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Correlation Function and Power Spectrum of Non-Stationary Shot Noise

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1 Introduction

Light detection at photodiodes involves the quantum mechanical process of absorption of photons and creation of free electrons. As a result an electric current $i(t)$, called the photocurrent, is produced in the photodiode circuit. Due to the discrete, random nature of the electron creation process the photocurrent is inherently noisy. The noise associated with the finite value of the electron charge is called shot noise.

In the simplest case the average value of the photocurrent is constant

$$\langle i(t) \rangle = I_0, \tag{1}$$

where $\langle \ldots \rangle$ denotes an ensemble averaging over the creation times of individual electrons. In this case the properties of the shot noise are well known. The one-sided power spectrum of the shot noise is given by

$$S(\omega) = 2\mathrm{e}I_0, \qquad \omega > 0, \tag{2}$$

where e is the charge on an electron.

A more complicated situation arises when the average value of the photocurrent is not constant, but is a function of time

$$\langle i(t) \rangle = I(t). \tag{3}$$

The shot noise in this case becomes non-stationary and depends on the particular form of the average photocurrent. In the case when the photocurrent varies "slowly" in the sense explained below the non-stationary noise is simple enough to study using traditional methods of random processes. It turns out that in almost all practical applications the average photocurrent is a slowly varying function of time.

The detection of light with modulated intensity inevitably entails non-stationary shot noise. Such noise was observed in the Garching prototype gravitational wave detector [4]. It was also demonstrated experimentally in [5] and [6]. An experiment specifically design to study non-stationary shot noise was conducted by the Australian group [7].

Early calculations of the non-stationary shot noise were distinguished by extraordinary complexity compared to the well-known case of stationary noise. One of the earliest attempts to correctly account for the non-stationary part of the noise was done in LIGO, see [2]. To our knowledge the only other attempt to present a systematic derivation of the non-stationary shot noise was done in [8]. The latter, however, does not approach the problem from a statistical point of view and is limited to the case of a pure sinusoidal dependence of the average photocurrent.

In this paper we present detailed calculations of the non-stationary shot noise from the point of view of random processes. We base our calculations on the properties of the Poisson pulses with non-stationary mean, see [1]. It is well-known that the calculations simplify if done entirely in frequency domain. Here we use the frequency domain representation of the instantaneous photocurrent very similar to the one described in [3]. We also discuss demodulation of the signal to determine the power spectrum of the demodulated noise.

2 Pulse Description of the Photocurrent

2.1 Poisson Pulses

The creation of an electron is a quantum mechanical process. As a result there is an uncertainty in the determination of the photon arrival time or the exact time of the electron creation. The number of electrons produced in the processes roughly corresponds to the number of photons absorbed. Typically, a large number of photons impinge on the photodiode; for an incident light power of $100\,\mathrm{mW}$ the number of photons absorbed in 1 sec is of the order of 10^{17}. Since we have a large number of electrons we can approach the subject from the point of view of random processes.

The random process of charge accumulation due to the generation of electrons at random times is a Poisson process. The associated current pulses are called Poisson pulses. These pulses are very narrow. Moreover, the number of pulses is large and the separation between the pulses is very small compared to resolution time of the photodiode. Hence we typically measure the average photocurrent $\langle i(t)\rangle$, where the time scale over which the averaging is performed is much greater than the width of an individual pulse or separation between the pulses. This time scale is determined by the frequency response of the photodiode. For modern semiconductor diodes this is typically of the order 10^{-11} sec. Therefore, the average photocurrent does not show fine structure due to single electron creation. However, the average photocurrent can vary at frequencies up to gigahertz.

We shall deal with the case where the time scale for noticeable changes in $I(t)$ is large compared to the difference between arrival times of the individual pulses. Thus there is a large number of electrons created over the time period $(t, t+\Delta t)$ during which $I(t)$ stays practically constant. The arrival times within this interval are, to a good approximation, equally distributed. The average number of the pulses in this interval is proportional to the value of the average photocurrent at that time. This is non-stationary Poisson process.

We shall assume that measurements only take place over a finite interval $(-T/2, T/2)$ of duration T, called the observation time. Let the electrons be produced at times

$$t_1, t_2, \ldots, t_N, \tag{4}$$

where N is the total number of electrons created in the time interval $(-T/2, T/2)$. We shall assume these are the same as the photon arrival times. The time series t_k is a random process. To a good approximation each arrival time t_k is independent of the others. At the moment an electron is created a pulse of electric current is produced in the photodiode circuit. Let $f(t-t_k)$ be the current due to a single electron created at $t = t_k$. The photocurrent is a sum of a large number of the pulses. We shall assume that the pulses arrive at different times but otherwise are indistinguishable

$$i(t) = \sum_{k=1}^{N} f(t-t_k). \tag{5}$$

The arrival times t_k are random variables taking values in the interval $(-T/2, T/2)$.

The total charge of each pulse is equal to the electron charge. Therefore

$$\int_{-T/2}^{T/2} f(t-t_k)\; dt = \mathrm{e}, \tag{6}$$

for all t_k.

Since the duration of the pulse is very short compared to Δt, we may think of a pulse as having a zero width. In other words, we may approximate $f(t)$ as proportional to the Dirac delta-function:

$$f(t) = \mathrm{e}\ \delta(t). \tag{7}$$

This narrow pulse approximation is a very good approximation for all calculations in the present paper.

The deviation from the narrow pulse approximation appear for very high frequencies. The photocurrent has to be sampled at a rate of the order of 100 GHz.

2.2 Finite-Time Fourier Transformation

Since we have a finite observation period $(-T/2, T/2)$, it is appropriate to use the finite-time Fourier transform

$$\tilde{F}(\omega) = \int\limits_{-T/2}^{T/2} F(t) e^{-i\omega t} dt. \tag{8}$$

The inverse Fourier transform is

$$F(t) = \int\limits_{-\infty}^{+\infty} \tilde{F}(\omega) e^{i\omega t}\ \frac{d\omega}{2\pi}, \tag{9}$$

for all t within the interval $(-T/2, T/2)$. In all the subsequent calculations the observation time is kept finite. At the end of the calculations we take the limit of large T.

The finite-time delta-function is defined by

$$2\pi\delta_T(\omega) = \int\limits_{-T/2}^{T/2} e^{-i\omega t} dt = T\ \frac{\sin(\omega T/2)}{(\omega T/2)}. \tag{10}$$

The finite-time delta-function is regular, in particular, $2\pi\delta_T(0) = T$. For stationary processes the correlation functions in the frequency domain will be proportional to $\delta_T(0)$. Thus the correlation functions diverge in the large T limit. However, the power spectrum obtained from the correlation function will be finite.

Let $\tilde{f}(\omega)$ be Fourier transform of the individual pulse. The Fourier transform of the photocurrent can be written in terms of the Fourier transform of a single pulse as

$$\tilde{i}(\omega) = \tilde{f}(\omega) \sum_{k=1}^{N} e^{-i\omega t_k}. \tag{11}$$

In the narrow pulse approximation we have

$$\tilde{f}(\omega) = \mathrm{e}. \tag{12}$$

Thus, in this approximation the photocurrent becomes particularly simple

$$\tilde{i}(\omega) = \mathrm{e}\ \sum_{k=1}^{N} e^{-i\omega t_k}. \tag{13}$$

2.3 Average Photocurrent and Noise

The pulses corresponding to individual electrons have very narrow width. As a result the photocurrent changes abruptly with time. The average photocurrent, however, is a smooth function of time

$$I(t) = \langle i(t) \rangle. \tag{14}$$

Here $\langle \ldots \rangle$ is ensemble averaging over the arrival times of the individual photons within the response time, to be defined precisely in Sec. 3.

We will need the Fourier transform of the average photocurrent

$$\tilde{I}(\omega) = \int_{-T/2}^{T/2} I(t)\; e^{-i\omega t} dt. \tag{15}$$

If $I(t)$ is constant, its Fourier transform is proportional to the finite-time delta-function.

The noise in the photocurrent, $n(t)$, is defined through the relation

$$i(t) = I(t) + n(t). \tag{16}$$

As a result the noise has zero average

$$\langle n(t) \rangle = 0. \tag{17}$$

Thus the correlation function of the photocurrent becomes a sum

$$\langle i(t)\; i(t') \rangle = I(t)\; I(t') + \langle n(t)\; n(t') \rangle. \tag{18}$$

The first term in the sum is a simple product, it is the totally uncorrelated part of the photocurrent. All the correlation between the photocurrents at different times comes from the noise.

In the frequency domain we have the corresponding expression for the average

$$\tilde{i}(\omega) = \tilde{I}(\omega) + \tilde{n}(\omega) \tag{19}$$

and the correlation function

$$\langle \tilde{i}(\omega)\; \tilde{i}^*(\omega') \rangle = \tilde{I}(\omega)\; \tilde{I}^*(\omega') + \langle \tilde{n}(\omega)\; \tilde{n}^*(\omega') \rangle. \tag{20}$$

If we remove the uncorrelated part of the photocurrent we obtain the correlation function of noise

$$\langle \tilde{n}(\omega)\; \tilde{n}^*(\omega') \rangle = \langle \tilde{i}(\omega)\; \tilde{i}^*(\omega') \rangle - \tilde{I}(\omega)\; \tilde{I}^*(\omega'). \tag{21}$$

2.4 Photocurrent Correlation Function

In this section we derive the expression for the auto-correlation of the Poisson pulses in terms of the averages.

Using the frequency domain representation of Sect.2.2 we obtain that the correlation function of the photocurrent is

$$\langle \tilde{i}(\omega)\; \tilde{i}^*(\omega') \rangle = \mathrm{e}^2 \sum_k \sum_m \langle e^{-i\omega t_k}\; e^{i\omega' t_m} \rangle. \tag{22}$$

By separating the uncorrelated part we obtain (see Appendix for details)

$$\langle \tilde{i}(\omega)\ \tilde{i}^*(\omega')\rangle \;=\; \mathrm{e}^2 \sum_k \langle e^{-i\omega t_k}\rangle \ \sum_m \langle e^{i\omega' t_m}\rangle \tag{23}$$

$$+\ \mathrm{e}^2 \sum_k \left[\langle e^{-i(\omega-\omega')t_k}\rangle - \langle e^{-i\omega t_k}\rangle\langle e^{i\omega' t_k}\rangle\right]. \tag{24}$$

Therefore the correlation function of the noise is

$$\langle \tilde{n}(\omega)\ \tilde{n}^*(\omega')\rangle = \mathrm{e}^2 \sum_k \left[\langle e^{-i(\omega-\omega')t_k}\rangle - \langle e^{-i\omega t_k}\rangle\langle e^{i\omega' t_k}\rangle\right]. \tag{25}$$

This expression for the correlation function of noise is a general result. It does not specify how the ensemble averaging is defined. The averages will be calculated explicitly in the following sections.

3 Ensemble Averaging

3.1 Probability Density

The average number of pulses in any given interval of time, $(t, t + dt)$, is proportional to the total charge $I(t)dt$ within that time interval. The probability of finding any particular pulse with arrival time t_k within the interval $(t, t + dt)$ is proportional to the total charge in the interval. Thus the probability density for the pulses is equal to the average photocurrent up to a constant

$$p(t) = \mathrm{const}\ I(t). \tag{26}$$

The constant can be found from normalization

$$\int_{-T/2}^{T/2} I(t)dt = \tilde{I}(0). \tag{27}$$

Here the integrated average photocurrent is equal to the total charge that has flowed through the photodiode

$$\tilde{I}(0) = \mathrm{e}N. \tag{28}$$

Thus we have found the normalization constant and therefore the probability distribution is

$$p(t) = \frac{I(t)}{\tilde{I}(0)}. \tag{29}$$

3.2 Correlation Function of Noise

The arrival times t_k are distributed within the entire observation time T according to the probability distribution $p(t)$. Therefore

$$\langle e^{-i\omega t_k}\rangle = \int_{-T/2}^{T/2} e^{-i\omega t_k}\ p(t_k)\ dt_k = \frac{\tilde{I}(\omega)}{\tilde{I}(0)}. \tag{30}$$

Now we know how to calculate the averages it is easy to see that the noise correlation function is

$$\langle \tilde{n}(\omega)\ \tilde{n}^*(\omega') \rangle = \mathrm{e} \left[\tilde{I}(\omega - \omega') - \frac{\tilde{I}(\omega)\tilde{I}^*(\omega')}{\tilde{I}(0)} \right]. \tag{31}$$

3.3 Power Spectrum

The correlation function leads directly to the power spectrum. We have to set $\omega = \omega'$ and take the appropriate limit. The one-sided power spectrum of the shot noise is given by

$$S_n(\omega) = 2 \lim_{T \to \infty} \frac{\langle \tilde{n}(\omega)\tilde{n}^*(\omega) \rangle}{T}. \tag{32}$$

The coefficient "2" accounts for the fact that we chose the one-sided $(\omega > 0)$ power spectrum.

Using Eq.31, we obtain

$$S_n(\omega) = 2\ \mathrm{e} \lim_{T \to \infty} \frac{\tilde{I}(0)}{T} \left[1 - \frac{|\tilde{I}(\omega)|^2}{|\tilde{I}(0)|^2} \right]. \tag{33}$$

This equation clearly reduces to the standard result

$$S(\omega) = 2\mathrm{e}I_0, \tag{34}$$

for constant average current. Indeed, if $I(t) = I_0$ then $\tilde{I}(0) = I_0 T$ and

$$\frac{\tilde{I}(\omega)}{\tilde{I}(0)} = I_0 \frac{\sin(\omega T/2)}{(\omega T/2)} \to 0 \tag{35}$$

as $T \to \infty$. This is the power spectrum of the raw noise, that is, the noise at the photodiode.

4 Applications

4.1 Modulated Photocurrent

The light intensity detected by the photodiodes is often modulated at some frequency Ω. Generally the light has harmonics at multiples of Ω. As the photoelectric emission rate is proportional to the light intensity, the average photocurrent is also modulated

$$I(t) = I_0 + I_1 \sin(\Omega t + \gamma) + I_2 \cos 2\Omega t, \tag{36}$$

Note that the instantaneous photocurrent $i(t)$ and hence its average value is always positive. This is due to the fact that the photocurrent is produced in the negatively biased photodiode. The charge can flow only in one direction.

The term at the modulation frequency, proportional to I_1, carries the signal. It is the component of the photocurrent that is demodulated by the mixer. In this note we are not interested in the signal part of the photocurrent. We consider only components of $I(t)$ that contribute to the noise. Thus we will be dealing with the photocurrent in the absence of signal

$$I(t) = I_0 + I_2 \cos 2\Omega t, \tag{37}$$

The Fourier transform of the photocurrent is

$$\tilde{I}(\omega) = I_0\ 2\pi\delta_T(\omega) + I_2\ \pi[\delta_T(\omega - 2\Omega) + \delta_T(\omega + 2\Omega)]. \tag{38}$$

4.2 Demodulation Signal

At the mixer the noise in the photocurrent $n(t)$ is multiplied by the demodulation signal $D(t)$. The demodulation signal is a square wave, however, it can be well approximated by a sinusoidal wave

$$D(t) = D_0 \sin(\Omega t + \beta). \tag{39}$$

Here D_0 is the amplitude of the demodulation signal and β is the phase of the demodulation. To get maximum signal the demodulation phase should be set to the phase of the signal

$$\beta = \gamma \mod \pi. \tag{40}$$

Since we assume there is no signal we leave the phase β arbitrary.

The Fourier transform of the demodulation signal is

$$\tilde{D}(\omega) = \frac{\pi}{i} D_0\ [e^{i\beta}\delta_T(\omega - \Omega) - e^{-i\beta}\delta_T(\omega + \Omega)]. \tag{41}$$

After the mixer the noise is

$$y(t) = D(t)\ n(t). \tag{42}$$

It is this noise that we are mostly interested in.

4.3 Demodulated Noise

In the Fourier domain the demodulated noise is given by the convolution

$$\tilde{y}(\omega) = \int_{-\infty}^{+\infty} \tilde{D}(\omega - \omega')\tilde{n}(\omega')\frac{d\omega'}{2\pi} \tag{43}$$

Demodulation effectively passes frequencies near the modulation frequency Ω. Therefore the noise near Ω shifts to low frequencies after demodulation

$$\tilde{y}(\omega) = \frac{D_0}{2i}\ [e^{i\beta}\tilde{n}(\omega - \Omega) - e^{-i\beta}\tilde{n}(\omega + \Omega)]. \tag{44}$$

To characterise the demodulated noise we need to find the correlation function of the demodulated noise $\langle\tilde{y}(\omega)\ \tilde{y}^*(\omega)\rangle$. Knowing the correlation function of raw noise we can find the correlation function of the demodulated noise

$$\langle\tilde{y}(\omega)\ \tilde{y}^*(\omega)\rangle = \frac{\mathrm{e}}{4}\ D_0^2\ [\ 2\tilde{I}(0) - \tilde{I}(2\Omega) - \tilde{I}(-2\Omega) + \ldots\ ], \tag{45}$$

where the dots stand for the terms that vanish in the large T limit.

4.4 Power Spectrum of the Demodulated Noise

The power spectrum of the demodulated noise can be obtained from the correlation function

$$S_y(\omega) = 2 \lim_{T\to\infty} \frac{\langle \tilde{y}(\omega)\ \tilde{y}^*(\omega)\rangle}{T}. \tag{46}$$

From Eq.(38) we can see that

$$\lim_{T\to\infty}[\tilde{I}(0)/T] = I_0, \tag{47}$$

$$\lim_{T\to\infty}[\tilde{I}(2\Omega)/T] = I_2/2. \tag{48}$$

Using these results we obtain

$$S_y(\omega) = \mathrm{e}\ D_0^2 \left(I_0 - \frac{I_2}{2}\cos 2\beta\right). \tag{49}$$

The important point here is that the shot noise consists of two contributions. One is proportional to I_0 and the other one is proportional to I_2. The first contribution is a stationary shot noise. The second contribution is due to the non-stationary nature of the noise.

Note a particular dependence of the power spectrum on the demodulated phase β. By adjusting the phase we can make the noise spectrum to vary within some range.

The most noticeable effect arises for maximum modulation, that is when $I_0 = I_2$. In this case the maximum power in the demodulated noise is

$$S_y(\omega) = \frac{3}{2}\,\mathrm{e}D_0^2\ I_0. \tag{50}$$

It corresponds to $\beta = \pi/2$. The minimum is obtained for $\beta = 0$ and is equal to

$$S_y(\omega) = \frac{1}{2}\,\mathrm{e}D_0^2\ I_0. \tag{51}$$

Similar dependence of the power spectrum on the demodulation phase was found experimentally in [7].

Appendix

A Derivation of the Photocurrent Correlation Function

In this section we derive the expression for the auto-correlation of the Poisson pulses in terms of the averages.

The auto-correlation function in frequency domain is

$$\langle\tilde{i}(\omega)\ \tilde{i}^*(\omega')\rangle = \mathrm{e}^2 \sum_k \sum_m \langle e^{-i\omega t_k}\ e^{i\omega' t_m}\rangle. \tag{52}$$

The first step in the calculation is to break the double sum into diagonal and off-diagonal sums

$$\langle\tilde{i}(\omega)\ \tilde{i}^*(\omega')\rangle = \mathrm{e}^2 \sum_k \sum_{m\neq k} \langle e^{-i\omega t_k} e^{i\omega' t_m}\rangle + \mathrm{e}^2 \sum_k \langle e^{-i(\omega-\omega')t_k}\rangle. \tag{53}$$

Now consider the first sum. Since all t_k are independent the average of the product factors

$$\sum_k \sum_{m\neq k} \langle e^{-i\omega t_k}\ e^{i\omega' t_m}\rangle = \sum_k \sum_{m\neq k} \langle e^{-i\omega t_k}\rangle\langle e^{i\omega' t_m}\rangle. \tag{54}$$

The last step is to complete this sum by adding and subtracting the diagonal terms to it. Now we can write the final result for the auto-correlation function as

$$\langle\tilde{i}(\omega)\ \tilde{i}^*(\omega')\rangle = \mathrm{e}^2 \sum_k \langle e^{-i\omega t_k}\rangle\ \sum_m \langle e^{i\omega' t_m}\rangle \tag{55}$$

$$+\ \mathrm{e}^2 \sum_k \left[\langle e^{-i(\omega-\omega')t_k}\rangle - \langle e^{-i\omega t_k}\rangle\langle e^{i\omega' t_k}\rangle\right]. \tag{56}$$

Note that the first part of the auto-correlation function is an uncorrelated part. Namely,

$$\mathrm{e}^2 \sum_k \langle e^{-i\omega t_k}\rangle \sum_m \langle e^{i\omega' t_m}\rangle = \langle\tilde{i}(\omega)\rangle\ \langle\tilde{i}^*(\omega')\rangle. \tag{57}$$

This term accounts for usual energy dissipation in the conducting medium.

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