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Limits on cross couplings in a suspended optic

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ABSTRACT

It has recently been observed that cross couplings between mechanical modes in a suspended optic cannot be completely eliminated by tuning force or sensor constants in the optic's controller. The residual cross couplings appear to be 90° out of phase from the modes being driven and are hence unaffected by torques or forces applied through the actuators. The origin of these residual cross couplings has so far been a mystery.

It turns out that such residual cross couplings are a natural consequence of the phase shifts introduced between modes by resonances. This behavior is very general and is independent of the mechanism that couples the modes of an optic. We should always expect to see some nonzero residual cross coupling, and its phase should be 90° once we have tuned the force constants to minimize its amplitude.

INTRODUCTION

You might think (as I did a few weeks ago) that you should be able to compensate for mechanical cross couplings between the modes of a suspended optic by tuning the force constants. If you excite the pendulum mode of an optic, for example, and that introduces some pitch motion, then you might expect to be able to tune the pendulum force constants so that they apply just enough pitch torque to counter the original pitch motion.

In fact, this is not the case. We can tune the force constants to minimize the cross couplings between modes, but we cannot eliminate them entirely. Moreover, these cross couplings appear to have a 90° phase shift between the mode being driven and the coupled mode [1]. These residual cross couplings apparently cannot be tuned away by adjusting either the force constants or the sensor constants [2].

What could give rise to cross couplings that apparently cannot be compensated for? One possibility is the phase shift introduced by mechanical resonances in the intrinsic cross couplings. You can only compensate for motion that is in phase with your actuator, and any component that is 90° out of phase with it will not be affected if you adjust the force constants. This paper describes the consequences of such a phase shift and how it can give rise to residual cross couplings.

DEFINITIONS

We may describe the position of a mirror by the positions of each of its magnets.

$$\vec{x} = (x_{UL}, x_{UR}, x_{LL}, x_{LR}),$$

where x_{UL} represents the position of the upper left magnet, etc., along the x-axis. (The axis of the mirror itself would be parallel to this x-axis when the mirror is perfectly aligned.) The mirror's mechanical modes can be represented by a set of vectors that describe, pitch, yaw and position, respectively as[1]

$$\hat{p} = \frac{1}{2}(+1,+1,-1,-1),$$

$$\hat{y} = \frac{1}{2}(-1,+1,-1,+1),$$

and

$$\hat{z} = \frac{1}{2}(+1,+1,+1,+1).$$

[1] Thanks to Stan Whitcomb for suggesting this vector-based notation.

We only need three vectors to describe the mirror, despite the fact that we have four magnets, because all four magnets lie in the same plane: the back surface of the mirror. The fourth basis vector needed to span a complete four dimensional space would be

$$\hat{s} = \frac{1}{2}(+1,-1,-1,+1),$$

which does not correspond to anything the mirror can do physically.

The state of the mirror can be expressed in terms of our three basis vectors as

$$\vec{x} = x_{POS}\hat{z} + x_{PIT}\hat{p} + x_{YAW}\hat{y},$$

where

$$x_{POS} = \vec{x} \cdot \hat{z},$$

$$x_{PIT} = \vec{x} \cdot \hat{p},$$

and

$$x_{YAW} = \vec{x} \cdot \hat{y}.$$

We may also describe the forces and torques on the mirror by the forces applied along the x-axis to each of the magnets.

$$\vec{f} = (f_{UL}, f_{UR}, f_{LL}, f_{LR})$$

The force on each magnet is the sum of forces applied by each channel in the controller. If S_{PIT} is the signal applied to the pitch channel, etc., then the force mirror would be

$$\vec{f} = S_{PIT}\vec{\Theta} + S_{YAW}\vec{\Phi} + S_{POS}\vec{Z},$$

where the force constant vectors for pitch, yaw, and position are

$$\vec{\Theta} = \left(+\Theta_{UL}, +\Theta_{UR}, -\Theta_{LR}, -\Theta_{LL}\right),$$

$$\vec{\Phi} = \left(-\Phi_{UL}, +\Phi_{UR}, -\Phi_{LR}, +\Phi_{LL}\right),$$

and

$$\vec{Z} = \left(+Z_{UL}, +Z_{UR}, +Z_{LR}, +Z_{LL}\right).$$

We can ignore the potato chip mode $\hat{s}$ and write the force in the pitch, yaw, and position basis.

$$\vec{f} = \frac{1}{2\sqrt{2}R}\left(\tau_{PIT}\hat{p} + \tau_{YAW}\hat{y}\right) + \frac{1}{2} f_{TOT}\hat{z},$$

where f_{TOT} is the total force on the mirror, τ_{PIT} is the torque applied to the mirror's pitch mode, τ_{YAW} is the torque applied to yaw, and R is the radius from the center of the mirror where the magnets are glued.

Now that we have enough vocabulary to describe the kinematics of the mirror, we need to address its dynamics.

THE INERTIA MATRIX

The position of the mirror will be related to the force applied to it (in frequency space) by

$$\begin{aligned} \vec{f} &= \mathbf{M}\ddot{\vec{x}} \\ &= -\omega^2 \mathbf{M}\vec{x} \end{aligned} \tag{1}$$

where $\mathbf{M}$ is some matrix. For a free mirror, this inertia matrix is easy to derive by the usual application of Newton's third law. In a basis where

$$\hat{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

the inertia matrix for a free mirror of mass m, length L, and radius R is

$$\mathbf{M} = \frac{m}{2}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The parameter α is related to the moment of inertia of the mirror about its pitch and yaw axes.

$$\alpha = \frac{1}{16}\left[1 + \frac{1}{3}\left(\frac{L}{R}\right)^2\right]$$

For a free mirror, the inertia matrix is diagonal. There are no intrinsic cross couplings between the modes.

For a suspended optic, we may simply modify the matrix for a free mirror to include restoring forces for each mode and cross couplings between the modes. Let's consider the case where there is coupling between pitch and position. In this case the inertia matrix becomes

$$\mathbf{M} = \frac{m}{2}\begin{bmatrix} \alpha K_{PIT} & 0 & \varepsilon_1 \\ 0 & \alpha K_{YAW} & 0 \\ \varepsilon_2 & 0 & K_{POS} \end{bmatrix}.$$

Here the K's are frequency dependent transfer functions that account for the resonances in each mode. (They should approach $K \approx 1$ at high frequency.) The cross coupling terms ε_1 and ε_2 describe the position to pitch and pitch to position coupling, respectively. Note that the K's and ε's are complex, to account for phase, and that both are probably frequency dependent.

CROSS COUPLING

If we try to excite only one mode, for example position, we set the drive signals in the other modes to zero, *i.e.*

$$S_{PIT} = S_{YAW} = 0,$$

but this alone does not guarantee that x_{PIT} and x_{YAW} will also vanish. We would like to be able to minimize the pitch and yaw motions when position is driven by tuning the position force constants in $\vec{Z}$

It is straightforward to calculate the motion in the modes in this formalism. We can rewrite Equation 1 with only the position channel being driven as

$$S_{POS}\vec{Z} = -\omega^2\left(x_{POS}\mathbf{M}\hat{z} + x_{PIT}\mathbf{M}\hat{p} + x_{YAW}\mathbf{M}\hat{y}\right). \quad (2)$$

We can solve for the motion in any mode we like by constructing a vector orthogonal to two of the three terms on the right side of Equation 2 and dotting it into each side. To solve for x_{PIT}, for example, we could isolate it like so,

$$S_{POS}\vec{Z}\cdot\left(\mathbf{M}\hat{y}\times\mathbf{M}\hat{z}\right) = -\omega^2 x_{PIT}\left(\mathbf{M}\hat{p}\right)\cdot\left(\mathbf{M}\hat{y}\times\mathbf{M}\hat{z}\right).$$

Solving for the pitch and position motions and taking their ratio gives us a position to pitch cross coupling of

$$\frac{x_{PIT}}{x_{POS}} = \frac{\vec{Z}\cdot\left(\mathbf{M}\hat{y}\times\mathbf{M}\hat{z}\right)}{\vec{Z}\cdot\left(\mathbf{M}\hat{p}\times\mathbf{M}\hat{y}\right)}.$$

We would like to introduce a pitch component to the force constants $\vec{Z}$ to compensate for this cross coupling, so we write

$$\vec{Z} = z_{POS}\hat{z} + z_{PIT}\hat{p}.$$

With this expression for the force constants, we can solve for the position to pitch cross coupling.

$$\begin{aligned}\frac{x_{PIT}}{x_{POS}} &= \frac{K_{POS}z_{PIT} - \varepsilon_1 z_{POS}}{\alpha K_{PIT}z_{POS} - \varepsilon_2 z_{PIT}} \\ &= \frac{K_{POS}\xi - \varepsilon_1}{-\varepsilon_2\xi + \alpha K_{PIT}}\end{aligned} \tag{3}$$

Here I have defined the variable $\xi = z_{PIT}/z_{POS}$. We have a constraint in our suspension controllers that we can only change the gains of the force constants, not their phases, so if z_{POS} and z_{PIT} are complex, they must have the same phase, and ξ must be real. We are at liberty to vary ξ to try and compensate for these cross couplings, but if the K's and ε's are complex, as we expect them to be, then we will not be able to eliminate this cross coupling entirely.

MINIMUM RESIDUAL CROSS COUPLING

Even if we cannot eliminate the cross couplings between modes, we can still minimize them. But how well can we do, and what will the phase be of the residual cross coupling? Equation 3 holds the answers.

First let's calculate the intrinsic cross coupling between position and pitch. If we introduce no pitch compensation to $\vec{Z}$, then $\xi = 0$, and the cross coupling is

$$\left(\frac{x_{PIT}}{x_{POS}}\right)_{IN} = -\frac{\varepsilon_1}{\alpha K_{PIT}}.$$

Now, minimizing the magnitude of Equation 3 is a lot of work. However, if the intrinsic cross coupling between pitch and position is weak enough that

$$\varepsilon_2\xi << \alpha K_{PIT},$$

we can approximate the position to pitch cross coupling as

$$\frac{x_{PIT}}{x_{POS}} = \frac{K_{POS}\xi - \varepsilon_1}{\alpha K_{PIT}}.$$

A little algebra shows that the magnitude of this cross coupling is minimized when

$$\xi = \frac{\mathrm{Re}\left\{K_{POS}\varepsilon_1\right\}}{\left|K_{POS}\right|^2},$$

and that the minimum residual cross coupling is

$$\left(\frac{x_{PIT}}{x_{POS}}\right)_{MIN} = i\,\frac{\mathrm{Im}\left\{K_{POS}\varepsilon_1^*\right\}}{\alpha K_{POS}^* K_{PIT}}.$$

Above both the position (pendulum) and pitch resonant frequencies, $K_{POS} \approx K_{PIT} \approx 1$, and the phase of this residual cross coupling is approximately 90°.

The magnitude of the minimum cross coupling is related to the magnitude of the intrinsic cross coupling by

$$\left|\frac{x_{PIT}}{x_{POS}}\right|_{MIN} = \left|\frac{x_{PIT}}{x_{POS}}\right|_{IN} \sin\psi,$$

where ψ is the phase difference between the off-diagonal coupling term ε_1 and the on-diagonal pendulum term K_{POS}.

Note that it doesn't take much phase to set a relatively large limit on your residual cross couplings. Even a phase shift of 5.7° (one tenth of a radian) implies that you can't reduce the cross coupling from its intrinsic value by more than a factor of ten.

CONCLUSIONS

This analysis shows one mechanism that can set a lower limit on the cross couplings in a suspended optic. In this formalism it is straightforward to predict both the magnitude and phase of the residual cross couplings, after they have been minimized by tuning of the force constants. I have kept this analysis very general in this paper and not treated any specific model for the intrinsic cross couplings.

In retrospect, these conclusions seem obvious. We should expect some phase shift between mechanical modes in any real system. If we were to excite an optic's pendulum mode and that were to lead to some pitch motion, then we could describe that pitch motion as having two components: one that is in phase with the pendulum motion, and one that is 90° out of phase (in quadrature). Since we can only compensate for motion that is in phase with our actuator, we can only tune the force constants on the pendulum mode to get rid of the pitch component that is in phase with the pendulum motion. That leaves us with some residual pitch motion that is 90° out of phase from the pendulum motion. We could compensate for all of the pitch motion if we could tune the phases of our pendulum force constants, as well as their magnitudes.

REFERENCES

[1] David Shoemaker, private communication.

[2] Eric Black, *Sensor constants and cross coupling in a large optic,* LIGO-T990093-00-D.