{1}{\lambda_1} & \frac{i}{\lambda_3} & \frac{-i}{\lambda_3} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{117}$$

$$\mathbf{H}_d = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & -i\lambda_3 & 0 \\ 0 & 0 & 0 & i\lambda_3 \end{bmatrix} , \tag{118}$$

and we obtain the exponential

$$\begin{bmatrix} \frac{\lambda_1^2\,\cos(\lambda_3 L)+\lambda_3^2\,\cosh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_1^3\,\sin(\lambda_3 L)+\lambda_3^3\,\sinh(\lambda_1 L)}{\lambda_1\,\lambda_3\,\left(\lambda_1^2+\lambda_3^2\right)} & \frac{\cosh(\lambda_1 L)-\cos(\lambda_3 L)}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_3\,\sinh(\lambda_1 L)-\lambda_1\,\sin(\lambda_3 L)}{\lambda_1\,\lambda_3\,\left(\lambda_1^2+\lambda_3^2\right)} \\ \frac{\lambda_1\,\lambda_3\,(-\lambda_1\,\sin(\lambda_3 L)+\lambda_3\,\sinh(\lambda_1 L))}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_1^2\,\cos(\lambda_3 L)+\lambda_3^2\,\cosh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_3\,\sin(\lambda_3 L)+\lambda_1\,\sinh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} & \frac{\cosh(\lambda_1 L)-\cos(\lambda_3 L)}{\lambda_1^2+\lambda_3^2} \\ \frac{\lambda_1^2\,\lambda_3^2\,(\cosh(\lambda_1 L)-\cos(\lambda_3 L))}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_1\,\lambda_3\,(\lambda_3\,\sinh(\lambda_1 L)-\lambda_1\,\sin(\lambda_3 L))}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_3^2\,\cos(\lambda_3 L)+\lambda_1^2\,\cosh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_3\,\sin(\lambda_3 L)+\lambda_1\,\sinh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} \\ \frac{\lambda_1^2\,\lambda_3^2\,(\lambda_3\,\sin(\lambda_3 L)+\lambda_1\,\sinh(\lambda_1 L))}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_1^2\,\lambda_3^2\,(\cosh(\lambda_1 L)-\cos(\lambda_3 L))}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_1^3\,\sinh(\lambda_1 L)-\lambda_3^3\,\sin(\lambda_3 L)}{\lambda_1^2+\lambda_3^2} & \frac{\lambda_3^2\,\cos(\lambda_3 L)+\lambda_1^2\,\cosh(\lambda_1 L)}{\lambda_1^2+\lambda_3^2} \end{bmatrix} \tag{119}$$

which gives the "evolution" matrix between start and end of each wire.

4.1.1 Approximate lagrangian: clamping in the CM

In order to define the multiple pendulum problem, we need to connect the wires with filters. As a first approximation, let us assume that both the wires are clamped in the CM. In this approximation, the boundary conditions are given by

$$\begin{aligned} y_{i+1}(0) &= y_i(L_i), \quad i \in [1,\, n-1] \\ y'_{i+1}(0) &= y'_i(L_i), \quad i \in [1,\, n-1] \end{aligned} \tag{120}$$

while the motion equations of the filters are, according to Eq. 60

$$\begin{aligned} &[EI_i y'''_i(L_i) - T_i y'_i(L_i)] - [EI_{i+1} y'''_{i+1}(0) - T_{i+1} y'_{i+1}(0)] = M_i \ddot{y}_i(L_i) \\ &EI_{i+1} y''_{i+1}(0) - EI_i y''_i(L_i) = J_i \ddot{y}'_i(L_i)\ . \end{aligned} \tag{121}$$

These equations define the impedance matrix for $i \in [1,\, n-1]$, as

$$\mathbf{Z}_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 \frac{J_i}{EI_{i+1}} & \frac{I_i}{I_{i+1}} & 0 \\ \omega^2 \frac{M_i}{EI_{i+1}} & \frac{T_{i+1}-T_i}{EI_{i+1}} & 0 & \frac{I_i}{I_{i+1}} \end{bmatrix} \tag{122}$$

(recall that in ψ the variables y'', y''' have been rescaled by EI/T). In terms of matrices Z_i and U_i the last filter state vector is given by

$$\psi_n(L) = \mathbf{U}_n(L_n) \cdot \mathbf{Z}_{n-1} \cdots \mathbf{Z_1} \cdot \mathbf{U}_1(L_1) \cdot \psi_1(0) \equiv \mathbf{\Phi}_{n,1} \psi_1(0) \tag{123}$$

and the equations of motion for the last filter

$$\begin{aligned} EI_n y'''_n(L_n) - T_n y'_n(L_n) &= -\omega^2 M_n y_n(L_n) \\ EI_n y''_i(L_n) &= \omega^2 J_n y'_n(L_n) \end{aligned} \tag{124}$$

allow to eliminate $y''_n(L_n)$, $y'''_n(L_n)$ and define a matrix $\mathbf{Z}_n$

$$\mathbf{Z}_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\omega^2 \frac{J_n}{EI_n} & 1 & 0 \\ \omega^2 \frac{M_n}{EI_n} & -\frac{T_n}{EI_n} & 0 & 1 \end{bmatrix} \tag{125}$$

such that

$$\begin{pmatrix} y_n(L_n) \\ y'_n(L_n) \\ 0 \\ 0 \end{pmatrix} = \mathbf{Z}_n \cdot \Phi_{n,1} \cdot \psi_1(0)\ . \tag{126}$$

4.2 "Real" lagrangian: clamping not in the CM

We must take care of the fact that the wires are not clamped in the CM of each filter: we have relations of the form

$$\begin{aligned} y_i^{CM} &= y_i(L_i) + (d_i - a_i) y_i'(L_i) \\ y_i^{CM} &= y_{i+1}(0) - (d_i + a_i) y_{i+1}'(L_i) \\ \theta_{x\,i}^{CM} &= -y_i'(L_i) \ , \end{aligned} \tag{127}$$

therefore the boundary conditions are

$$\begin{aligned} & y_{i+1}(0) = y_i(L_i) + 2d_i y_i'(L), \quad i \in [1,\, n-1] \\ & y_{i+1}'(0) = y_i'(L_i), \quad i \in [1,\, n-1] \end{aligned} \tag{128}$$

while the motion equations need to be written in terms of the coordinates of the CM, for $i \in [1,\, n-1]$

$$\begin{aligned} & \left[T_{i+1} y_{i+1}'(0) - EI_{i+1} y_{i+1}'''(0)\right] - \left[T_i y_i'(L_i) - EI_i y_i'''(L_i)\right] = -\omega^2 M_i \left[y_i(L_i) + (d_i - a_i) y_i'(L_i)\right] \ , \\ & \left[T_{i+1} y_{i+1}'(0) - EI_{i+1} y_{i+1}'''(0)\right](d_i + a_i) + \left[T_i y_i'(L_i) - EI_i y_i'''(L_i)\right](d_i - a_i) + \\ & \qquad + EI_{i+1} y_{i+1}''(0) - EI_i y_i''(L_i) = +J_i \ddot{y}_i'(L_i) = -\omega^2 J_i y_i'(L_i) \ ; \end{aligned} \tag{129}$$

Recheck all the signs!

in the last equations we have included the concentrated momentum applied by the wires and the momentum of the trasversal forces. The symbols J_i stand for inertia momentum of the filter i around axis x, because the symbol I_i is already in use for the inertia momentum of the wire.

The motion equations and the boundary conditions can be collected in the "impedance" matrix

$$\mathbf{Z}_i = \begin{bmatrix} 1 & 2d_i & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \omega^2 \frac{M_i(d_i+a_i)}{EI_{i+1}} & \frac{\omega^2\left[\left(d_i^2-a_i^2\right)M_i - J_i\right] - 2d_i T_i}{EI_{i+1}} & \frac{I_i}{I_{i+1}} & 2d_i \frac{I_i}{I_{i+1}} \\ \omega^2 \frac{M_i}{EI_{i+1}} & \frac{T_{i+1} - T_i + \omega^2 M_i (d_i - a_i)}{EI_{i+1}} & 0 & \frac{I_i}{I_{i+1}} \end{bmatrix} \tag{130}$$

which gives the "state vector" of the lower clamp in terms of the one of the upper: it can be used as before for all the filters except the last one. The last filter upper clamp obeys the equations

$$\begin{aligned} & \left[T_i y_i'(L_i) - EI_i y_i'''(L_i)\right] = +\omega^2 M_i \left[y_i(L_i) + (d_i - a_i) y_i'(L_i)\right] \ , \\ & \left[T_i y_i'(L_i) - EI_i y_i'''(L_i)\right](d_i - a_i) - EI_i y_i''(L_i) = -\omega^2 J_i y_i'(L_i) \ ; \end{aligned} \tag{131}$$

which allows to eliminate $y_i''(L_i)$ and $y_i'''(L_i)$ defining a Z_n matrix [2]

[2]Note that Z_n can be put in triangular form simply interchanging y'' and y''' in the state vectors, and $\det Z_n = 1$

$$\mathbf{Z}_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{T_i(d_i-a_i)+\omega^2 J_i}{EI_i} & 1 & (d_i - a_i) \\ \omega^2 \frac{M_i}{EI_i} & \frac{\omega^2 M_i(d_i-a_i)-T_i}{EI_i} & 0 & 1 \end{bmatrix} \tag{132}$$

such that

$$\begin{pmatrix} y_n(L_n) \\ y_n'(L_n) \\ 0 \\ 0 \end{pmatrix} = \mathbf{Z}_n \cdot \Phi_{n,1} \cdot \psi_1(0) \tag{133}$$

where $\Phi_{n,1}$ is defined as in Eq. 108.

The equation 133 defines a system of 4 equations in 4 unknowns: $y_n(L_n)$, $y_n'(L_n)$, $y_1''(0)$, $y_1'''(0)$ which can be solved as before in terms of $y_1(0)$, $y_1'(0)$, the coordinate and "angle" of the suspension point.

5 Transfer functions for real pendola

In this section we show simply the transfer functions obtained, for a 9 stages VIRGO pendulum

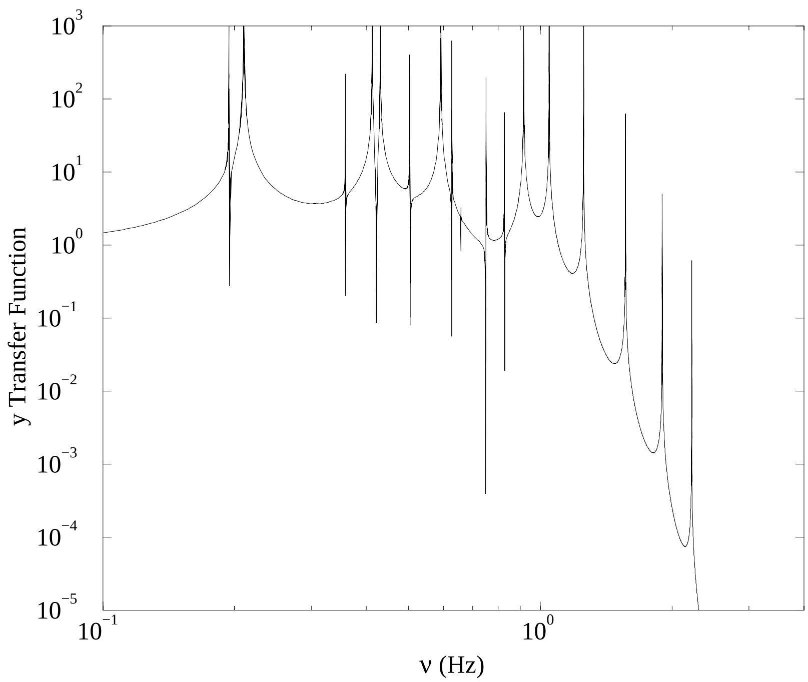

Figure 11: Y transfer function

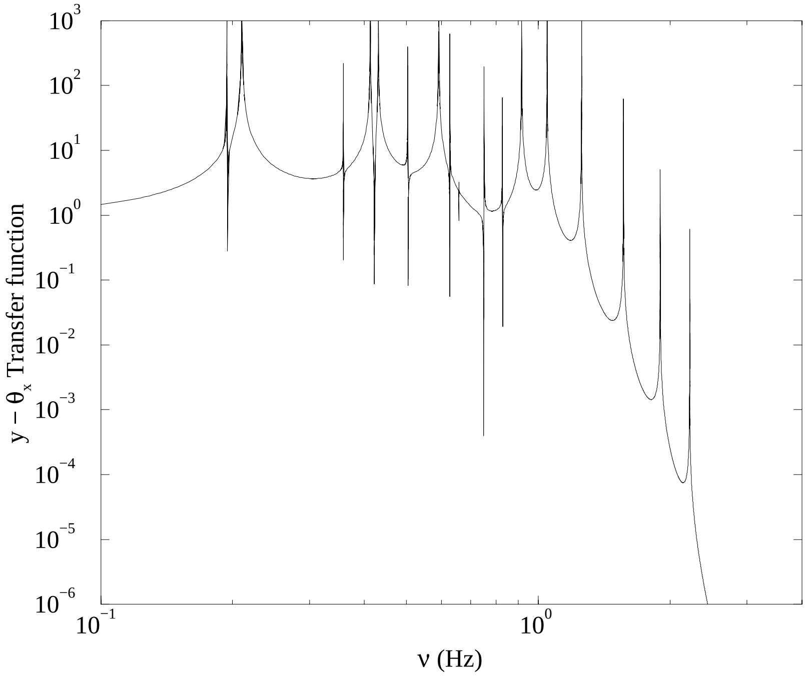

Figure 12: θ_x transfer function

6 Compressed beam

The mathematical treatment for the compressed beam is completely analogous to the one for the beam under tension.

We have again a potential energy of the form (note the change in sign of the tension, which is now a compression)

$$U = \frac{1}{2}\int_0^l \left[(EIy'')^2 - T\,(y')^2 \right] dz \ ; \tag{134}$$

again in adimensional units

$$U = \frac{Tl}{2}\frac{1}{k^2 l^2}\int_0^1 \left[(y'')^2 - k^2\,(y')^2 \right] dz \ , \tag{135}$$

where

$$k^2 \equiv \frac{T\,l^2}{EI} \tag{136}$$

The motion equation is

$$y'''' + k^2 y'' = 0 \tag{137}$$

which admits for $k \neq 0$ the general solution ($\zeta = z/l$)

$$y(\zeta) = \mathbf{c}^T \cdot \begin{pmatrix} \sin k\zeta \\ \cos k\zeta \\ k\zeta \\ 1 \end{pmatrix} . \tag{138}$$

Proceeding as before and defining the boundary condition vector

$$\mathbf{x}^T = (y(0),\ y'(0),\ y(1),\ y'(1)) \ : \tag{139}$$

we obtain the potential energy

$$U = \frac{E\,I}{2\,l}\mathbf{x}^T \cdot \mathbf{W} \cdot \mathbf{x} \tag{140}$$

where

$$\begin{aligned} \mathbf{W} &= \frac{k^2}{l^2\,[k\cos k/2 - 2\sin k/2]} \times \\ & \begin{bmatrix} -\left(k\ \cos(\frac{k}{2})\right) & -\sin(\frac{k}{2}) & k\ \cos(\frac{k}{2}) & -\sin(\frac{k}{2}) \\ -\sin(\frac{k}{2}) & \frac{(k\ \cos(k)-\sin(k))}{2\,k\ \sin(\frac{k}{2})} & \sin(\frac{k}{2}) & -\frac{(k-\sin(k))}{2\,k\ \sin(\frac{k}{2})} \\ k\ \cos(\frac{k}{2}) & \sin(\frac{k}{2}) & -\left(k\ \cos(\frac{k}{2})\right) & \sin(\frac{k}{2}) \\ -\sin(\frac{k}{2}) & \frac{(-k+\sin(k))}{2\,k\ \sin(\frac{k}{2})} & \sin(\frac{k}{2}) & \frac{(k\ \cos(k)-\sin(k))}{2\,k\ \sin(\frac{k}{2})} \end{bmatrix} . \end{aligned} \tag{141}$$

An alternate form is

$$\begin{aligned} \mathbf{W} &= \frac{k^3}{l^2\ [k - 2\tan k/2]} \times \\ & \begin{bmatrix} -1 & -\frac{1}{k}\tan\left(\frac{k}{2}\right) & 1 & -\frac{1}{k}\tan\left(\frac{k}{2}\right) \\ -\frac{1}{k}\tan\left(\frac{k}{2}\right) & \frac{1}{k}\left(\frac{1}{\tan k} - \frac{1}{k}\right) & \frac{1}{k}\tan\left(\frac{k}{2}\right) & -\frac{1}{k}\left(\frac{1}{\sin k} - \frac{1}{k}\right) \\ 1 & \frac{1}{k}\tan\left(\frac{k}{2}\right) & -1 & \frac{1}{k}\tan\left(\frac{k}{2}\right) \\ -\frac{1}{k}\tan\left(\frac{k}{2}\right) & -\frac{1}{k}\left(\frac{1}{\sin k} - \frac{1}{k}\right) & \frac{1}{k}\tan\left(\frac{k}{2}\right) & \frac{1}{k}\left(\frac{1}{\tan k} - \frac{1}{k}\right) \end{bmatrix} \end{aligned} \tag{142}$$

6.1 Simplified model: free end

Before considering the realistic setup rigid beam, let us go back to the single flexural joint and assume that its upper end is free, that is, the upper end is subject to forces but not couples.

This corresponds to the boundary condition $y''(1) = 0$, that is ...

6.2 Inverted pendulum values

In the inverted pendulum setup projected for the Pisa SuperAttenuator the flexible joint at the base of each of the three supporting columns will have the following characteristics

$$
\begin{aligned}
d &= 2r = 27 \times 10^{-3} m \qquad \Rightarrow EI \simeq 5200 N\, m^2 \\
l &= 8d = 0.216 m \\
T &\simeq 650 * 9.81 \simeq 6400 N \; .
\end{aligned} \tag{143}
$$

In consequence of these values, one has

$$
k^2 = \frac{Tl^2}{EI} \simeq 0.057 \qquad\qquad k \simeq 0.24 \; . \tag{144}
$$

We are in a regime opposite to the one considered for the beams: the reason is that the inertial momemtum I is much larger thanks to a larger beam section. We can expand the expression for W close to $k = 0$, obtaining

$$
\mathbf{W} = \frac{2}{l^2} \begin{bmatrix} 6 & 3 & -6 & 3 \\ 3 & 2 & -3 & 1 \\ -6 & -3 & 6 & -3 \\ 3 & 1 & -3 & 2 \end{bmatrix} + \frac{k^2}{5\, l^2} \begin{bmatrix} -6 & -\frac{1}{2} & 6 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{2}{3} & \frac{1}{2} & \frac{1}{6} \\ 6 & \frac{1}{2} & -6 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{2} & -\frac{2}{3} \end{bmatrix} + O(k^4) \; ; \tag{145}
$$

the $k = 0$ term could have been obtained also setting the pressure $T = 0$.
The potential energy can be written in terms of coordinates and angles as

$$
\begin{aligned}
U &= \frac{EI}{2\, l} \left[\mathbf{y}^T \cdot \mathbf{W} \cdot \mathbf{y} + \mathbf{x}^T \cdot \mathbf{W} \cdot \mathbf{x} \right] \\
\mathbf{x}^T &= (x(0),\, l\theta_y(0), x(1), l\theta_y(1)) \\
\mathbf{y}^T &= (y(0),\, -l\theta_x(0), y(1), -l\theta_x(1))
\end{aligned} \tag{146}
$$

Dropping the k^2 term, the potential energy of the elastic line can be therefore written as

$$
\begin{aligned}
U_0 = \frac{E\, I}{2\, l} \Big\{ & \frac{12}{l^2} \left[y(1) - y(0) \right]^2 + \frac{12}{l^2} \left[x(1) - x(0) \right]^2 \\
& + \frac{12}{l} \left[\theta_x(0) + \theta_x(1) \right] \left[y(1) - y(0) \right] - \frac{12}{l} \left[\theta_y(0) + \theta_y(1) \right] \left[x(1) - x(0) \right] \\
& + 3 \left[\theta_x(0) + \theta_x(1) \right]^2 + 3 \left[\theta_y(0) + \theta_y(1) \right]^2 \\
& + \left[\theta_x(0) - \theta_x(1) \right]^2 + \left[\theta_y(0) - \theta_y(1) \right]^2 \Big\} \; .
\end{aligned} \tag{147}
$$

We need to take into account that on top of the joint l it is fixed a rigid beam of length $L - l$ and mass m, as shown in Fig. 13. We are interested in writing the potential in terms of the coordinates $X(1), Y(1)$ on top of the beam.

To this end we note that forces are applied on top of the rigid beam (and in the CM of the beam itself), but not couples.

It is easy to convince oneself that the coordinates of the top of the flexural joint are given by

$$
\begin{aligned}
x(1) &= X(1) - (L - l)\theta_y(1) \\
y(1) &= Y(1) + (L - l)\theta_x(1) \; .
\end{aligned} \tag{148}
$$

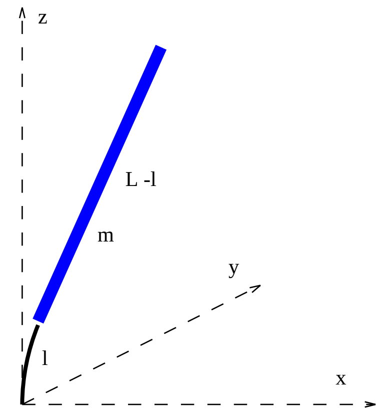

Figure 13: Inverted pendulum cantilever

7 Experimental setup for rocking modes

In this section we consider a simplified lagrangian, to account for the study of the rocking modes of a single filter. The experimental setup is shown in Fig. 14

the filter (of mass m) is suspended to a beam of length L_1 and radius r_1, and it is loaded with a mass M, suspended to a second beam of length and radius L_2, r_2.

The suspension points do not coincide with the filter's center of mass, but they are separated by a distance $2\,d$, and it is assumed that the CM is at a distance a from the midpoint between the suspension points, as in Fig. 15

after linearization, the coordinates $\mathbf{x}^{1,2}$ of the suspension points are, in terms of the coordinates $\mathbf{x}$ of the CM and the rocking angles of the filter

$$\begin{array}{ll} x_1 = x + (d-a)\theta_y & x_2 = x - (d+a)\theta_y \\ y_1 = y - (d-a)\theta_x & y_2 = y + (d+a)\theta_y \ ; \end{array} \tag{149}$$

the signs of angles are positive when the rotation around the corresponding axis is positive according to the right-hand rule.

We further assume that the suspension point of the first beam, $\mathbf{x}_0$, is clamped, and initially we neglect the horizontal and rocking motions of the loading mass M.

The horizontal and rotation lagrangian for the filter in this simplified model, up to terms $O(T)^{1/2}$ in the expansion in powers of the tension, is given by

$$\begin{aligned} \mathcal{L} \;=\; & \frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)+\frac{J_x}{2}\dot{\theta}_x^2+\frac{J_y}{2}\dot{\theta}_y^2 \\ & -\frac{(m+M)\,g}{2\,L_1}\left[(x_1-x_0)^2+(y_1-y_0)^2\right] \\ & -\frac{M\,g}{2\,L_2}\left[(x_2-x_M)^2+(y_2-y_M)^2\right] \\ & -\frac{H_1}{2}\left\{\left[\frac{x_1-x_0}{L_1}+\theta_y\right]^2+\left[\frac{x_1-x_0}{L_1}\right]^2\right. \end{aligned}$$

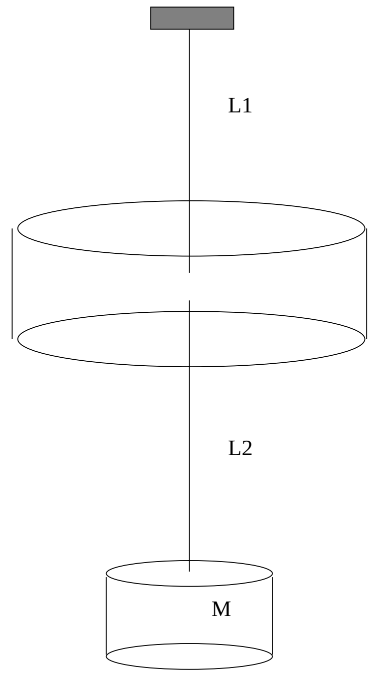

Figure 14: Experimental setup

$$\left[\frac{y_1 - y_0}{L_1} - \theta_x\right]^2 + \left[\frac{y_1 - y_0}{L_1}\right]\Bigg\}$$
$$-\frac{H_2}{2}\left\{\left[\frac{x_M - x_2}{L_2}\right]^2 + \left[\frac{x_M - x_2}{L_2} + \theta_y\right]^2\right.$$
$$\left.\left[\frac{y_M - y_2}{L_2}\right]^2 + \left[\frac{y_M - y_2}{L_2} - \theta_x\right]^2\right\} \tag{150}$$

where the coordinates $x_{1,2}$, $y_{1,2}$ should be expressed in terms of the coordinates x, y of the filter CM and the angles $\theta_{x,y}$. We have left expressed the coordinates of the suspension point and of the clamping point on the mass M.

The flexural stiffnesses of the two wires are

$$H_1 = \sqrt{(M+m)gEI_1} \qquad H_2 = \sqrt{MgEI_2} \tag{151}$$

where

$$I_{1,2} = \frac{\pi r_{1,2}^4}{4} \tag{152}$$

are the inertia momenta of the wires.

We assume for the beam parameters the values

$$L_{1,2} = 0.66m$$

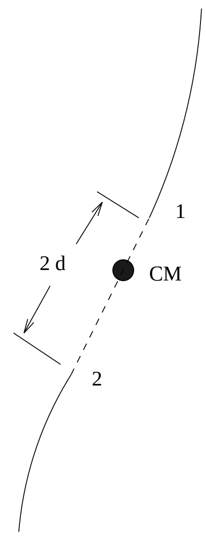

Figure 15: CM position relative to suspension points

$$
\begin{aligned}
2r_{1,2} &= 3.5mm \\
E &= 2 \times 10^{11} Pascal
\end{aligned} \tag{153}
$$

We inserting numerical values taken from the note [4], namely

$$
\begin{aligned}
m &= 142Kg \\
J_x &= 8.63Kgm^2 \\
J_y &= 8.42Kgm^2 \\
2d &= 0.01m \ ;
\end{aligned} \tag{154}
$$

the momenta of inertia correspond to the filter loaded with the magnetic antispring box, which was effectively present in the experimental setup. We further set the loading mass to

$$
M = 650Kg \ . \tag{155}
$$

To compute the eigenfrequencies of the (weakly) coupled x, θ_y motion, we first write the equations of motion, in the frequency domain, as a linear system

$$
\mathbf{A}\left(\omega^2\right) \cdot \begin{pmatrix} x \\ \theta_y \end{pmatrix} = 0 \ . \tag{156}
$$

The equation

$$
\det\mathbf{A}\left(\omega^2\right) = 0 \tag{157}
$$

allows to determine the two frequencies. The eigenvectors are then obtained by solving the degenerate linear system in correspondence of each eigenfrequency.

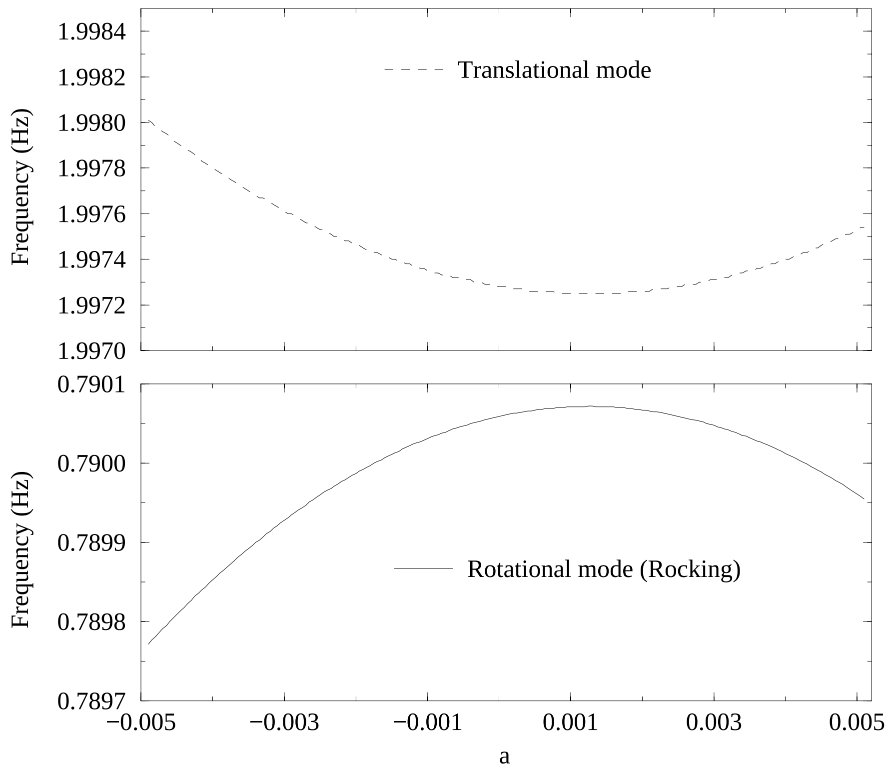

Figure 16: Eigenfrequencies of the translational and rocking motions

We plot in Fig. 16 the two eigenfrequencies $\nu = \frac{\omega}{2\pi}$ in dependence of the position of the CM, parameterized by a: we call the two frequencies "translational" and "rocking" because they are respectively dominated by an x or θ_y motion.

The eigenfrequencies depend very little on the CM position a.

A Small oscillations

Given a quadratic lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\mathbf{x}} \cdot \mathbf{T} \cdot \dot{\mathbf{x}} - \frac{1}{2}\mathbf{x} \cdot \mathbf{V} \cdot \mathbf{x} \tag{158}$$

we have the motion equations

$$\mathbf{T} \cdot \ddot{\mathbf{x}} + \mathbf{V} \cdot \mathbf{x} = 0 \ ; \tag{159}$$

in Fourier transform we look for eigenvectors $\mathbf{x}^k$ wuch that

$$-\omega_i^2 \mathbf{T} \cdot \mathbf{x}^k + \mathbf{V} \cdot \mathbf{x} = 0 \ . \tag{160}$$

We have therefore to find eigenvalues and eigenvectors for the problem

$$\mathbf{T}^{-1} \cdot \mathbf{V} \cdot \mathbf{x}^k = \omega_i^2 \mathbf{x}^k \ ; \tag{161}$$

assuming the problem solved, we look next for the solution of the forced problem

$$\mathbf{T} \cdot \ddot{\mathbf{x}} + \mathbf{V} \cdot \mathbf{x} = \mathbf{f} \tag{162}$$

where $\mathbf{f}$ is a vector of external forces. The solution is assumed as a superposition of normal modes

$$\mathbf{x} = \sum_k c_k \mathbf{x}^k \tag{163}$$

and gives rise to the equation, in Fourier transform

$$\sum_k c_k \left[-\omega^2 \mathbf{T} + \mathbf{V} \right] \cdot \mathbf{x}^k = \mathbf{f} \tag{164}$$

that is

$$\sum_k c_k \left[-\omega^2 + \omega_k^2 \right] \mathbf{T} \cdot \mathbf{x}^k = \mathbf{f} \ . \tag{165}$$

The orthogonality of the eigenvectors with respect to the dot product defined by the tensor $\mathbf{T}$ allows to invert the relation

$$c_k = \frac{1}{\omega_k^2 - \omega^2} \frac{\left(\mathbf{x}^k \cdot \mathbf{f} \right)}{\left(\mathbf{x}^k \cdot \mathbf{T} \cdot \mathbf{x}^k \right)} \ . \tag{166}$$

We assume from now on that the eigenvectors have been normalized in such a way to have

$$\left(\mathbf{x}^k \cdot \mathbf{T} \cdot \mathbf{x}^l \right) = \delta_{k\,l} \ : \tag{167}$$

we obtain the solution

$$\mathbf{x} = \sum_k \mathbf{x}^k \frac{\left(\mathbf{x}^k \cdot \mathbf{f} \right)}{\omega_k^2 - \omega^2} \tag{168}$$

The transfer function matrix between different points of the system is defined therefore as

$$H_{i,j}(\omega) = \sum_k \frac{x_i^k x_j^k}{\omega_k^2 - \omega^2} \frac{1}{\left(\mathbf{x}^k \cdot \mathbf{T} \cdot \mathbf{x}^k \right)} \tag{169}$$

and its symmetry is a consequence of linearity and it is similar to the Onsager relations in thermodynamics.

B Damped oscillator

The equation for a damped oscillator is

$$m\ddot{x} + \gamma \dot{x} + kx = 0 \ ; \tag{170}$$

the general solution is in terms of damped oscillations, in the form

$$x(t) = \exp\left(-t/\tau\right)\exp\left(i\omega_0 t\right) \ , \tag{171}$$

where

$$\omega_0 = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4\,m^2}}, \qquad \tau = \frac{2\,m}{\gamma} \ . \tag{172}$$

In terms of the quality factor Q, defined as

$$Q \equiv \tau\omega_0 \tag{173}$$

one has

$$\omega_0^2 = \frac{k}{m}\frac{1}{1+\frac{1}{Q^2}} \simeq \frac{k}{m}\left(1 - \frac{1}{Q^2}\right) \ . \tag{174}$$

The transfer function of the damped oscillator is obtained considering the forced arrangement, in Fourier space

$$-\omega^2 m\tilde{x}\left(\omega\right) + i\omega\gamma\tilde{x}\left(\omega\right) + k\tilde{x}\left(\omega\right) = k\,\tilde{x_0}\left(\omega\right) \ , \tag{175}$$

hence one has

$$\frac{\tilde{x}\left(\omega\right)}{\tilde{x}_0\left(\omega\right)} = \frac{1}{-m\omega^2 + i\omega\gamma + k} \tag{176}$$

[TUTTO DA RIVEDERE!!!]

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[3] R. Del Fabbro, *Super Attenuator Dynamics, Part II*, VIRGO note NTS95-004, 1995.

[4] R. Del Fabbro, *Parameters of the SuperAttenuator version 1996: input data for DYN program*, 6 Marzo 1996.