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LIGO calibration accuracy

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We derive formula for the calibration accuracy required for the initial LIGO detectors for classical matched filtering purposes. The (happy) conclusion is that to first order, amplitude and phase errors in the calibration do not change the measured Signal-to-Noise Ratio (SNR). Thus, the requirements placed on calibration accuracy by the detection problem (in a single detector, using matched filtering) are not severe. However such errors could affect the correlation of signals between different detectors.

I. INTRODUCTION

The LIGO IFO has a voltage output $v(t)$ which in the Fourier domain can be written as $\tilde{v}(f)$. The range of frequency here is $f \in (-\infty, \infty)$. This is related to the differential displacement $\Delta L(t)$ of the interferometer arms via the *calibration equation:*

$$\tilde{\Delta} L(f) = L R(f) \tilde{v}(f). \tag{1.1}$$

Here $L = 4$ km is the length of one arm, and $R(f)$ is the calibration function, whose units are 1/voltage or equivalently strain/voltage. Because both the voltage and the arm displacement are real, this satisfies $\tilde{R}(-f) = \tilde{R}^*(f)$. This function is simply the transfer function of the interferometer. The differential strain may be written in frequency space as $\tilde{h}(f) = \tilde{\Delta} L/L = R(f)\tilde{v}(f)$.

Let us distinguish between the amplitude and the phase of $R(f)$ by writing

$$R(f) = \rho(f) \mathrm{e}^{i\theta(f)}, \tag{1.2}$$

where $\rho > 0$ and $-\pi \le \theta < \pi$ are real functions. The reality condition above implies that $\rho(f) = \rho(-f)$ and $\theta(f) = -\theta(-f)$

Let us now consider the ways in which errors in R, or equivalently, errors in ρ and θ, affect the SNR achievable in matched filtering experiments. Let us denote the error in measuring R by

$$\delta R = \delta\rho \mathrm{e}^{i\theta} + \rho \mathrm{e}^{i\theta} i \delta\theta, \tag{1.3}$$

where both $\delta\rho$ and $\delta\theta$ are functions of frequency. Once again, the reality conditions imply that $\delta\rho(f) = \delta\rho(-f)$ and $\delta\theta(f) = -\delta\theta(-f)$.

These errors affect both the estimate of the signal, and the estimate of the noise. Note that we assume here that the IFO is stable over time, in other words, that there is a measured transfer function $R(f)$ that does not change with time, and that the error in this transfer function $\delta R(f)$ arises because the calibration process is not perfectly accurate (for example, the geometry of a driving coil or the field gradient of a permanent magnet are not exactly determined). We will let $R(f)$ denote the *true* value of the response function, and $R(f) + \delta R(f)$ denote the *measured* value of the response function. In general, if W is any quantity, we will use

$$\delta W \equiv W_{\text{measured}} - W_{\text{true}} \tag{1.4}$$

to denote this difference.

The signal output of a matched filter may be written as

$$S = \int df \tilde{h}(f) \tilde{Q}(f), \tag{1.5}$$

where here and elsewhere all frequency integrals are from $-\infty$ to ∞. The *optimal filter* for the waveform that we are searching for is denoted by $\tilde{Q}$. Let us assume that this waveform is known exactly (it is already known to high-enough post-Newtonian accuracy for the "detection" problem, so this is a reasonable approximation) and denote it by $\tilde{h}_I(f)$, where the subscript "I" stands for "inspiral". This waveform satisfies the reality condition $\tilde{h}_I(f) = \tilde{h}_I^*(-f)$.

The optimal choice of filter function is given by

$$\tilde{Q}(f) = \frac{\tilde{h}_I^*(f)}{P(f)} \tag{1.6}$$

where $P(f)$ is the noise power spectrum of the interferometer (in units of strain^2/Hz). (Note that shifting the coalescence or arrival time of the inspiral waveform to $t = t_0$ simply introduces a factor of $\exp(-2\pi i f t_0)$ into $\tilde{Q}$, and the signal S then depends upon t_0). Here and elsewhere all power spectra are "two-sided". This noise power spectrum is determined experimentally from a precise measurement of the IFO output voltage power spectrum, and therefore any error in $R(f)$ gives rise to an error in the estimate of $P(f)$. Since $|\tilde{h}(f)|^2 = |R(f)|^2|\tilde{v}(f)|^2 = \rho^2(f)|\tilde{v}(f)|^2$ this arises only from errors in determining the magnitude of R and not from errors in the phase. (Raab has pointed out that if the interferometer response is not stable, as I have assumed, then a changing phase response with time might also affect the estimates of the power spectrum.) In particular,

$$P(f) = \rho^2(f)P_v(f), \tag{1.7}$$

where $P_v(f)$ denotes the voltage power spectrum of the IFO output.

Using these relations, *the signal that is experimentally determined, using the measured (but erroneous) calibration to construct both the measured strain and the optimal filter* may be expressed in terms of the IFO output as

$$S_{\text{measured}} = \int df \frac{\mathrm{e}^{i(\theta(f)+\delta\theta(f))}\tilde{v}(f)\tilde{h}_I^*(f)}{(\rho(f)+\delta\rho(f))P_v(f)}. \tag{1.8}$$

In comparison to the true signal which would have been constructed if there were no measurement errors, this differs by an amount

$$\delta S = \int df \left(i\delta\theta - \frac{\delta\rho}{\rho}\right) \frac{\mathrm{e}^{i\theta(f)}\tilde{v}(f)\tilde{h}_I^*(f)}{\rho(f)P_v(f)}. \tag{1.9}$$

The expected value of this quantity, in the presence of the signal, may be easily determined. In the presence of a signal the expected value of the IFO output is given by $\rho(f)\mathrm{e}^{i\theta(f)}\langle\tilde{v}(f)\rangle = \tilde{h}_I(f)$. Thus, in constructing the expected value of the error $\langle\delta S\rangle$ one may set

$$\langle\tilde{v}(f)\rangle = \frac{\tilde{h}_I(f)}{\rho(f)}\mathrm{e}^{-i\theta(f)}. \tag{1.10}$$

Note that this equation holds for $\langle\tilde{v}\rangle$ in both "true" and "measured" quantities; the universe does not care about the errors δR! So in the presence of the signal, the expected error arising from errors in calibration is

$$\langle\delta S\rangle = \int df \left(i\delta\theta - \frac{\delta\rho}{\rho}\right) \frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}. \tag{1.11}$$

We will make use of this equation shortly.

The quantity which is actually measured in an experiment is the ratio of signal to noise. The noise is defined by

$$N \equiv S - \langle S\rangle, \tag{1.12}$$

and has variance

$$\langle N^2\rangle_{\text{true}} = \int df \frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}. \tag{1.13}$$

Errors in the calibration affect the noise, though only through the amplitude ρ. Because of the calibration error, the measured noise^2 is

$$\langle N^2 \rangle_{\rm measured} = \int df \frac{|\tilde{h}_I(f)|^2}{(\rho(f)+\delta\rho(f))^2 P_v(f)}. \tag{1.14}$$

The difference between these is the error in the noise^2 arising from the calibration error:

$$\delta\langle N^2 \rangle = \int df \left(-2\frac{\delta\rho}{\rho}\right) \frac{|\tilde{h}_I(f)|^2}{\rho^2(f) P_v(f)} \tag{1.15}$$

From here on, we drop the angle brackets - which denote ensemble averages - to simplify the notation in what follows. The (expected value of the) loss in the signal-to-noise ratio due to calibration errors may now be determined. It is given by:

$$\begin{aligned}
\delta\left(\frac{S}{N}\right) &= \left(\frac{S}{N}\right)\left(\frac{\delta S}{S} - \frac{\delta N}{N}\right) \\
&= \left(\frac{S}{N}\right)\left(\frac{\delta S}{S} - \frac{1}{2}\frac{\delta(N^2)}{N^2}\right) \\
&= \left(\frac{S}{N}\right)\left[\frac{\int df \left(i\delta\theta - \frac{\delta\rho}{\rho}\right)\frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}}{\int df \frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}} - \frac{1}{2}\frac{\int df \left(-2\frac{\delta\rho}{\rho}\right)\frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}}{\int df \frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}}\right] \\
&= \left(\frac{S}{N}\right)\frac{\int df\, (i\delta\theta)\frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}}{\int df \frac{|\tilde{h}_I(f)|^2}{\rho^2(f)P_v(f)}}
\end{aligned} \tag{1.16}$$

Now consider the numerator of this expression. This integral in the numerator vanishes because the integral is from $f=-\infty$ to ∞, $\delta\theta(f)$ is an odd function of frequency, and $|\tilde{h}_I(f)|^2/\rho^2 P_v(f)$ is an even function of frequency! Thus, to lowest order in $\delta\rho$ and $\delta\theta$, the errors in SNR arising from calibration errors *vanish*. These errors only appear at second order:

$$\delta\left(\frac{S}{N}\right) = {\rm O}(\delta\theta)^2 + {\rm O}(\frac{\delta\rho}{\rho})^2 + {\rm O}(\delta\theta\,\frac{\delta\rho}{\rho}). \tag{1.17}$$

The good news is as follows: *To lowest order, the errors in the calibration amplitude $\delta\rho$ and phase $\delta\theta$ drop out of the S/N ratio.* Since these errors only enter at second order, the constraints on the calibration for the binary inspiral detection problem are probably not very severe or difficult to achieve. For example, suppose that we do not wish to lose more than 10% of the expected event rate, as a result of calibration errors. Because the event rate is proportional to the volume of the universe which is observed by the detector, and the SNR is inversely proportional to the distance to the source, this requirement implies that we should not lose more than 3.5% of the SNR, which in turn implies that $\delta\rho/\rho \lesssim \sqrt{0.035} = 19\%$, and $\delta\theta \lesssim \sqrt{0.035} = 0.19$ radians $= 11$ degrees. So for the detection of binary inspiral, errors in the calibration are probably not a significant concern.

Let me temper this optimism slightly. Although the amplitude and phase errors do not affect the detection problem to lowest order, they *do* have an effect on the subsequent science at linear order; in particular the amplitude errors linearly affect our ability to infer the distance to a source. Ideally, these errors should be small enough (a few percent) so that the distance to a source can be determined with at most a few percent error. This would help to locate sources, and also to pin down the Hubble expansion rate.

The phase errors $\delta\theta$ also have a linear effect on the subsequent science, and affect the detection problem in a different way. The matched filtering to search for binary inspiral will maximize the output of a pair of filters corresponding to coalescence phases of 0 and $\pi/2$, by maximizing over the coalescence phase. The effect of this maximization is that any constant phase error $\delta\theta$ will *not* reduce the SNR to lowest order, but it will result in an error in the determination of the coalescence phase and could reduce the likelyhood of recognizing the signal in a correlation experiment with another detector. In similar fashion, a phase error for which $\delta\theta$ is a linear function of frequency will also not reduce the SNR to lowest order, but will result in a shift in the coalescence time t_0 of the binary system. Although it does not reduce the detection probability in a single detector to lowest order, it does increase the odds that the chirp will fail to correlate correctly in time with the same chirp observed in a different detector. This type of phase error will also decrease the precision with which a given source can be located on the sky; the sky location lies on a cone defined by the difference in arrival times at two separated detector sites.

For these reasons, it would be desirable if the errors in the phase $\delta\theta$ have no constant offset, and no linear trend. Although this does not (to lowest order) increase the SNR, it will assist correlating the signals with those of other detectors.

In this short memo, I have only examined the effects of calibration errors on the detection problem for binary inspiral. Let me conclude with a few (less rigorous) comments about the effects of calibration error on searches for other types of sources:

1. **Stochastic background detection:** In this case, the optimal filter has a bandpass from about 40 Hz to 300 Hz. Calibration errors outside of this range are not important. Because these experiments are done by correlating the outputs of a detector pair, what matters most is the relative error in the calibration of the pair. For example, in correlating two signals at angular frequency ω with the optimal filter, the ratio of measured signal-to-noise (proportional to $h_1 h_2$) to true signal-to-noise is $\cos(\delta\theta_1 - \delta\theta_2)$ which is second order in the phase difference $\delta\theta_1 - \delta\theta_2$. In similar fashion, the loss of signal-to-noise due to magnitude errors is second order in the error of $\rho_1 \rho_2$.
2. **Searchs for CW (pulsar) sources**: In this case, the frequency range of interest is extremely narrow; of order one Hz. Errors in the magnitude and phase do not matter, provided that these errors do not vary significantly over a 1 Hz bandwidth. In other words, what is important here is keeping the variations of $\delta\theta$ and $\delta\rho$ small over a 1 Hz bandwidth; their actual values are irrelevant.
3. **Searches for binary inspiral using time-frequency methods**: In this case, once again, errors in the calibration probably are not terribly important. Typically these methods construct points in time-frequency space and then use pattern recognition techniques to detect curves in the sets of points which might correspond to binary inspiral. These methods should not be very sensitive to either phase or magnitude errors. In fact, one can often reconstruct the SNR of the optimal filter from a properly weighted set of the time-frequency space measurements, which suggests that the errors in these techniques might also be second order in the calibration errors.
4. **Searches for transient sources**: The waveforms of these types of sources (supernovae, for example) are not well understood. Hence the techniques that will probably be used are not likely to be significantly affected by calibration magnitude or phase errors.
5. **Searchs for black-hole coalescence and ringdown**: The search for black-hole ringdown is a special case of optimal filtering, so the errors should only enter at second order. The coalescence signal is poorly enough understood so that this should just be a special case of the "transient source".

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